PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 11, November 2007, Pages 3733–3744
S 0002-9939(07)08894-6
Article electronically published on August 2, 2007

INDEX ESTIMATES FOR MINIMAL SURFACES AND k-CONVEXITY

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(Communicated by Jon G. Wolfson)

ABSTRACT. We prove Morse index estimates for the area functional for minimal surfaces that are solutions to the free boundary problem in k-convex domains in manifolds of nonnegative complex sectional curvature.

1. INTRODUCTION

In this paper we prove Morse index estimates for the area functional for minimal surfaces in Riemannian manifolds that are solutions to a free boundary problem. Morse index estimates for minimal surfaces are in general difficult to obtain, and can often only be estimated in very special cases. The minimal surfaces we consider are critical points of the area functional on the space of maps of the two-dimensional disk D into a Riemannian manifold N where the boundary of the disk ∂D is mapped to specified embedded submanifold M. Critical points of this variational problem are minimal surfaces that meet the submanifold M orthogonally along the boundary of the surface. This free boundary problem was well studied classically in $\mathbb{R}^3$ in case one seeks an area minimizing disk. Existence theory in the general setting of finding solutions predicted by Morse theory based on the topology of the mapping space has been studied, for example, in [J], [F1], [F2]. For geometric applications, it is important to understand the relations between the curvature of the ambient manifold N, the second fundamental form of the constraint submanifold M, and the Morse index of solutions. This turns out to be a challenging problem for which only partial results are known. The second variation is easier to understand in the case where M is a hypersurface, since any normal variation of the minimal surface is automatically tangent to M along the boundary of the surface.

Definition 1. A compact hypersurface M in an n-dimensional Riemannian manifold N is *k-convex*, $1 \leq k \leq n-1$, if the sum of any k principal curvatures of M with respect to the inward pointing unit normal is positive. A *k-convex domain* is a domain with smooth k-convex boundary.

A 1-convex domain is a convex domain. An $(n-1)$-convex hypersurface is a hypersurface of positive mean curvature. An example of a k-convex hypersurface

Received by the editors July 26, 2006.
2000 *Mathematics Subject Classification.* Primary 58E12; Secondary 53C21.
The author was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

is the boundary of a sufficiently small tubular neighborhood of a compact $(k-1)$-dimensional submanifold in a Riemannian manifold. Note that k-convexity implies $(k+1)$-convexity.

Moore and Schulte [MS] proved that any minimal disk that is a solution to the free boundary problem in an $(n-2)$-convex domain in $\mathbb{R}^n$ must be unstable. In fact, it is interesting to observe that the same argument can be used to show that any minimal submanifold Σ^l of dimension l that is a solution to the free boundary problem in a k-convex domain in $\mathbb{R}^n$ is unstable if $l \leq n-k$. In [F1] we obtained estimates on the index of minimal disks that are solutions to the free boundary problem in two-convex domains in $\mathbb{R}^n$ or in manifolds of positive isotropic curvature.

Theorem ([F1], Theorem 2.7, 2.8)**.** *Let Σ be a minimal disk in a domain N in $\mathbb{R}^n$ or in an n-dimensional manifold of positive isotropic curvature, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If ∂N is two-convex, then Σ has index at least $[\frac{n-2}{2}]$.*

Positive isotropic curvature is a notion of curvature that arises naturally when studying stability of minimal surfaces. If N is an n-dimensional Riemannian manifold, the inner product on any tangent space T_pN can be extended to the complexified tangent space $T_pN \otimes \mathbb{C}$ as a complex bilinear form $(\cdot,\cdot)$ or as a Hermitian inner product $\langle\cdot,\cdot\rangle$. The relationship between these extensions is given by $\langle v, w\rangle = (v, \bar{w})$ for v, $w \in T_pN \otimes \mathbb{C}$. The curvature tensor extends to complex vectors by linearity, and the *complex sectional curvature* of a two-dimensional subspace π of $T_pN \otimes \mathbb{C}$ is defined by $K(\pi) = \langle R(v,w)\bar{w}, v\rangle$, where $\{v, w\}$ is any unitary basis of π. A subspace π is said to be *isotropic* if every vector $v \in \pi$ has square zero; that is, $(v, v) = 0$.

Definition 2. A Riemannian manifold N has *positive isotropic curvature* (PIC) if $K(\pi) > 0$ for every isotropic two-plane $\pi \subset T_pN \otimes \mathbb{C}$ for all $p \in N$.

