PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 11, November 2007, Pages 3515–3520
S 0002-9939(07)08890-9
Article electronically published on August 15, 2007

A SHARP RESULT ON m-COVERS

HAO PAN AND ZHI-WEI SUN

(Communicated by Wen-Ching Winnie Li)

ABSTRACT. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be a finite system of residue classes which forms an m-cover of $\mathbb{Z}$ (i.e., every integer belongs to at least m members of A). In this paper we show the following sharp result: For any positive integers $m_1, \ldots, m_k$ and $\theta \in [0, 1)$, if there is $I \subseteq \{1, \ldots, k\}$ such that the fractional part of $\sum_{s\in I} m_s/n_s$ is θ, then there are at least 2^m such subsets of $\{1, \ldots, k\}$. This extends an earlier result of M. Z. Zhang and an extension by Z. W. Sun. Also, we generalize the above result to m-covers of the integral ring of any algebraic number field with a power integral basis.

1. Introduction

For an integer a and a positive integer n, we simply let $a(n)$ represent the set $a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod n\}$. Following Sun [S95, S96] we call a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of such sets an *m-cover* of $\mathbb{Z}$ (where $m \in \{1, 2, 3, \ldots\}$) if every integer lies in at least m members of (1.1). We use the term *cover* (or covering system) instead of 1-cover. For problems and results in this area, the reader may consult [G04, pp. 383–390], [PS] and [S05]. P. Erdős [E97] once said: *"Perhaps my favorite problem of all concerns covering systems."*

Example 1.1. For each integer $m \geqslant 1$, there is an m-cover of $\mathbb{Z}$ that is not the union of two covers of $\mathbb{Z}$. To wit, we let $p_1, \ldots, p_r$ be distinct primes with $r \geqslant 2m-1$, and set $N = p_1 \cdots p_r$. Clearly $A_* = \{\prod_{s\in I} p_s(N)\}_{I\subseteq\{1,\ldots,r\},\,|I|\geqslant m}$ does not cover any integer relatively prime to N. Let $a_1, \ldots, a_n$ be the list of those integers in $\{0, 1, \ldots, N-1\}$ not covered by A_* with each occurring exactly m times. If $x \in \mathbb{Z}$ is covered by A_*, then $x \in \bigcap_{s\in I} 0(p_s)$ for some $I \subseteq \{1, \ldots, r\}$ with $|I| \geqslant m$. Therefore

$$A_0 = \{0(p_1), \ldots, 0(p_r), a_1(N), \ldots, a_n(N)\}$$

forms an m-cover of $\mathbb{Z}$. Suppose that $I_1 \cup I_2 = \{1, \ldots, r\}$, $J_1 \cup J_2 = \{1, \ldots, n\}$ and $I_1 \cap I_2 = J_1 \cap J_2 = \emptyset$. For $i = 1, 2$ let A_i be the system consisting of those $0(p_s)$ with $s \in I_i$ and those $a_t(N)$ with $t \in J_i$. We claim that A_1 or A_2 is not a cover of $\mathbb{Z}$. Without loss of generality, let us assume that $|I_1| \leqslant |I_2|$. Since $2|I_2| \geqslant |I_1| + |I_2| > 2(m-1)$, we have $|I_2| \geqslant m$ and hence $\prod_{s\in I_2} p_s$ is covered

Received by the editors January 3, 2006 and, in revised form, June 3, 2006 and August 25, 2006.

2000 *Mathematics Subject Classification.* Primary 11B25; Secondary 11B75, 11D68, 11R04.

The second author is responsible for communications and is supported by the National Science Fund for Distinguished Young Scholars (No. 10425103) in China.

by A_*. Therefore $\prod_{s\in I_2} p_s \not\in \bigcup_{t=1}^n a_t(N)$. Clearly $\prod_{s\in I_2} p_s$ is not covered by $\{0(p_s)\}_{s\in I_1}$ either. Thus A_1 does not form a cover of $\mathbb{Z}$.

By means of the Riemann zeta function, in 1989 M. Z. Zhang [Z89] proved that if (1.1) forms a cover of $\mathbb{Z}$, then $\sum_{s\in I} 1/n_s$ is a positive integer for some $I \subseteq \{1,\ldots,k\}$.

