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ON FINITELY INJECTIVE MODULES AND LOCALLY PURE-INJECTIVE MODULES OVER PRÜFER DOMAINS

LUIGI SALCE

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ABSTRACT. Over Matlis valuation domains there exist finitely injective modules which are not direct sums of injective modules, as well as complete locally pure-injective modules which are not the completion of a direct sum of pure-injective modules. Over Prüfer domains which are either almost maximal, or h-local Matlis, finitely injective torsion modules and complete torsion-free locally pure-injective modules correspond to each other under the Matlis equivalence. Almost maximal Prüfer domains are characterized by the property that every torsion-free complete module is locally pure-injective. It is derived that semi-Dedekind domains are Dedekind.

INTRODUCTION

A result in the Ph.D. thesis of H. Bass, whose proof can be found in [3], states that a ring is left Noetherian if and only if the class of the injective modules is closed under taking direct sums. Ramamurthi and Rangaswamy [15] noted that left Noetherian rings are also characterized by the stronger condition that finitely injective modules are injective; recall that a module M is finitely injective if every finite set of elements of M is contained in an injective submodule; obviously, direct sums of injective modules are "trivial" examples of finitely injective modules.

Thus the following question naturally arises: if R is a non-Noetherian ring, is the class of the direct sums of injective modules strictly contained in the class of finitely injective modules? In other words, does a finitely injective R-module exist which fails to be a direct sum of injective modules? Note that such a module must be at least $\aleph_1$-generated, since a countably generated finitely injective module is necessarily a direct sum of injectives (see Proposition 1.1).

The first goal of this paper is to show that the answer to the above question is positive when R is a non-Noetherian Matlis valuation domain (recall that a domain is Matlis if the projective dimension of the field of quotients Q of R is ≤ 1).

The tool to obtain the required finitely injective module is a construction that was first performed by Hill [11] to exhibit $\aleph_1$-separable p-groups which are not direct sums of countable groups. Later on, this construction was modified by P. A. Griffith [9] to obtain flat $\aleph_1$-separable torsionless modules over non-left perfect rings,

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which fail to be direct sums of countably generated modules (here $\aleph_1$-separable means that each countable subset belongs to a countably generated direct summand). B. Zimmermann-Huisgen [20] generalized Hill's and Griffith's constructions (and similar ones by Gruson-Jensen [10] and W. Zimmermann [18]) to prove that all $\aleph_1$-separable modules over an arbitrary associative unital ring R are trivial (i.e., direct sums of countably generated modules) if and only if R has pure global dimension zero, namely, every pure inclusion of left R-modules splits.

We will adapt the construction in [20] to the setting of divisible modules over a non-Noetherian almost maximal Matlis valuation domain R, obtaining $\aleph_1$-generated finitely injective torsion modules whose only injective summands are isomorphic to finite direct sums of Q/R. Recall that over such a domain R all divisible modules are finitely injective (see [6] and [16]). Then we will extend the above result eliminating the almost maximal assumption.

Using the Matlis equivalence between h-divisible torsion modules and torsion-free complete modules, we obtain an example of a complete locally pure-injective torsion-free module over a non-Noetherian Matlis valuation domain, which fails to be the completion of a direct sum of pure-injective modules (see [19] for a study of locally pure-injective modules). Furthermore, we characterize almost maximal Prüfer domains R by means of the property that all torsion-free complete R-modules are locally pure-injective. As a subproduct of this result we derive that the semi-Dedekind domains, introduced by S. B. Lee in [13], are in fact Dedekind domains.

The latter goal of the paper is to show that, over Prüfer domains which are either almost maximal, or h-local Matlis, finitely injective torsion modules correspond, under the Matlis equivalence, to complete torsion-free locally pure-injectve modules.

1. Non-trivial finitely injective modules

It is useful to introduce some definitions. B. Zimmermann-Huisgen [20] calls a module over an arbitrary ring $\aleph_1$-separable if each countable subset belongs to a countably generated direct summand (this notion differs from that defined in [4, p. 87]); she also calls an $\aleph_1$-separable module "trivial" if it is a direct sum of countably generated submodules. Inspired by this definition, we say that a finitely injective module is *trivial* if it is a direct sum of injective modules. Given an infinite cardinal κ, a module D is said to be *κ-$\oplus$ injective* if every subset of D of cardinality $< \kappa$ is contained in a submodule D' of D which is a direct sum of injective modules. If, furthermore, D' can be chosen being a summand of D, then D is said to be *separable κ-$\oplus$ injective.*

A finitely injective module D is trivial exactly if it is κ-$\oplus$ injective for all κ. Note that a separable $\aleph_0$-$\oplus$ injective module is nothing else than a finitely injective module. The next result is similar to Proposition 2.5 in [19] (which deals with locally pure injective modules).

