PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 11, November 2007, Pages 3613–3619
S 0002-9939(07)08874-0
Article electronically published on June 29, 2007

# LINEAR MAPS PRESERVING THE SET OF FREDHOLM OPERATORS

MOSTAFA MBEKHTA

(Communicated by Joseph A. Ball)

Abstract. Let $H$ be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on $H$. In this paper we characterize surjective linear maps $\phi : F\mathcal{B}(H) \to \mathcal{B}(H)$ preserving the set of Fredholm operators in both directions. As an application we prove that $\phi$ preserves the essential spectrum if and only if the ideal of all compact operators is invariant under $\phi$ and the induced linear map $\varphi$ on the Calkin algebra is either an automorphism, or an anti-automorphism. Moreover, we have, either $ind(\phi(T)) = ind(T)$ or $ind(\phi(T)) = -ind(T)$ for every Fredholm operator $T$.

## 1. Introduction

Let $H$ be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on $H$ and $\mathcal{K}(H) \subset \mathcal{B}(H)$ be the closed ideal of all compact operators. For an operator $T \in \mathcal{B}(H)$ we write $T^{tr}$ for the transpose of $T$ with respect to an arbitrary but fixed orthonormal basis of $H$, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. The spectrum of $T$ is denoted by $\sigma(T)$.

An operator $T \in \mathcal{B}(H)$ is called *Fredholm* if $\mathrm{R}(T)$ is closed and

$$\max\{\dim \mathrm{N}(T), \mathrm{codim}\mathrm{R}(T)\} < \infty.$$

Let $\Phi(H) \subset \mathcal{B}(H)$ be the set of all Fredholm operators. We denote the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ by $\mathcal{C}(H)$. Let $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ be the quotient map. It is well known (Atkinson's theorem) that :

$$T \in \Phi(H) \iff \pi(T) \text{ is invertible in } \mathcal{C}(H). \tag{1.1}$$

For Fredholm theory and Calkin algebras, see for instance, [9], [12].

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras over the complex field. A bijective linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a *Jordan isomorphism* if $\phi(a^2) = (\phi(a))^2$ for every $a \in \mathcal{A}$, or equivalently $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all $a$ and $b \in \mathcal{A}$. It is obvious that every isomorphism and every anti-isomorphism is a Jordan isomorphism (a bijective linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called an anti-isomorphism if $\phi(ab) = \phi(b)\phi(a)$ for all $a$ and $b$ in $\mathcal{A}$).

---

Received by the editors April 4, 2006 and, in revised form, August 18, 2006.
2000 *Mathematics Subject Classification.* Primary 47B48, 47A10, 46H05.
*Key words and phrases.* Fredholm operators, Calkin algebra, linear preservers.
The work of the author is partially supported by "Action integrée Franco-Marocaine, Programme Volubilis, N° MA/03/64" and by I+D MEC project MTM 2004-03882.

In the last two decades there has been considerable interest in the so-called linear preserver problems (see the survey articles [3], [5], [7], [15], [4], [13]). The goal of studying linear preservers is to give structural characterizations of linear maps on algebras having some special properties such as leaving invariant a certain subset of the algebra, or leaving invariant a certain function on the algebra. For instance, in this paper the author considered surjective linear maps leaving invariant the set of Fredholm operators in $\mathcal{B}(H)$, and leaving invariant the essential spectrum of each element in $\mathcal{B}(H)$.

One of the most famous problems in this direction is Kaplansky's problem [16]: Let $\phi$ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that $\phi$ is a Jordan isomorphism?

This problem was first solved in the finite-dimensional case. J. Dieudonné [11] and Marcus and Purves [19] proved that every unital invertibility-preserving linear map on a complex matrix algebra is either an inner automorphism or an inner anti-automorphism.

This result was later extended to the algebra of all bounded linear operators on a Banach space by A.R. Sourour [22] and to von Neumann algebras by B. Aupetit [2].

Many other linear preserver problems, such as the problem of characterizing linear maps preserving idempotents or nilpotents or commutativity, that were first solved for matrix algebras, have recently been extended to the infinite-dimensional case.

Recently, M. Mbekhta, L. Rodman and P. Šemrl [18] studied bijective linear maps on $\mathcal{B}(H)$ that preserve generalized invertibility in both directions. Observe that every $n \times n$ complex matrix has a generalized inverse, and therefore, every linear map on a matrix algebra preserves generalized invertibility in both directions. So, we have here an example of a linear preserver problem which makes sense only in the infinite-dimensional case.