Let $m_1,\ldots,m_k$ be any positive integers. If (1.1) is a cover of $\mathbb{Z}$, then $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s=1}^k$ is also a cover of $\mathbb{Z}$, and hence Theorem 2 of [S95] indicates that for any $J \subseteq \{1,\ldots,k\}$ there is an $I \subseteq \{1,\ldots,k\}$ with $I \neq J$ such that $\{\sum_{s\in I} m_s/n_s\} = \{\sum_{s\in J} m_s/n_s\}$, where $\{\alpha\}$ denotes the fractional part of a real number α. When $J = \emptyset$ and $m_1 = \cdots = m_k = 1$, this yields Zhang's result. In 1999 Z. W. Sun [S99] proved further that if (1.1) forms an m-cover of $\mathbb{Z}$, then for any $J \subseteq \{1,\ldots,k\}$ we have

$$\left|\left\{I \subseteq \{1,\ldots,k\} : I \neq J \text{ and } \left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} = \left\{\sum_{s\in J}\frac{m_s}{n_s}\right\}\right\}\right| \geqslant m.$$

In this paper we will show the following sharp result.

Theorem 1.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$, and let $m_1,\ldots,m_k$ be any integers. Then for any $0 \leqslant \theta < 1$ the set*

$$I_A(\theta) = \left\{I \subseteq \{1,\ldots,k\} : \left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} = \theta\right\} \tag{1.2}$$

has at least 2^m elements if it is nonempty.

Remark 1.1. Clearly m copies of $0(1)$ form an m-cover of $\mathbb{Z}$. This shows that the lower bound in Theorem 1.1 is best possible.

Corollary 1.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$, and let $m_1,\ldots,m_k$ be any integers. Then $|S(A)| \leqslant 2^{k-m}$, where*

$$S(A) = \left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} : I \subseteq \{1,\ldots,k\}\right\}. \tag{1.3}$$

Proof. As $|I_A(\theta)| \geqslant 2^m$ for all $\theta \in S(A)$, we have

$$|S(A)|2^m \leqslant |\{I : I \subseteq \{1,\ldots,k\}\}| = 2^k$$

and hence $|S(A)| \leqslant 2^{k-m}$. □

Remark 1.2. Sun [S95, S96] showed that if $m_1,\ldots,m_k$ are relatively prime to $n_1,\ldots,n_k$, respectively, then (1.1) forms an m-cover of $\mathbb{Z}$ whenever it covers $|S(A)|$ consecutive integers at least m times.

Corollary 1.2. *Suppose that* (1.1) *forms an m-cover of $\mathbb{Z}$ but $\{a_s(n_s)\}_{s=1}^{k-1}$ does not. If the covering function $w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|$ is periodic modulo n_k, then for any $r = 0,\ldots,n_k - 1$ we have*

$$\left|\left\{I \subseteq \{1,\ldots,k-1\} : \left\{\sum_{s\in I}\frac{1}{n_s}\right\} = \frac{r}{n_k}\right\}\right| \geqslant 2^{m-1}. \tag{1.4}$$

Proof. By Theorem 1 of Sun [S07],

$$\left|\left\{\left\lfloor\sum_{s\in I}\frac{1}{n_s}\right\rfloor : I \subseteq \{1,\ldots,k-1\} \text{ and } \left\{\sum_{s\in I}\frac{1}{n_s}\right\} = \frac{r}{n_k}\right\}\right| \geqslant m.$$

In particular, $\{\sum_{s\in I} 1/n_s\} = r/n_k$ for some $I \subseteq \{1,\dots,k-1\}$, and hence (1.4) holds in the case $m = 1$. For $A_k = \{a_s(n_s)\}_{s=1}^{k-1}$, clearly $w_{A_k}(x) \geqslant m-1$ for all $x \in \mathbb{Z}$. In the case $m > 1$, we obtain (1.4) by applying Theorem 1.1 to A_k with $m_1 = \dots = m_{k-1} = 1$ and $\theta = r/n_k$. □

Remark 1.3. When n_k is divisible by all the moduli $n_1,\dots,n_k$, Corollary 1.2 was stated by the second author in [S03, Theorem 2.5]. When $w_A(x) = m$ for all $x \in \mathbb{Z}$, the following result stronger than (1.4) (with $r \in \{0,\dots,n_k-1\}$) was proved in [S97]:

$$\left|\left\{I \subseteq \{1,\dots,k-1\} : \sum_{s\in I} \frac{1}{n_s} = n + \frac{r}{n_k}\right\}\right| \geqslant \binom{m-1}{n}$$

for every $n = 0,\dots,m-1$.