Proposition 1.1. *A finitely injective module is $\aleph_1$-$\oplus$ injective.*

Proof. Let M be a finitely injective module and $(x_n)_{n\in\omega}$ a sequence of elements in M. We argue by induction. Let $E_1 \oplus \cdots \oplus E_n \oplus N = M$, where $E_1 \oplus \cdots \oplus E_n$ is a finite direct sum of injective submodules of M containing $x_1,\dots,\ x_n$. Let $x_{n+1} = y + z$, with $y \in E_1 \oplus \cdots \oplus E_n$ and $z \in N$. Since N is finitely injective, z belongs to an injective submodule E_{n+1} of N, thus $E_1 \oplus \cdots \oplus E_{n+1}$ contains $x_1,\dots,$ x_{n+1}. There follows that $\bigoplus_{n\in\omega} E_n$ contains $(x_n)_{n\in\omega}$. □

An immediate consequence of Proposition 1.1 is that countably generated finitely injective modules are trivial, i.e., direct sums of injective modules.

Most of this section is devoted to proving the following.

Theorem 1.2. *Let R be a non-Noetherian Matlis almost maximal valuation domain. Then there exists a non-trivial separable $\aleph_1$-$\oplus$ injective module D.*

Proof. Let $0 < Rr_1 < Rr_2 < \cdots < Rr_n < \cdots$ be a strictly increasing sequence of principal ideals of R, and let $J = \bigcup_{n\in\omega} Rr_n$. For each ordinal $\sigma \le \aleph_1$ set

$$S_\sigma = \bigoplus_{\rho<\sigma} K_\rho, \quad P_\sigma = \prod_{\rho<\sigma} K_\rho,$$

where $K_\rho = K = Q/R$ for all ρ.

First we construct a countably generated submodule T_ω of P_ω containing S_ω, which is isomorphic to $\bigoplus_{\aleph_0} K$ such that S_ω is not a direct summand of T_ω.

Let $\mathbf{x} = (x_n)_{n\in\omega} \in P_\omega$ be the element defined by setting, for each $n \in \omega$, $x_n = r_n^{-1} + R$. Then $\mathrm{Ann}_R(\mathbf{x}) = Rr_1$, hence $R\mathbf{x} \cong R/Rr_1$, and an easy computation shows that $\mathrm{Ann}_R(\mathbf{x} + S_\omega) = J$. Let us set

$$T_\omega = S_\omega + E_\omega,$$

where E_ω denotes the injective envelope of the cyclic module $R\mathbf{x}$ inside the injective module P_ω. E_ω is countably generated, since $E_\omega = E(R\mathbf{x}) \cong Q/Rr_1 \cong K$. Therefore, T_ω is countably generated. Moreover, since E_ω is uniserial and $\mathrm{Ann}_R(\mathbf{x} + S_\omega) = J$, we have

$$T_\omega/S_\omega \cong E_\omega/(E_\omega \cap S_\omega) = E(R\mathbf{x})/J\mathbf{x} \cong Q/J.$$

It follows that $\mathrm{p.d.}\, T_\omega/S_\omega = 2$ (see [8, VI, Exercise 3.3]; as usual, $\mathrm{p.d.}\, M$ denotes the projective dimension of the module M). On the other hand, $\mathrm{p.d.}\, T_\omega = 1$; in fact, every finitely generated submodule of T_ω is contained in a submodule of the form $K_1 \oplus \cdots \oplus K_n + E_\omega$, which is isomorphic to $K_1 \oplus \cdots \oplus K_n \oplus K$, since $\mathrm{Ann}_R(\mathbf{x} + K_1 \oplus \cdots \oplus K_n) = Rr_{n+1}$. There it follows that T_ω is coherent, hence is isomorphic to $\bigoplus_{\aleph_0} K$. (N.B.: It is easily seen that the module T_ω is isomorphic to the module T constructed in [1, Lemma 3.4]. However, it is important to construct T_ω as above, in order to adapt Griffith's technique.)

Let us now fix a ladder system on the set $\lim(\aleph_1)$ of the limit ordinals in $\aleph_1$ (see [4, p. 40]); that is, for each limit ordinal $\lambda < \aleph_1$, we fix an increasing sequence $J(\lambda)$ of ordinals: $\sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots < \lambda$, whose supremum is λ. We denote by $S_{J(\lambda)}$ and $P_{J(\lambda)}$, respectively, the two modules $\bigoplus_{\rho\in J(\lambda)} K_\rho$ and $\prod_{\rho\in J(\lambda)} K_\rho$.