In this paper, we study linear maps preserving the set of Fredholm operators in $\mathcal{B}(H)$. Here also, we have an example of a linear preserver problem which makes sense only in the infinite-dimensional case. Indeed, it is easy to see that $\dim H < \infty \iff \Phi(H) = \mathcal{B}(H)$.

## 2. Linear maps preserving the set of Fredholm operators

We say that a linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves the set of Fredholm operators in both directions* if $T \in \Phi(H) \iff \phi(T) \in \Phi(H)$.

We say that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is surjective up to compact operators if for every $T \in \mathcal{B}(H)$ there exists $T' \in \mathcal{B}(H)$ such that $T - \phi(T') \in \mathcal{K}(H)$. It is clear that if $\phi$ is surjective, then it is surjective up to compact operators.

2.1. **Theorem.** *Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators. Then the following conditions are equivalent:*

*(1) $\phi$ preserves the set of Fredholm operators in both directions;*

*(2) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$, is the composition of either an automorphism or an anti-automorphism and left multiplication by an invertible element in $\mathcal{C}(H)$.*

*In particular $\varphi$ is continuous.*

The proof of Theorem 2.1 uses the following lemmas.

2.2. **Lemma.** *Let $K \in \mathcal{B}(H)$ be an operator. Then*

$$K \in \mathcal{K}(H) \iff T + K \in \Phi(H), \ \ \forall T \in \Phi(H).$$

*Proof.* If $K \in \mathcal{K}(H)$, then $\pi(K) = 0$. Hence, by (1.1), for all $T \in \Phi(H)$, $\pi(T+K) = \pi(T)$ is invertible in $\mathcal{C}(H)$, and consequently $T + K \in \Phi(H)$.

Conversely, suppose that $T + K \in \Phi(H)$, for every $T \in \Phi(H)$. Then $\pi(T) + \pi(K) = \pi(T + K)$ is invertible for all $\pi(T)$ invertible in $\mathcal{C}(H)$. Thus, from [17, Theorem 2.5], it follows that $\pi(K)$ belongs to the radical of $\mathcal{C}(H)$. Finally, since $\mathcal{C}(H)$ is a semi-simple Banach algebra, $\pi(K) = 0$ and $K$ is compact. □

2.3. **Lemma.** *Suppose that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective up to compact operators linear map. If $\phi$ preserves the set of Fredholm operators in both directions, then we have*

*(i) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$;*
*(ii)* $\mathrm{N}(\phi) \subseteq \mathcal{K}(H)$.

*Proof.* ($i$) Let $K \in \mathcal{K}(H)$; we will prove that $\phi(K)$ is in $\mathcal{K}(H)$. Suppose $T \in \Phi(H)$. Since $\phi$ is surjective up to compact operators, there exists $T' \in \mathcal{B}(H)$ and $K' \in \mathcal{K}(H)$, for which $T = \phi(T') + K'$. Hence, $\phi(T') = T - K' \in \Phi(H)$ and thus $T' \in \Phi(H)$. Now, $T + \phi(K) = \phi(T') + K' + \phi(K) = \phi(T' + K) + K' \in \Phi(H)$. Thus, for every $T \in \Phi(H)$, $\quad T + \phi(K) \in \Phi(H)$. It follows from Lemma 2.2 that $\phi(K) \in \mathcal{K}(H)$, which establishes that $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$.

($ii$) If $K \in \mathrm{N}(\phi)$ and $T \in \Phi(H)$, then $\phi(T + K) = \phi(T) \in \Phi(H)$. Thus, for all $T \in \Phi(H)$, $T + K \in \Phi(H)$. Hence by Lemma 2.2, $K \in \mathcal{K}(H)$. □

*Proof of Theorem 2.1.* (1) $\Longrightarrow$ (2): By Lemma 2.3 ($i$), we have that $\mathcal{K}(H)$ is invariant under $\phi$; thus $\phi$ induces a linear map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ such that $\varphi \circ \pi = \pi \circ \phi$. Clearly, $\varphi$ is surjective, since $\phi$ is surjective up to compact operators. Also, because $I \in \Phi(H)$, $\phi(I)$ is Fredholm and $\pi(\phi(I))$ is invertible in $\mathcal{C}(H)$. Hence there exists a Fredholm operator $S \in \mathcal{B}(H)$ such that $\pi(S)\pi(\phi(I)) = \pi(\phi(I))\pi(S) = \pi(I)$. Define the map