This curvature condition is nonvacuous only for $n \geq 4$, since in dimensions less than four there are no two-dimensional isotropic subspaces. Positive isotropic curvature first derived its importance from the following beautiful theorem of Micallef and Moore [MM]: *any compact simply connected manifold of PIC is homeomorphic to the standard sphere S^n*. The classical conditions of pointwise quarter-pinched sectional curvatures and positive curvature operator are easily seen to imply PIC, and so in particular, this is a generalization of the classical sphere theorem. Just as positive sectional curvature is well suited to studying the stability of geodesics, positive isotropic curvature is in a similar way ideally suited to studying the stability of minimal surfaces. In fact, there is a very interesting analogy between second variation theory for geodesics in manifolds of positive sectional curvature (e.g. the theorems of Synge, Bonnet, Frankel [Fr1], [Fr2]), and second variation theory for minimal surfaces in manifolds of positive isotropic curvature ([MM], [F1], [F2], [F3], [FW]). The proof of the sphere theorem of Micallef and Moore involves an amplified version of the celebrated existence theory for minimal two-spheres of Sacks-Uhlenbeck [SU] together with estimates giving lower bounds on the Morse index for the area function of any minimal two-sphere in a manifold of PIC. The index estimates for minimal two-spheres of Micallef and Moore are based upon a complexified formula for the second variation of area, and Grothendieck's splitting theorem for holomorphic vector bundles over the Riemann sphere. The index estimates of [F1] also use the complexified formula for the second variation of area, together with a construction of holomorphic variations of the minimal disk satisfying certain boundary conditions. A key tool in obtaining the index estimates of [F1] is the use of a Riemann-Hilbert problem, which can be shown to have a certain index. This turns out to be the natural problem to consider for understanding the second variation theory. It can be applied to produce a real $n-2$-dimensional space of holomorphic sections whose real parts are pointwise orthonormal along the boundary of the disk.

In this paper, we show that the index estimates of [F1] can be improved. Moreover, we show they can be generalized to k-convex domains. We obtain a whole range of estimates, depending on the degree of convexity of the constraint submanifold. The main results of the paper are

Theorem 1. *Let Σ be a minimal disk in a domain N in $\mathbb{R}^n$, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If ∂N is k-convex, then Σ has index at least $n-k-1$.*

These estimates are sharp. In section 3 we give examples that illustrate this. In the case of curved ambient manifolds, we obtain the following:

Theorem 2. *Let Σ be a minimal disk in a k-convex domain N in an n-dimensional manifold of nonnegative isotropic curvature, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If $k \leq n-3$, then Σ is unstable. If k is even, then Σ is unstable if $k \leq n-2$.*

It is not clear what the sharp estimates should be in the PIC case. The two-convex condition is well adapted to PIC. However, in the k-convex case, index estimates for the minimal disks hold as in Theorem 1 when the ambient manifold has nonnegative complex sectional curvature (Theorem 3).

The final section of the paper contains some concluding remarks and a survey of various know results on the topology of k-convex hypersurfaces.

2. Index estimates

Complex second variation formula. Let N be an orientable Riemannian manifold, and let M be an embedded submanifold. Let Σ_0 be a compact connected orientable surface with nonempty boundary $\partial\Sigma_0$. Suppose $u : \Sigma_0 \to N$ with $u(\partial\Sigma_0) \subset M$ is a minimal immersion such that $u(\Sigma_0)$ meets M orthogonally along $u(\partial\Sigma_0)$. The tangent bundle TN of N pulls back to a smooth vector bundle $u^*(TN)$ over Σ_0, and the Riemannian metric and Levi-Civita connection ∇ on TN pull back to a metric $\langle\cdot,\cdot\rangle$ and connection ∇ on $u^*(TN)$. We denote by $\nabla^\top$ and $\nabla^\perp$ the tangential and normal components to $u(\Sigma_0)$. Let (x, y) be local isothermal coordinates on Σ_0 with the induced metric, so the area form $da = \lambda^2 dxdy$. The second variation of area along a smooth normal variation of $u(\Sigma_0)$ with normal variation field $V \in \Gamma(u^*(\mathcal{N}))$, where $\mathcal{N}$ is the normal bundle of $u(\Sigma_0)$ in N, is given by:

$$\delta^2\mathcal{A}(V,V) = \int_{\Sigma_0} [|\nabla V|^2 - 2|\nabla^\top V|^2 - \langle R(V), V\rangle]\, da + \int_{\partial\Sigma_0} \langle \nabla_V V, \nu\rangle\, ds$$

where $|\nabla V|^2 = \frac{1}{\lambda^2}(|\nabla_{\frac{\partial}{\partial x}} V|^2 + |\nabla_{\frac{\partial}{\partial y}} V|^2)$,

$$R(V) = \frac{1}{\lambda^2}\left[R(V, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + R(V, \frac{\partial u}{\partial y})\frac{\partial u}{\partial y}\right]$$

and ν is the outward pointing unit normal of $\partial\Sigma_0$ along u. Now consider the complexified bundle $u^*(TN)\otimes\mathbb{C}$. The metric extends to $u^*(TN)\otimes\mathbb{C}$ as a complex bilinear form $(\cdot,\cdot)$ or as a Hermitian inner product $\langle\cdot,\cdot\rangle$. The connection ∇ extends to a complex linear connection on $u^*(TN)\otimes\mathbb{C}$, and the curvature R extends by linearity. The index form $\delta^2\mathcal{A}(V,W)$ extends to a Hermitian symmetric bilinear form I on sections of the complexified normal bundle $\mathbf{E}=u^*(\mathcal{N})\otimes\mathbb{C}$:

$$I(V,V)=\int_{\Sigma_0}[|\nabla V|^2-2|\nabla^\top V|^2-\langle R(V),V\rangle]\,da+\int_{\partial\Sigma_0}\langle\nabla_V\bar{V},\nu\rangle\,ds.$$