For an algebraic number field K, let O_K be the ring of algebraic integers in K. For $\alpha,\beta \in O_K$, we set

$$\alpha + \beta O_K = \{\alpha + \beta\omega : \omega \in O_K\}$$

and call it a residue class in O_K. For a finite system

$$\mathcal{A} = \{\alpha_s + \beta_s O_K\}_{s=1}^k \tag{1.5}$$

of such residue classes, if $|\{1 \leqslant s \leqslant k : x \in \alpha_s + \beta_s O_K\}| \geqslant m$ for all $x \in O_K$ (where $m \in \{1,2,3,\dots\}$), then we call $\mathcal{A}$ an m-cover of O_K. Covers of the ring $\mathbb{Z}[\sqrt{-2}] = O_{\mathbb{Q}(\sqrt{-2})}$ were investigated by J. H. Jordan [J68].

An algebraic number field K of degree n is said to have a *power integral basis* if there is $\gamma \in O_K$ such that $1,\gamma,\dots,\gamma^{n-1}$ form a basis of O_K over $\mathbb{Z}$. It is well known that all quadratic fields and cyclotomic fields have power integral bases.

Here is a generalization of Theorem 1.1.

Theorem 1.2. *Let K be an algebraic number field with a power integral basis. Suppose that* (1.5) *forms an m-cover of O_K, and let $\omega_1,\dots,\omega_k \in O_K$. Then, for any $\mu \in K$, the set*

$$\left\{I \subseteq \{1,\dots,k\} : \sum_{s\in I} \frac{\omega_s}{\beta_s} \in \mu + O_K\right\}$$

is empty or it has at least 2^m elements.

Remark 1.4. We conjecture that the requirement in Theorem 1.2 that K has a power integral basis can be cancelled.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* (1.1) *be an m-cover of $\mathbb{Z}$. Let $m_1,\dots,m_k \in \mathbb{Z}$ and define $S(A)$ as in* (1.3)*. Then, for any given $\theta \in S(A)$, there exists $t \in \{1,\dots,k\}$ such that both θ and $\{\theta - m_t/n_t\}$ lie in $S(A_t)$, where $A_t = \{a_s(n_s)\}_{1\leqslant s\leqslant k,\ s\neq t}$.*

Proof. Choose a maximal $J \subseteq \{1,\dots,k\}$ such that $\{\sum_{s\in J} m_s/n_s\} = \theta$. As (1.1) is a cover of $\mathbb{Z}$, by [S95, Theorem 2] or [S99, Theorem 1(i)] there exists $I \subseteq \{1,\dots,k\}$ for which $I \neq J$ and $\{\sum_{s\in I} m_s/n_s\} = \theta$. Note that $J \not\subseteq I$ and hence $t \in J \setminus I$ for some $1 \leqslant t \leqslant k$. Clearly $\theta = \{\sum_{s\in I} m_s/n_s\} \in S(A_t)$ and also $\{\theta - m_t/n_t\} = \{\sum_{s\in J\setminus\{t\}} m_s/n_s\} \in S(A_t)$. This concludes the proof. □

Proof of Theorem 1.1. We use induction on m.

The $m=1$ case, as mentioned above, has been handled in [S95, S99].

Now let $m>1$ and assume that Theorem 1.1 holds for smaller positive integers. Let $\theta \in S(A)$. In light of Lemma 2.1, there is $t \in \{1,\dots,k\}$ such that both θ and $\theta' = \{\theta - m_t/n_t\}$ lie in $S(A_t)$. As A_t forms an $(m-1)$-cover of $\mathbb{Z}$, by the induction hypothesis we have $|I_{A_t}(\theta)| \geqslant 2^{m-1}$ and $|I_{A_t}(\theta')| \geqslant 2^{m-1}$. Observe that

$$I_A(\theta) = I_{A_t}(\theta) \cup \{I \cup \{t\} : I \in I_{A_t}(\theta')\}.$$

Therefore

$$|I_A(\theta)| = |I_{A_t}(\theta)| + |I_{A_t}(\theta')| \geqslant 2^{m-1} + 2^{m-1} = 2^m.$$

We are done. □

3. Proof of Theorem 1.2

At first we give a lemma on algebraic number fields with power integral bases.

Lemma 3.1. *Let K be an algebraic number field with a power integral basis $1,\gamma,\dots,\gamma^{n-1}$. For any $\mu = \sum_{r=0}^{n-1} \mu_r\gamma^r \in K$ with $\mu_0,\dots,\mu_{n-1} \in \mathbb{Q}$, we have*

$$\mu \in O_K \iff \psi(\mu), \psi(\mu\gamma), \dots, \psi(\mu\gamma^{n-1}) \in \mathbb{Z},$$

where $\psi(\mu)$ denotes the last coordinate μ_{n-1} of μ.