$$\psi = L_{\pi(S)} \circ \varphi : \mathcal{C}(H) \to \mathcal{C}(H),$$

where $L_{\pi(S)}$ is the left multiplication by $\pi(S)$ . Then $\psi(\pi(T)) = \pi(S\phi(T))$ for all $T \in \mathcal{B}(H)$. Now it is easy to see that $\psi$ is surjective, $\psi(\pi(I)) = \pi(I)$ (that is, $\psi$ is unital) and preserves the set of invertible elements of $\mathcal{C}(H)$. Therefore the continuity of $\psi$ follows from [1, Corollary 5.5.2].

We prove now that $\psi$ is injective. Since $\pi(S)$ is invertible in $\mathcal{C}(H)$,

$$\mathrm{N}(\psi) = \mathrm{N}(\varphi) = \pi(\mathrm{N}(\phi)),$$

and by part ($ii$) of Lemma 2.3, we obtain $\mathrm{N}(\psi) = \{0\}$. Thus $\psi$ is injective. Consequently, $\psi$ is bijective.

Now, by [2, Theorem 1.2], $\psi$ preserves the set of orthogonal idempotents in $\mathcal{C}(H)$. It is easy to see that $\mathcal{C}(H)$ is a $C^*$-algebra of real rank zero [8]; that is, the set of all finite real linear combinations of orthogonal projections is dense in the set of all Hermitian elements of $\mathcal{C}(H)$. Then, by [13, Theorem 4.1], $\psi$ is a Jordan automorphism. Recall further that an algebra $\mathcal{A}$ is a prime algebra if for every pair $a, b \in \mathcal{A}$, the relation $a\mathcal{A}b = \{0\}$ implies that $a = 0$ or $b = 0$. Standard arguments yield that $\mathcal{C}(H)$ is a prime algebra. It is well known that every Jordan automorphism of a prime algebra is an automorphism or an anti-automorphism [14]. Thus, $\psi$ is either an automorphism, or an anti-automorphism, as desired.

Finally, since $\pi(S)$ is invertible in $\mathcal{C}(H)$, we have

$$\varphi = L^{-1}_{\pi(S)} \circ \psi = L_{\pi(S)^{-1}} \circ \psi = L_{\pi(\phi(I))} \circ \psi.$$

(2) $\Longrightarrow$ (1): Notice that, from the hypothesis (2), it follows immediately that $\varphi$ preserves the invertible elements of $\mathcal{C}(H)$. Thus, using (1.1), the following equivalences hold:

$$\begin{array}{rcl} T \in \Phi(H) & \Longleftrightarrow & \pi(T) \text{ is invertible in } \mathcal{C}(H) \\ & \Longleftrightarrow & \varphi(\pi(T)) = \pi(\phi(T)) \text{ is invertible in } \mathcal{C}(H) \\ & \Longleftrightarrow & \phi(T) \in \Phi(H). \end{array}$$

Therefore $\phi$ preserves the set of Fredholm operators in both directions. Hence the proof of the theorem is complete. □

2.4. *Remark.* For a fixed $A, B \in \Phi(H)$ and $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$ a linear map, consider the following two mappings defined by $T \mapsto ATB + \chi(T)$ and $T \mapsto AT^{tr}B + \chi(T)$, $\quad T \in \mathcal{B}(H)$. Then both mappings are surjective up to compact operators and preserve the set of Fredholm operators in both directions.

2.5. **Question.** *Does a map satisfying the hypothesis of Theorem 2.1 necessarily have the same form as the mappings in the above remark?*

## 3. Essential spectrum

If $T \in \mathcal{B}(H)$ we define the *essential spectrum* of $T$, denoted $\sigma_e(T)$, as the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{C}(H)$ (i.e., $\sigma_e(T) = \sigma(\pi(T))$). Then, from (1.1), we have

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \quad T - \lambda I \notin \Phi(H)\}.$$

3.1. **Lemma.** *Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators. Then the following are equivalent:*

*(i) $\phi$ preserves the set of Fredholm operators in both directions and $\phi(I) = I - K$, where $K \in \mathcal{K}(H)$;*