As in [MM], [F1] this can be written as

$$\begin{aligned}I(V,V)=&\int_{\Sigma_0}4[\,|\nabla^\perp_{\frac{\partial}{\partial\bar{z}}}V|^2-|\nabla^\top_{\frac{\partial}{\partial z}}V|^2-\langle R(V,\frac{\partial u}{\partial z})\frac{\partial u}{\partial\bar{z}},V\rangle]\,dxdy\\&+\int_{\partial\Sigma_0}\langle\nabla_V\bar{V},\nu\rangle\,ds-i\int_{\partial\Sigma_0}\langle\nabla_{\frac{d}{ds}}V,V\rangle\,ds\end{aligned}$$

where $\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial\bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$. Setting $V=U+iW$, the boundary terms are:

$$\begin{aligned}I(V,V)=&\int_{\Sigma_0}4[\,|\nabla^\perp_{\frac{\partial}{\partial\bar{z}}}V|^2-|\nabla^\top_{\frac{\partial}{\partial z}}V|^2-\langle R(V,\frac{\partial u}{\partial z})\frac{\partial u}{\partial\bar{z}},V\rangle]\,dxdy\\&+\int_{\partial\Sigma_0}[\langle\nabla_U U,\nu\rangle+\langle\nabla_W W,\nu\rangle-2\langle\nabla_{\frac{d}{ds}}U,W\rangle]\,ds.\end{aligned}\tag{2.1}$$

This formula suggests that we look for sections $V\in\Gamma(\mathbf{E})$ that satisfy $\nabla^\perp_{\frac{\partial}{\partial\bar{z}}}V=0$ on Σ_0 with $\operatorname{Im}V=0$ on $\partial\Sigma_0$. For such variations, the first term in the interior vanishes, and the last boundary term vanishes in this formula. The remaining terms each have a sign under suitable curvature assumption on N, and convexity assumption on M.

Holomorphic sections. The bundle $u^*(\mathcal{N})$ over Σ_0 is topologically trivial ([McS], Proposition 2.66, since $SO(n-2)$ is connected). Let $s_1,\dots,s_{n-2}$ be a global basis of smooth sections of $u^*(\mathcal{N})$. With respect to this basis, any section V of the complexified bundle $\mathbf{E}=u^*(\mathcal{N})\otimes\mathbb{C}$ can be written $V=\sum_{j=1}^{n-2}f_js_j$ for some complex-valued functions f_j, $j=1,\dots,n-2$ on Σ_0. There is a unique holomorphic structure on $\mathbf{E}$ with respect to which the $\bar\partial$-operator on $\mathbf{E}$ is given by $\bar\partial\omega=(\nabla^\perp_{\frac{\partial}{\partial\bar{z}}}\omega)d\bar{z}$. The complex second variation formula above suggests we look for holomorphic sections on Σ_0 that are real on the boundary $\partial\Sigma_0$; i.e. solutions of the following problem:

$$(P)\begin{cases}\bar\partial V=0 & \text{on } \Sigma_0,\\ \operatorname{Im} V=0 & \text{on } \partial\Sigma_0.\end{cases}$$

We have $\bar\partial s_j=\sum_{k=1}^{n-2}a_{jk}s_k$ for some smooth $(0,1)$-forms a_{jk} on Σ_0. Therefore (P) can be rewritten as

$$(P)\begin{cases}\bar\partial f_j+\sum_{k=1}^{n-2}a_{kj}f_k=0 & \text{on } \Sigma_0\\ \operatorname{Im} f_j=0 & \text{on } \partial\Sigma_0\end{cases}\quad\text{for } j=1,\dots,n-2.$$

Consider the operator

$$A : \mathcal{D} \to L^2(\Sigma_0, \mathbb{C}^{n-2})$$

defined by

$$(Af)_j = \bar{\partial} f_j + \sum_{k=1}^{n-2} a_{kj} f_k$$

on the domain $\mathcal{D} = \{f \in H^1(\Sigma_0, \mathbb{C}^{n-2}) : \operatorname{Im} f = 0 \text{ on } \partial\Sigma_0\}$. Since Σ_0 is compact with boundary, and A is an elliptic operator with elliptic boundary condition $\operatorname{Im} f = 0$ on $\partial\Sigma_0$, A is a Fredholm operator on the domain $\mathcal{D}$. The zero order term $\sum_{k=1}^{n-2} a_{kj} f_k$ defines a compact operator from $\mathcal{D}$ to $L^2_{(0,1)}(\Sigma_0)$, the space of $(0,1)$-forms on Σ_0 of class L^2. Hence, the index of A is equal to the index of the Fredholm operator $Bf = \bar{\partial} f$ defined on the domain $\mathcal{D}$. First note that any element of the kernel of B is a holomorphic function on Σ_0 that is real on $\partial\Sigma_0$ and therefore must be constant. Hence, $\dim(\operatorname{Ker} B) = n-2$. We would like to compute the dimension of the cokernel of B. Integrating by parts, we have

$$\begin{aligned}
\operatorname{Re}(\overline{\partial} f, \omega)_{L^2} &= \operatorname{Re} \int_{\Sigma_0} \overline{\partial} f \wedge *\overline{\omega} \\
&= -\operatorname{Re} \int_{\Sigma_0} f \overline{\partial} * \overline{\omega} + \operatorname{Re} \int_{\partial\Sigma_0} f * \overline{\omega} \\
&= -\operatorname{Re}(f, *\partial * \omega)_{L^2} + \operatorname{Re} \int_{\partial\Sigma_0} f * \overline{\omega}.
\end{aligned}$$