Proof. If $\mu \in O_K$, then $\mu, \mu\gamma, \dots, \mu\gamma^{n-1} \in O_K$ and hence $\psi(\mu\gamma^j) \in \mathbb{Z}$ for every $j = 0,\dots,n-1$.

Now assume that $\psi(\mu\gamma^j) \in \mathbb{Z}$ for all $j=0,\dots,n-1$. We want to show that $\mu \in O_K$ (i.e., $\mu_0,\dots,\mu_{n-1} \in \mathbb{Z}$). Clearly $\mu_{n-1} = \psi(\mu\gamma^0) \in \mathbb{Z}$. If $0 \leqslant r < n-1$ and $\mu_{r+1},\dots,\mu_{n-1} \in \mathbb{Z}$, then

$$\begin{aligned}\mu_r &= \psi(\mu_0\gamma^{n-1-r} + \mu_1\gamma^{n-r} + \dots + \mu_r\gamma^{n-1}) \\ &= \psi(\mu\gamma^{n-1-r}) - \psi(\mu_{r+1}\gamma^n + \dots + \mu_{n-1}\gamma^{2n-2-r})\end{aligned}$$

and hence $\mu_r \in \mathbb{Z}$ since $\mu_{r+1}\gamma^n + \dots + \mu_{n-1}\gamma^{2n-2-r} \in O_K$. So, by induction, $\mu_r \in \mathbb{Z}$ for all $r = 0,\dots,n-1$. We are done. □

Proof of Theorem 1.2. In the spirit of the proof of Theorem 1.1, it suffices to handle the case $m=1$. That is, we should prove that for any $J \subseteq \{1,\dots,k\}$ there is $I \subseteq \{1,\dots,k\}$ with $I \neq J$ such that $\sum_{s\in I}\omega_s/\beta_s - \sum_{s\in J}\omega_s/\beta_s \in O_K$.

Let $\{1,\gamma,\dots,\gamma^{n-1}\}$ be a power integral basis of K, and define ψ as in Lemma 3.1.

Let $x_0,\dots,x_{n-1} \in \mathbb{Z}$ and $x = \sum_{r=0}^{n-1} x_r\gamma^r$. Since $-x \in O_K$ is covered by $\mathcal{A} = \{\alpha_s + \beta_s O_K\}_{s=1}^k$, we have

$$\begin{aligned}0 &= \prod_{s=1}^{k}\left(1 - e^{2\pi i\psi(\omega_s(x+\alpha_s)/\beta_s)}\right) = \sum_{I\subseteq\{1,\dots,k\}} (-1)^{|I|} \prod_{s\in I} e^{2\pi i\psi(\omega_s(x+\alpha_s)/\beta_s)} \\ &= \sum_{I\subseteq\{1,\dots,k\}} (-1)^{|I|} \prod_{s\in I} e^{2\pi i(\psi(\omega_s\alpha_s/\beta_s) + \sum_{r=0}^{n-1} x_r\psi(\omega_s\gamma^r/\beta_s))} \\ &= \sum_{I\subseteq\{1,\dots,k\}} (-1)^{|I|} e^{2\pi i\psi(\sum_{s\in I}\omega_s\alpha_s/\beta_s)} \prod_{r=0}^{n-1} e^{2\pi i x_r\psi(\sum_{s\in I}\omega_s\gamma^r/\beta_s)} \\ &= \sum_{\theta_0\in S_0} e^{2\pi i x_0\theta_0} \sum_{\theta_1\in S_1} e^{2\pi i x_1\theta_1} \cdots \sum_{\theta_{n-1}\in S_{n-1}} e^{2\pi i x_{n-1}\theta_{n-1}} f(\theta_0,\dots,\theta_{n-1}),\end{aligned}$$

where

$$S_r = \left\{ \left\{ \psi\left(\sum_{s\in I} \frac{\omega_s \gamma^r}{\beta_s} \right) \right\} : I \subseteq \{1, \ldots, k\} \right\}$$

and

$$f(\theta_0, \ldots, \theta_{n-1}) = \sum_{\substack{I\subseteq\{1,\ldots,k\} \\ \{\psi(\sum_{s\in I} \omega_s\gamma^r/\beta_s)\}=\theta_r \\ \text{for all } r=0,\ldots,n-1}} (-1)^{|I|} e^{2\pi i \psi(\sum_{s\in I} \omega_s \alpha_s/\beta_s)}.$$