*(ii) $\sigma_e(\phi(T)) = \sigma_e(T)$ for all $T \in \mathcal{B}(H)$.*

*Proof.* $(i) \Longrightarrow (ii)$: Since $\pi(\phi(I)) = \pi(I)$ it follows from Theorem 2.1 that the induced linear map $\varphi$ on the Calkin algebra is an automorphism or an anti-automorphism. Consequently, for all $T \in \mathcal{B}(H)$, we have

$$\sigma_e(\phi(T)) = \sigma(\pi(\phi(T))) = \sigma(\varphi(\pi(T))) = \sigma(\pi(T)) = \sigma_e(T).$$

$(ii) \Longrightarrow (i)$: Assume $(ii)$ holds. Then clearly $\phi$ preserves the set of Fredholm operators in both directions. We only need to show that $\phi(I) - I \in \mathcal{K}(H)$. Put $K = \phi(I) - I$. Let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H)$ and $K' \in \mathcal{K}(H)$ for which $T = \phi(T') + K'$ ($\phi$ is surjective up to compact operators). Then $\sigma_e(T) = \sigma_e(T')$ and

$$\begin{array}{rcl} \sigma_e(T+K) & = & \sigma_e(T + \phi(I) - I) = \sigma_e(T + \phi(I)) - 1 \\ & = & \sigma_e(\phi(T') + \phi(I) + K') - 1 = \sigma_e(\phi(T' + I)) - 1 \\ & = & \sigma_e(T' + I) - 1 = \sigma_e(T') \\ & = & \sigma_e(T). \end{array}$$

This gives $\sigma_e(T+K) = \sigma_e(T)$ for all $T \in \mathcal{B}(H)$. It follows from [24, Corollary 3.5] (see also [1], page 95) that $\pi(K)$ is in the radical of $\mathcal{C}(H)$. Hence $\pi(K) = 0$, since $\mathcal{C}(H)$ is a semi-simple Banach algebra. This proves that $K$ is compact. $\square$

We say that a linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves the essential spectrum* if $\sigma_e(\phi(T)) = \sigma_e(T)$ for every $T \in \mathcal{B}(H)$.

The following theorem is an immediate consequence of Lemma 3.1 and Theorem 2.1.

3.2. **Theorem.** *Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators. Then the following are equivalent:*

*(1) $\phi$ preserves the set of Fredholm operators in both directions and $\phi(I) = I - K$, where $K \in \mathcal{K}(H)$;*

*(2) $\phi$ preserves the essential spectrum;*

*(3) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$, is either an automorphism or an anti-automorphism.*

*In particular, $\varphi$ is continuous and unital.*

Let $ind : \Phi(H) \to \mathbb{Z}$ be the *"classical" index*, defined by

$$ind(T) = \dim N(T) - \mathrm{codim} R(T), \quad T \in \Phi(H).$$

We denote by $\mathcal{G} \subset \mathcal{C}(H)$ the group of invertible elements in the Calkin algebra. Then, by Atkinson's theorem, we have $\mathcal{G} = \pi(\Phi(H))$. Let $\mathcal{G}_0$ be the connected component of the identity in $\mathcal{G}$. It is well known (see [12, Proposition 2.9]) that $\mathcal{G}_0$ is an open and closed normal subgroup of $\mathcal{G}$ and the quotient group, *the abstract index group*, $\mathcal{G}/\mathcal{G}_0$ is discrete. The quotient homomorphism $\gamma$ from $\mathcal{G}$ onto $\mathcal{G}/\mathcal{G}_0$ is called *the abstract index* of $\mathcal{C}(H)$. Note that [12, Theorem 5.35] shows that the two notions, "classical" index and abstract index, are essentially the same.

3.3. **Theorem.** *Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective up to compact operators linear map. Suppose that $\phi$ preserves the essential spectrum. Then, either*

$$ind(\phi(T)) = ind(T) \quad \textit{or} \quad ind(\phi(T)) = -ind(T), \qquad \textit{for all } T \in \Phi(H).$$

*Proof.* According to the above theorem, $\phi$ induces the map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, either an automorphism, or an anti-automorphism. Clearly, we have

$$\varphi(\mathcal{G}) = \mathcal{G}.$$

Moreover, from [12, Theorem 2.14], we have $\mathcal{G}_0 = \mathrm{Exp}(\mathcal{C}(H))$, where $\mathrm{Exp}(\mathcal{C}(H)) = \{\prod_{i=0}^n e^{t_i} : \quad n \in \mathbb{N}, \quad t_i \in \mathcal{C}(H), \quad i \in [0,n]\}$ is the subgroup generated by the exponential elements of $\mathcal{C}(H)$. Now, it is not difficult to show that

$$\varphi(\mathcal{G}_0) = \mathcal{G}_0.$$

Thus $\varphi$ induces a quotient automorphism or anti-automorphism

$$\tilde{\varphi} : \mathcal{G}/\mathcal{G}_0 \to \mathcal{G}/\mathcal{G}_0.$$

There is no loss of generality in assuming that $\varphi$ is an automorphism from the multiplicative group $\mathcal{G}$ onto $\mathcal{G}$, since otherwise we can replace $\varphi$ by the automorphism $\pi(T) \mapsto (\varphi(\pi(T))^*, \quad \pi(T) \in \mathcal{G}$.