The L^2 adjoint B^* of B is defined on the domain

$$\mathcal{F} = \{\omega = (\omega^1, \ldots, \omega^{n-2}) : \omega^j \in H^1_{(0,1)}(\Sigma_0) \text{ and } \operatorname{Re} *\overline{\omega}|_{\partial\Sigma_0} = 0\},$$

and is given by $B^*\omega = - * \partial * \omega$. Suppose $\omega \in \operatorname{Ker} B^*$. Then each component of $*\overline{\omega}$ is a holomorphic 1-form that is imaginary on $\partial\Sigma_0$. It is a classical result that the real dimension of the space of holomorphic 1-forms on Σ_0 that are imaginary on $\partial\Sigma_0$ is $2g + c - 1$, where g is the genus of Σ_0 and c is the number of connected components of $\partial\Sigma_0$. Therefore, $\dim \operatorname{Ker} B^* = (n-2)(2g+c-1)$.

For the remainder of the paper we restrict to the case where Σ_0 is the two-dimensional disk D. In this case we can see that $\operatorname{Ker} B^* = \{0\}$ as follows. We have $*\overline{\omega} = \partial h$ for some holomorphic function h on D. If T is a vector field tangent to ∂D, then $0 = \operatorname{Re} *\overline{\omega}(T) = \operatorname{Re} \partial h(T) = \operatorname{Re} dh(T) = \operatorname{Re} T(h)$. Therefore, $\operatorname{Re} h$ is constant on ∂D. Since h is holomorphic, this implies that h is constant on D. Therefore, $*\omega = 0$. Hence,

$$\text{index } A = \text{index } B = \dim \operatorname{Ker} B - \dim \operatorname{Ker} B^* = n-2,$$

and it follows that the space of holomorphic sections that are real on the boundary,

$$\mathcal{H} = \{V \in \Gamma(\mathbf{E}) : \bar{\partial} V = 0 \text{ on } D,\ \operatorname{Im} V = 0 \text{ on } \partial D\},$$

has dimension at least $n-2$ over $\mathbb{R}$.

Index estimates. In this section we prove the main index estimates. We restrict to the case where $u : D \to N$ is a minimal immersion with $u(\partial D) \subset \partial N$, meeting ∂N orthogonally along the boundary of the disk. Let $\Sigma = u(D)$. In this case, where the constraint submanifold $M = \partial N$ has codimension one, any variation that is normal to the surface is automatically tangent to the constraint submanifold along the boundary of the surface. Hence, any section of the normal bundle is an admissible variation.

Theorem 1. *Let Σ be a minimal disk in a domain N in $\mathbb{R}^n$, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If ∂N is k-convex, then Σ has index at least $n-k-1$.*

Proof. Define a Hermitian symmetric bilinear form $I_0 : \Gamma(\mathbf{E}) \times \Gamma(\mathbf{E}) \to \mathbb{R}$ by

$$I_0(V,V) = \int_D 4\,|\nabla^{\perp}_{\frac{\partial}{\partial \bar z}} V|^2\, dxdy + \int_{\partial D} [\langle \nabla_U U, \nu\rangle + \langle \nabla_W W, \nu\rangle - 2\langle \nabla_{\frac{d}{ds}} U, W\rangle]\, ds$$

for $V = U + iW \in \Gamma(\mathbf{E})$. Observe that $I(V,V) \le I_0(V,V)$ for all $V \in \Gamma(\mathbf{E})$. If $V_1, V_2 \in \mathcal{H}$, then $\frac{\partial}{\partial \bar z}(V_1,V_2) = 0$ and (V_1,V_2) is real on ∂D. Therefore (V_1,V_2) is a real constant. Thus, the form $(\cdot,\cdot)$ is a symmetric bilinear form that is real valued on $\mathcal{H}$. Moreover, $(\cdot,\cdot)$ is positive definite on $\mathcal{H}$, since if $V \in \mathcal{H}$ and $(V,V) = 0$, then V is holomorphic and zero on ∂D and so V must be zero. The real part, $\operatorname{Re} I_0$, of the Hermitian form I_0 is a symmetric bilinear form on $\mathcal{H}$. We may therefore choose a basis $\{V_1,\dots,V_{n-2}\}$ of $\mathcal{H}$ that is orthonormal with respect to $(\cdot,\cdot)$ and that diagonalizes $\operatorname{Re} I_0$. Suppose $V_j = U_j + iW_j$. Note in particular, that $\{U_1,\dots,U_{n-2}\}$ are pointwise orthonormal on ∂D. Then,