For each $r = 0, \ldots, n-1$, if $\sum_{\theta_r \in S_r} e^{2\pi i x_r \theta_r} F(\theta_r) = 0$ for all $x_r = 0, \ldots, |S_r| - 1$, then $F(\theta_r) = 0$ for every $\theta_r \in S_r$, because the Vandermonde determinant $\det(e^{2\pi i x_r \theta_r})_{0\leqslant x_r < |S_r|,\, \theta_r \in S_r}$ does not vanish. So, by the above, we have $f(\theta_0, \ldots, \theta_{n-1}) = 0$ for all $\theta_0 \in S_0, \ldots, \theta_{n-1} \in S_{n-1}$.

Now suppose that $\mu \in K$ and $\sum_{s\in J} \omega_s/\beta_s \in \mu + O_K$ for a unique subset J of $\{1, \ldots, k\}$. We want to deduce a contradiction.

Set $\theta_r = \{\psi(\mu\gamma^r)\}$ for $r = 0, \ldots, n-1$. For any $I \subseteq \{1, \ldots, k\}$ we have

$$\begin{aligned} &\left\{ \psi\left(\sum_{s\in I} \frac{\omega_s\gamma^r}{\beta_s} \right) \right\} = \theta_r \quad \text{for all } r = 0, \ldots, n-1 \\ \iff &\psi\left(\left(\sum_{s\in I} \frac{\omega_s}{\beta_s} - \mu \right) \gamma^r \right) \in \mathbb{Z} \quad \text{for all } r = 0, \ldots, n-1 \\ \iff &\sum_{s\in I} \frac{\omega_s}{\beta_s} \in \mu + O_K \qquad \text{(by Lemma 3.1)} \\ \iff &I = J. \end{aligned}$$

Thus the expression of $f(\theta_0, \ldots, \theta_{n-1})$ only contains one summand, and therefore

$$0 = f(\theta_0, \ldots, \theta_{n-1}) = (-1)^{|J|} e^{2\pi i \psi(\sum_{s\in J} \omega_s\alpha_s/\beta_s)} \neq 0,$$

which is a contradiction.

The proof of Theorem 1.2 is now complete. □

Acknowledgment

The authors are indebted to the referee for his/her valuable suggestions.

References

[E97] P. Erdős, *Some of my favorite problems and results,* in: The Mathematics of Paul Erdős, I, 47–67, Algorithms Combin., 13, Springer, Berlin, 1997. MR1425174 (98e:11002)

[G04] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd edition, Springer, New York, 2004. MR2076335 (2005h:11003)

[J68] J. H. Jordan, *A covering class of residues with odd moduli*, Acta Arith. **13** (1968), 335–338. MR0220657 (36:3709)

[PS] Š. Porubský and J. Schönheim, *Covering systems of Paul Erdös: Past, present and future*, in: Paul Erdös and his Mathematics. I (edited by G. Halász, L. Lovász, M. Simonvits, V. T. Sós), Bolyai Soc. Math. Studies 11, Budapest, 2002, pp. 581–627. MR1954716 (2004d:11006)

[S95] Z. W. Sun, *Covering the integers by arithmetic sequences*, Acta Arith. **72** (1995), 109–129. MR1347259 (96k:11013)

[S96] Z. W. Sun, *Covering the integers by arithmetic sequences. II*, Trans. Amer. Math. Soc. **348** (1996), 4279–4320. MR1360231 (97c:11011)

[S97] Z. W. Sun, *Exact m-covers and the linear form $\sum_{s=1}^k x_s/n_s$*, Acta Arith. **81** (1997), 175–198. MR1456240 (98h:11019)

[S99] Z. W. Sun, *On covering multiplicity*, Proc. Amer. Math. Soc. **127** (1999), 1293–1300. MR1486752 (99h:11012)

[S03] Z. W. Sun, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 51–60. MR1988872 (2004i:11017)

[S05] Z. W. Sun, *On the range of a covering function*, J. Number Theory **111** (2005), 190–196. MR2124049 (2005m:11015)

[S07] Z. W. Sun, *A connection between covers of the integers and unit fractions*, Adv. in Appl. Math., **38** (2007), 267–274.

[Z89] M. Z. Zhang, *A note on covering systems of residue classes*, Sichuan Daxue Xuebao (Nat. Sci. Ed.) **26** (1989), Special Issue, 185–188. MR1059702 (92c:11003)

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: `haopan79@yahoo.com.cn`

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: `zwsun@nju.edu.cn`