On the other hand, there is an isomorphism $\alpha$ from the multiplicative group $\mathcal{G}/\mathcal{G}_0$ onto the additive group $\mathbb{Z}$ (see [12], page 134).

Then it is apparent that the following diagram is commutative:

$$\begin{array}{ccccccc}
\Phi(H) & \xrightarrow{\pi} & \mathcal{G} & \xrightarrow{\gamma} & \mathcal{G}/\mathcal{G}_0 & \xrightarrow{\alpha} & \mathbb{Z} \\
\phi\big\downarrow & & \varphi\big\downarrow & & \tilde{\varphi}\big\downarrow & & \big\downarrow\beta \\
\Phi(H) & \xrightarrow[\pi]{} & \mathcal{G} & \xrightarrow[\gamma]{} & \mathcal{G}/\mathcal{G}_0 & \xrightarrow[\alpha]{} & \mathbb{Z}
\end{array}$$

and so we have

$$\varphi \circ \pi = \pi \circ \phi, \quad \tilde{\varphi} \circ \gamma = \gamma \circ \varphi \quad \text{and} \quad \beta = \alpha \circ \tilde{\varphi} \circ \alpha^{-1}.$$

Thus, $\beta : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ is an automorphism. But, it is well known that the automorphisms from the additive group $\mathbb{Z}$ onto $\mathbb{Z}$ are of the form $\iota$ or $-\iota$, where $\iota(n) = n,\ n \in \mathbb{Z}$. Therefore, either $\beta(n) = n$ or $\beta(n) = -n$ for every $n \in \mathbb{Z}$.

Observe first that if $T \in \Phi(H)$, then

$$ind(T) = (\alpha \circ \gamma \circ \pi)(T).$$

Now, by the above observation, if $T \in \Phi(H)$, we have $\phi(T) \in \Phi(H)$ and

$$\begin{aligned}
ind(\phi(T)) &= (\alpha \circ \gamma \circ \pi)(\phi(T)) \\
&= (\alpha \circ \gamma \circ \pi \circ \phi)(T) \\
&= (\alpha \circ \gamma \circ \varphi \circ \pi)(T) \\
&= (\alpha \circ \tilde{\varphi} \circ \gamma \circ \pi)(T) \\
&= (\beta \circ \alpha \circ \gamma \circ \pi)(T) \\
&= \beta[(\alpha \circ \gamma \circ \pi)(T)].
\end{aligned}$$

Thus, if $\beta(n) = n$ for every $n \in \mathbb{Z}$, then

$$ind(\phi(T)) = (\alpha \circ \gamma \circ \pi)(T) = ind(T),$$

and, if $\beta(n) = -n$ for every $n \in \mathbb{Z}$, then

$$ind(\phi(T)) = -(\alpha \circ \gamma \circ \pi)(T) = -ind(T).$$

Consequently, either

$$ind(\phi(T)) = ind(T) \quad \text{or} \quad ind(\phi(T)) = -ind(T), \qquad \text{for all } \ T \in \Phi(H).$$

Hence the proof of the theorem is complete. □

We conclude this paper by a natural conjecture that we have been unable to answer.

3.4. **Conjecture.** *Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators.*

*Then the following conditions are equivalent:*

*(I) $\phi$ preserves the essential spectrum;*

*(II) there exists $\psi : \mathcal{B}(H) \to \mathcal{B}(H)$ either an automorphism or an anti-automorphism and there exists $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$ a linear map such that $\phi(T) = \psi(T) + \chi(T)$ for every $T \in \mathcal{B}(H)$;*

*(III) either*

*(i) $\phi(T) = ATA^{-1} + \chi(T)$ for every $T \in \mathcal{B}(H)$, where $A$ is an invertible operator in $\mathcal{B}(H)$ and $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$ is a linear map, or*