$$I_0(V_j,V_j) = \int_{\partial D} \langle \nabla_{U_j} U_j, \nu\rangle\, ds.$$

Since $\{U_1,\dots,U_k\}$ are pointwise orthonormal and tangent to ∂N along the boundary of D, ν is the outward unit normal vector to ∂N, and ∂N is k-convex,

$$\sum_{l=1}^{k} I_0(V_{j_l},V_{j_l}) = \sum_{l=1}^{k} \int_{\partial D} \langle \nabla_{U_{j_l}} U_{j_l}, \nu\rangle\, ds < 0$$

for any distinct indices $1 \le j_1,\dots,j_k \le n-2$. Therefore, at most $k-1$ of $I(V_j,V_j)$, $j = 1,\dots,n-2$, can be greater than or equal to zero. Set $p = n-k-1$. By reordering the basis if necessary, we may assume that $I(V_j,V_j) < 0$ for $j = 1,\dots,p$. Since $\{V_1,\dots,V_{n-2}\}$ diagonalizes $\operatorname{Re} I_0$, for $a_j \in \mathbb{R}$,

$$\begin{aligned} I_0\,(\sum_{j=1}^{p} a_j V_j, \sum_{j=1}^{p} a_j V_j\,) &= \operatorname{Re} I_0\,(\sum_{j=1}^{p} a_j V_j, \sum_{j=1}^{p} a_j V_j\,) \\ &= \sum_{j=1}^{p} a_j^2 \operatorname{Re} I_0(V_j,V_j) \\ &= \sum_{j=1}^{p} a_j^2\, I_0(V_j,V_j) \\ &< 0. \end{aligned}$$

Therefore, $\{V_1,\dots,V_p\}$ span a real p-dimensional subspace of $\Gamma(\mathbf{E})$ on which the form I_0 is negative definite. Now,

$$I_0(V_j,V_j) = I_0(U_j,U_j) + I_0(W_j,W_j).$$

However, I_0 is positive semidefinite on the space of sections of $\mathbf{E}$ that are zero on the boundary ∂D. Therefore, $I_0(W_j,W_j) \ge 0$, and so we must have $I_0(U_j,U_j) < 0$,

$1 \leq j \leq p$. Similary, for $a_j \in \mathbb{R}$,

$$\begin{aligned} 0 > I_0\,(\sum_{j=1}^{p} a_j V_j, \sum_{j=1}^{p} a_j V_j\,) \\ &= I_0\,(\sum_{j=1}^{p} a_j U_j, \sum_{j=1}^{p} a_j U_j\,) + I_0\,(\sum_{j=1}^{p} a_j W_j, \sum_{j=1}^{p} a_j W_j\,) \\ &\geq I_0\,(\sum_{j=1}^{p} a_j U_j, \sum_{j=1}^{p} a_j U_j\,) \end{aligned}$$

since $\sum_{j=1}^{p} a_j W_j = 0$ on ∂D. Note that $\{U_1, \ldots, U_p\}$ are linearly independent over $\mathbb{R}$, for example, since they are pointwise orthonormal along ∂D. Therefore, $\{U_1, \ldots, U_p\}$ span a real p-dimensional subspace on which I_0 is negative definite. But since I_0 dominates the index form I, $\{U_1, \ldots, U_p\}$ span a real p-dimensional subspace of $\Gamma(u^*(\mathcal{N}))$ on which the second variation of area is negative. $\square$

If the case of an ambient manifold of nonnegative isotropic curvature, we are able to prove instability. We note that the condition of two-convexity is better adapted to isotropic curvature, and in this case, the index estimate of [F1], Theorem 2.8 gives the stronger result that the index must be at least $[\frac{n-2}{2}]$.

Theorem 2. *Let Σ be a minimal disk in a k-convex domain N in an n-dimensional manifold of nonnegative isotropic curvature, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If $k \leq n-3$, then Σ is unstable. If k is even, then Σ is unstable if $k \leq n-2$.*

Proof. As in the proof of Theorem 1, $(\cdot,\cdot)$ is a real valued positive definite symmetric bilinear form on $\mathcal{H}$. The form $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by:

$$B(V_1, V_2) = \int_{\partial D} \langle \nabla_{\frac{d}{ds}} V_1, V_2 \rangle \, ds = \int_{\partial D} \langle \nabla_{\frac{d}{ds}} U_1, U_2 \rangle \, ds$$

where $V_j = U_j + iW_j$, is skew-symmetric. We may choose an ordered orthonormal basis $V_1, \ldots, V_{n-2}$ for $\mathcal{H}$ with respect to $(\cdot,\cdot)$ so that B is in standard form. In particular, $B(V_{2j-1}, V_{2j}) \geq 0$ for $j = 1, 2, \ldots$ and $B(V_j, V_l) = 0$ if $l \neq j+1$. Let $X_j = V_{2j-1} + iV_{2j}$ for $j = 1, 2, \ldots$. The sections $X_1, X_2, \ldots$ of the complexified normal bundle $\mathbf{E}$ are holomorphic, isotropic, and satisfy $-iB(X_j, X_j) \leq 0$. The fact that the sections are isotropic follows since $V_1, \ldots, V_{n-2}$ are orthonormal with respect to $(\cdot,\cdot)$. Also, observe that since u is conformal, $(\frac{\partial u}{\partial z}, \frac{\partial u}{\partial z}) = 0$ and so $\frac{\partial u}{\partial z}$ is isotropic. Moreover, X_j and $\frac{\partial u}{\partial z}$ span an isotropic two-plane, since $(X_j, \frac{\partial u}{\partial z}) = 0$. Therefore, $\langle R(X_j, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar{z}}, X_j \rangle \geq 0$, since the manifold has nonnegative isotropic curvature. By the complexified second variation formula (2.1),

$$\begin{aligned} I(X_j, X_j) &= \int_D 4[\,|\nabla^{\perp}_{\frac{\partial}{\partial \bar{z}}} X_j|^2 - |\nabla^{\top}_{\frac{\partial}{\partial z}} X_j|^2 - \langle R(X_j, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar{z}}, X_j \rangle]\, dxdy \\ &\quad + \int_{\partial D} [\langle \nabla_{U_{2j-1}} U_{2j-1}, \nu \rangle + \langle \nabla_{U_{2j}} U_{2j}, \nu \rangle - i\langle \nabla_{\frac{d}{ds}} X_j, X_j \rangle]\, ds \\ &\leq \int_{\partial D} [\langle \nabla_{U_{2j-1}} U_{2j-1}, \nu \rangle + \langle \nabla_{U_{2j}} U_{2j}, \nu \rangle]\, ds. \end{aligned}$$

If k is even and $k \leq n-2$, then

$$\sum_{j=1}^{k/2} I(X_j, X_j) \leq \sum_{j=1}^{k} \int_{\partial D} \langle \nabla_{U_j} U_j, \nu \rangle \, ds < 0$$

since $\{U_1, \ldots, U_{n-2}\}$ are pointwise orthonormal and tangent to ∂N along ∂D, and ∂N is k-convex. Therefore, there exists j, $1 \leq j \leq k/2$, such that $I(X_j, X_j) < 0$. It follows that Σ is unstable. If k is odd and $k \leq n-3$, then

$$\sum_{j=1}^{(k+1)/2} I(X_j, X_j) \leq \sum_{j=1}^{k+1} \int_{\partial D} \langle \nabla_{U_j} U_j, \nu \rangle \, ds < 0$$

since ∂N is k-convex. Therefore, for some j, $1 \leq j \leq (k+1)/2$, $I(V_j, V_j) < 0$, and Σ is unstable. $\square$

Under the stronger assumption of positive complex sectional curvature, we get index estimates as in Theorem 1,

Theorem 3. *Let Σ be a minimal disk in a domain N in an n-dimensional manifold of nonnegative complex sectional curvature, with $\partial\Sigma \subset \partial N$ and meeting ∂N orthogonally. If ∂N is k-convex, then Σ has index at least $n-k-1$.*

Similar index estimates hold for geodesics in k-convex domains in manifolds of positive sectional curvature that are solutions to the free endpoint problem for geodesics, following the method of Frankel (see [Fr1], [Fr2], [SW]).

Theorem 4. *Let N be a k-convex domain in an n-dimensional manifold of nonnegative sectional curvature. Suppose $\gamma : [0,1] \to N$ is a geodesic in N with $\gamma(0)$, $\gamma(1) \in \partial N$, and meeting ∂N orthogonally at the endpoints. Then γ has index at least $n-k$ (as a critical point of the length function on the space of paths in the manifold with endpoints in ∂N).*

Proof. The second variation of length for a normal variation V of γ is given by

$$I(V,V) = \int_0^1 (|\nabla_{\gamma'} V|^2 - \langle R(V,\gamma')\gamma', V\rangle) dt + \langle \nabla_V V, \gamma' \rangle \Big|_0^1.$$

Any $V(0) \in T_{\gamma(0)}\partial N$ parallel translates along γ to produce a parallel vector field $V(t)$ along γ. Since $V(t)$ remains orthogonal to γ, and ∂N has codimension one, $V(1) \in T_{\gamma(1)}\partial N$. Let $\mathcal{P}$ denote the vector space of parallel vector fields along γ that are tangent to ∂N at the endpoints. Since $\langle \cdot, \cdot \rangle$ is constant on $\mathcal{P}$, we may choose a basis $V_1, \ldots, V_{n-1}$ of $\mathcal{P}$ that is orthonormal with respect to $\langle \cdot, \cdot \rangle$ and diagonalizes the index form I. But,

$$\sum_{l=1}^{k} I(V_{j_l}, V_{j_l}) \leq \sum_{l=1}^{k} \langle \nabla_{V_{j_l}} V_{j_l}, \gamma' \rangle \Big|_0^1 < 0$$

for any distinct indices $1 \leq j_1, \ldots, j_k \leq n-1$, since ∂N is k-convex. Therefore, at most $k-1$ of $I(V_j, V_j)$, $j = 1, \ldots, n-1$, can be greater than or equal to zero. By reordering the basis if necessary, we may assume that $I(V_j, V_j) < 0$ for $j = 1, \ldots, n-k$. Since $\{V_1, \ldots, V_{n-1}\}$ diagonalizes I, the index form I is negative definite on the linear span of $V_1, \ldots, V_{n-k}$. Therefore, the index of γ is at least $n-k$. $\square$

3. Examples

In this section we give examples that show that the index estimates of Theorem 1 are sharp.