*(ii) $\phi(T) = BT^{tr}B^{-1} + \chi(T)$ for every $T \in \mathcal{B}(H)$, where $B$ is an invertible operator in $\mathcal{B}(H)$ and $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$ is a linear map.*

3.5. *Remark.* The implications $(II) \iff (III) \Longrightarrow (I)$ are true. Indeed, the implications $(III) \Longrightarrow (II) \Longrightarrow (I)$ are clear and $(II) \Longrightarrow (III)$ follows from the fundamental isomorphism theorem ([20, Theorem 2.5.19], see also [10]). Therefore, it remains to prove the implication $(I) \Longrightarrow (II)$ or $(I) \Longrightarrow (III)$.

## References


[1] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991. MR1083349 (92c:46001)

[2] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc. **62** (2000), 917-924. MR1794294 (2001h:46078)

[3] B. Aupetit, *Sur les transformations qui conservent le spectre*, Banach Algebra '97, Walter de Gruyter, Berlin, (1998), 55-78. MR1656598 (99m:46114)

[4] L. Beasley, C.-K. Li, and S. Pierce, *Miscellaneous preserver problems. A survey of linear preserver problems*, Linear and Multilinear Algebra **33** (1992), 109–119. MR1346786

[5] M. Brešar and P. Šemrl, *Linear maps preserving the spectral radius*, J. Funct. Anal. **142** (1996), 360-368. MR1423038 (97i:47070)

[6] M. Brešar and P. Šemrl, *Linear preservers on* $\mathcal{B}(X)$, Banach Center Publications **38** (1997), 49-58. MR1457000 (99c:47044)

[7] M. Brešar and P. Šemrl, *Invertibility preserving maps preserve idempotents*, Michigan Math. J. **45** (1998), 483-488. MR1653251 (2000j:47119)

[8] L.G. Brown and G.K. Pedersen, $C^*$*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131-149. MR1120918 (92m:46086)

[9] S.R. Caradus, W.E. Pfaffenberger and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*, Marcel Dekker, New York, 1974. MR0415345 (54:3434)

[10] P.R. Chernoff, *Representations, automorphisms, and derivations of some operator algebras*, J. Funct. Anal. **12** (1973), 257-289. MR0350442 (50:2934)

[11] J. Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables* , Arch. Math (Basel) 1 (1949), 282-287. MR0029360 (10:586l)

[12] R.G. Douglas, *Banach Algebra Techniques in Operator Theory,* Academic Press, 1972. MR0361893 (50:14335)

[13] A. Guterman, C.-K. Li, and P. Šemrl, *Some general techniques on linear preserver problems*, Linear Algebra Appl. **315** (2000), 61-81. MR1774960 (2002a:15003)

[14] I.N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331-341. MR0076751 (17:938f)

[15] A. Jafarian and A.R. Sourour, *Spectrum preserving linear maps*, J. Funct. Anal. **66** (1986), 255-261. MR832991 (87m:47011)

[16] I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, Amer. Math. Soc., Providence, 1970. MR0312283 (47:845)

[17] A. Lebow and M. Schechter, *Semigroups of operators and measures of non-compactness*, J. Funct. Anal. **7** (1971), 1-26. MR0273422 (42:8301)

[18] M. Mbekhta, L. Rodman, and P. Šemrl, *Linear maps preserving generalized invertibility*, Integral Equations Operator Theory **55** (2006), 93-109. MR2226639

[19] M. Marcus and R. Purves, *Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions*, Canad. J. Math. **11** (1959), 383-396. MR0105425 (21:4167)

[20] C.E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton, 1960. MR0115101 (22:5903)

[21] A.R. Sourour, *The Gleason-Kahane-Żelazko theorem and its generalizations*, Banach Center Publications **30** (1994), 327-331. MR1285618 (95i:46066)

[22] A.R. Sourour, *Invertibility preserving linear maps on* $\mathcal{L}(X)$, Trans. Amer. Math. Soc. **348** (1996), 13-30. MR1311919 (96f:47069)

[23] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, John Wiley and Sons, 2nd ed., 1980. MR564653 (81b:46001)

[24] J. Zemànek, *Properties of the spectral radius in Banach algebras*, Banach Center Publications **8** (1982), 579-595. MR738318 (85d:46064)



Université de Lille I, UFR de Mathématiques, 59655 Villeneuve d'Ascq Cedex, France
*E-mail address*: `mostafa.mbekhta@math.univ-lille1.fr`