Example 1. Let $K = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-k+1}^2 = 1\} = S^{n-k} \times \mathbb{R}^{k-1}$. We may cut off and close off K to obtain a smooth compact k-convex hypersurface bounding a weakly convex region N in $\mathbb{R}^n$.

Lemma 5. *Let $N \subset \mathbb{R}^n$ be as above, and let Σ be the flat (minimal) disk intersecting ∂N orthogonally given by*

$$\Sigma = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 \leq 1 \text{ and } x_3 = x_4 = \cdots = x_n = 0\}.$$

Then Σ has index equal to $n - k - 1$.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Then e_1, e_2 are tangent to Σ, and any section V of the normal bundle to Σ can be written $V = \sum_{i=3}^n \varphi^i e_i$, for some smooth functions φ^i. Let $\frac{\partial}{\partial r}$ denote the unit radial vector in the x_1x_2-plane. Since Σ is totally geodesic, the second variation of area for normal variations reduces to

$$\begin{aligned} \delta^2\mathcal{A}(V,V) &= \int_\Sigma |\nabla V|^2 \, dx_1dx_2 \, + \int_{\partial\Sigma} \langle \nabla_V V, \frac{\partial}{\partial r}\rangle \, d\theta \\ &= -\int_\Sigma \langle \Delta V, V\rangle \, dx_1dx_2 \, + \int_{\partial\Sigma} [\langle \nabla_V V, \frac{\partial}{\partial r}\rangle + \langle \nabla_{\frac{\partial}{\partial r}} V, V\rangle] \, d\theta \\ &= \int_\Sigma \sum_{i=3}^n (\Delta\varphi^i)\varphi^i \, dx_1dx_2 + \int_{\partial\Sigma} \left(\sum_{i=3}^{n-k+1} (\varphi^i)^2 + \sum_{i=3}^n \frac{\partial\varphi^i}{\partial r}\varphi^i \right) d\theta. \end{aligned} \tag{3.1}$$

We may consider the disk Σ as a solution to the free boundary problem in the three-dimensional ball $x_1^2 + x_2^2 + x_i^2 \leq 1$, for any $3 \leq i \leq n-k+1$. However observe that any flat equitorial disk has index 1 for the free boundary problem in the unit ball in $\mathbb{R}^3$. The index is clearly at least one, since if X is a parallel unit normal to the disk, then $\delta^2\mathcal{A}(X,X) = \int_{\partial\Sigma} \langle \nabla_X X, \frac{\partial}{\partial r}\rangle < 0$. Suppose the index was greater than or equal to two. Then there would exist a two-dimensional subspace $\mathcal{S}$ of normal variations containing X for which the second variation of area was negative. Let $Y \in \mathcal{S}$ be a normal variation orthogonal to X, $\int_\Sigma \langle X, Y\rangle \, dx_1dx_2 = 0$. In particular, Y has zero average $\int_\Sigma Y \, dx_1dx_2 = 0$. But, any flat equitorial disk in the unit ball in $\mathbb{R}^3$ minimizes area subject to the constraint that it separates the volume of the ball in half. This implies that the second variation of area is nonnegative for variations that have zero average. That is, $\delta^2\mathcal{A}(Y,Y) \geq 0$, a contradiction.

From equation (3.1) we see that the index form decomposes into a sum of the index forms for the three-dimensional problems corresponding to each direction $i = 3, \ldots, n$. That is, the Jacobi operator decouples into scalar operators on the component functions. It follows that the index is equal to the sum of the indices of the three-dimensional problems. To see this, suppose that V_i is an eigenfunction for the three-dimensional problem (corresponding to the x_i-direction). Then V_i, extended to $\mathbb{R}^n$ with zeros in the extended components, is an eigenfunction for the original problem in $\mathbb{R}^n$. Conversely, if $V = \sum_{i=3}^n \varphi^i e_i$ is an eigenfunction for the problem in $\mathbb{R}^n$, then $\Delta\varphi^i = \lambda\varphi^i$ on Σ for $3 \le i \le n$, with boundary condition $\varphi^i + \frac{\partial\varphi^i}{\partial r} = 0$ on $\partial\Sigma$ for $3 \le i \le n-k+1$ and $\frac{\partial\varphi^i}{\partial r} = 0$ on $\partial\Sigma$ for $n-k+2 \le i \le n$. Therefore, if the projection V_i of V into the three-dimensional space corresponding to the x_i-direction is nonzero, then V_i is an eigenfunction for the corresponding three-dimensional problem.

From above, the index is equal to 1 for the three-dimensional problems corresponding to $3 \le i \le n-k+1$. For the remaining directions $n-k+2 \le i \le n$ the index is zero. These correspond to the flat directions, and any solution to the free boundary problem in the cylinder $x_1^2+x_2^2 \le 1$ is stable, since $\delta^2 A(V,V) \ge \int_\Sigma |\nabla V|^2\, da \ge 0$. Therefore, by adding up the indices, we obtain that the index of Σ is $n-k-1$. $\square$

4. Topology of k-convex hypersurfaces

We expect that the index estimates of Theorem 1 should have topological implications for k-convex hypersurfaces. However, this requires an existence theory for minimal surfaces that meet the constraint hypersurface on the inside, along the boundary of the surface. Two-convexity is a natural condition for preventing a minimal surface from hitting the boundary of the domain. The index estimates of [F1] (Theorem 2.7, 2.8), together with an existence theory for minimal disks inside two convex domains ([F2], Theorem 2.1), imply that if N is a simply connected two-convex domain in $\mathbb{R}^n$ or in an n-dimensional manifold of PIC, then N is contractible. The boundary of a sufficiently small tubular neighborhood of a curve is two-convex, and in [F2] we conjectured that every two-convex hypersurface is of this type. Since then, Huisken and Sinestrari [HS] have used the mean curvature flow to show that any compact two-convex hypersurface in $\mathbb{R}^n$ is diffeomorphic to a connected sum of S^{n-1} and finitely many copies of $S^1 \times S^{n-2}$.

Using a geodesic argument (as in [Fr1], [Fr2]), Lawson ([L], Theorem 1) proved that if N^n is a compact connected manifold of positive Ricci curvature, and the mean curvature of ∂N is nonnegative (i.e. $(n-1)$-convex), then ∂N is connected and the map $i_* : \pi(\partial N) \to \pi(N)$ induced by the inclusion is surjective. Using a minimal surface argument, Moore and Schulte [MS] proved that if M is an $(n-2)$-convex hypersurface in $\mathbb{R}^n$, then the unbounded component of $\mathbb{R}^n - M$ is simply connected. More generally, results of H. Wu [Wu] and Sha [Sh1] (see also [I]) using Morse theory prove that if N is a compact manifold of k-nonnegative curvature and has k-convex boundary, then N has the homotopy type of a CW-complex with cells of dimension $\le k-1$. In particular, they prove that $H_i(N;\mathbb{Z}) = \pi_i(N) = 0$ for $i \ge k$; $H_i(N,\partial N;\mathbb{Z}) = \pi_i(N,\partial N) = 0$ for $i \le n-k$. In the two-convex case, by [F2] Theorem 1.3, we obtained that $H_i(N;\mathbb{Z}) = \pi_i(N) = 0$ for $i = 1,\dots,n$ and $H_i(N,\partial N;\mathbb{Z}) = \pi_i(N,\partial N) = 0$ for $i \le n-1$ if N is a simply connected two-convex domain in $\mathbb{R}^n$ or in an n-dimensional manifold of PIC. However note that this result requires that N be simply connected, which is not assumed by Sha or Wu. Sha [Sh2] proved a converse to [Sh1], that if N is a compact connected manifold with nonempty boundary, and is furthermore a handlebody with handles only of dimension $\le k-1$, then N admits a Riemannian metric with *positive* sectional curvature and k-convex boundary.

Finally, we observe that the argument of a theorem of Mercuri and Noronha ([MN] Theorem 1) for hypersurfaces in Euclidean space with nonnegative isotropic curvature gives the following result on the vanishing of homology groups for k-convex hypersurfaces.

Theorem 6. *Let M be a compact k-convex hypersurface in $\mathbb{R}^{n+1}$. Then the homology groups $H_i(M;\mathbb{Z}) = 0$ for $k \leq i \leq n-k$.*

Proof. Let $f : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a compact, connected manifold M. Let ξ be a unit normal vector such that $\pm\xi$ are regular values of the Gauss map $\Phi : M^n \to S^n \subset \mathbb{R}^{n+1}$. Then the height function $h_\xi : M \to \mathbb{R}$ given by $h_\xi(x) = \langle f(x), \xi\rangle$ is a Morse function with critical points $\Phi^{-1}(\pm\xi)$. At such a point, the hessian of h_ξ is given, up to a sign, by the Weingarten operator A_ξ. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_xM that diagonalizes A_ξ, say, $A_\xi e_i = \lambda_i e_i$. If M is k-convex, then

$$\lambda_{i_1} + \cdots + \lambda_{i_k} > 0$$

for all distinct indices $1 \leq i_1, \dots, i_k \leq n$. Then we conclude that all but at most $k-1$ of the λ_i's have the same sign and hence the index of a critical point of h_ξ has to be one of $0,\ 1, \dots, k-1,\ n-k-1, \dots, n$. By the standard Morse Theory, M has the homotopy type of a CW-complex, with no cells of dimension i for $k \leq i \leq n-k$. Therefore the homology groups $H_i(M;\mathbb{Z}) = 0$ for $k \leq i \leq n-k$. $\square$

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