Let us define an element $\mathbf{x}^\lambda = (x^\lambda_\sigma)_{\sigma<\lambda} \in P_\lambda$ in the following way:

$$x^\lambda_\sigma = \begin{cases} r_n^{-1} + R & \text{if } \sigma = \sigma_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

Let us set

$$T_\lambda = S_{J(\lambda)} + E_\lambda,$$

where E_λ denotes the injective envelope of the cyclic module $R\mathbf{x}^\lambda$ inside the injective module $P_{J(\lambda)}$ (we look at $\mathbf{x}^\lambda$ as an element of $P_{J(\lambda)}$ in the obvious way). The module T_λ is a "clone" (B. Zimmermann-Huisgen calls the similar module in her construction a "carbon copy" [20]) of the module T_ω defined above. In particular, $T_\lambda/S_{J(\lambda)} \cong Q/J$; hence, $\mathrm{p.d.}\, T_\lambda/S_{J(\lambda)} = 2$.

We are now in a position to define the desired finitely injective module D. Let us set

$$D = S_{\aleph_1} + \sum_{\lambda \in \lim(\aleph_1)} E_\lambda.$$

We will prove that D is separable $\aleph_1$-$\oplus$ injective and is not a direct sum of injective modules. Actually, we will prove more: the only injective summands of D are isomorphic to $(Q/R)^n$ $(n \in \mathbb{N})$. Note that D is the union of an increasing chain of submodules: $D = \bigcup_{\sigma<\aleph_1} D_\sigma$, where, for each $\sigma < \aleph_1$,

$$D_\sigma = S_\sigma + \sum_{\lambda \in \lim(\sigma)} E_\lambda.$$

Each module D_σ is divisible, countably generated and coherent; hence, $D_\sigma \cong \bigoplus_{\aleph_0} K$ for all $\sigma < \aleph_1$. We need the following technical lemmas.

Lemma 1.3. *If $\lambda \in \lim(\aleph_1)$, then $D \cap P_\lambda = D_\lambda + E_\lambda$.*

Proof. The inclusion $D \cap P_\lambda \geq D_\lambda + E_\lambda$ is obvious. Conversely, let $x = \sum_{1\leq i\leq n} k_{\rho_i} + \sum_{1\leq j\leq m} e_{\lambda_j} \in D \cap P_\lambda$, where $k_{\rho_i} \in K_{\rho_i}$, $e_{\lambda_j} \in E_{\lambda_j}$, and $\lambda_1 < \cdots < \lambda_m$ are limit ordinals. Since $J(\lambda_m)$ has finite intersection with $J(\lambda_j)$ for each j, $x \in P_\lambda$ implies that $\lambda_m \leq \lambda$. Then $\sum_{1\leq j\leq m} e_{\lambda_j} \in P_\lambda$ and consequently also $\sum_{1\leq i\leq n} k_{\rho_i} \in P_\lambda$. Therefore, $\rho_i < \lambda$ for all i; hence, $\sum_{1\leq i\leq n} k_{\rho_i} + \sum_{1\leq j\leq m-1} e_{\lambda_j} \in D_\lambda$ and $e_{\lambda_m} \in E_\lambda$, so that $x \in D_\lambda + E_\lambda$ as desired. $\square$

Lemma 1.4. *If $\lambda \in \lim(\aleph_1)$, then*
(1) $D_\lambda = (D_\lambda \cap \prod_{\rho\in\lambda\setminus J(\lambda)} K_\rho) \oplus S_{J(\lambda)}$;
(2) $D_{\lambda+1} = (D_\lambda \cap \prod_{\rho\in\lambda\setminus J(\lambda)} K_\rho) \oplus T_\lambda \oplus K_\lambda$;
(3) $D_{\lambda+1} = D \cap \prod_{\rho\leq\lambda} K_\rho$.

Proof. (1) Using the notation in the proof of Lemma 1.3, let $x = \sum_{1\leq i\leq n} k_{\rho_i} + \sum_{1\leq j\leq m} e_{\lambda_j} \in D_\lambda$, where $\rho_1 < \cdots < \rho_m < \lambda$, and $\lambda_1 < \cdots < \lambda_m < \lambda$ are limit ordinals. Since $J(\lambda)$ has finite intersection with $J(\lambda_j)$ for each j, each e_{λ_j} has only finitely many components in $S_{J(\lambda)}$. Let y denote the element of $S_{J(\lambda)}$ which is the sum of all of these components for $1 \leq j \leq m$, and of all the elements k_{ρ_i} such that $\rho_i \in J(\lambda)$. Then $x - y \in D_\lambda \cap \prod_{\rho\in\lambda\setminus J(\lambda)} K_\rho$, so we are done.

(2) $D_{\lambda+1} = (D_\lambda + E_\lambda) \oplus K_\lambda = (((D_\lambda \cap \prod_{\rho\in\lambda\setminus J(\lambda)} K_\rho) \oplus S_{J(\lambda)}) + E_\lambda) \oplus K_\lambda = (D_\lambda \cap \prod_{\rho\in\lambda\setminus J(\lambda)} K_\rho) \oplus T_\lambda \oplus K_\lambda$.

(3) Using Lemma 1.3 and the equality $P_{\lambda+1} = P_\lambda \oplus K_\lambda$, we get $D_{\lambda+1} \leq D \cap (P_\lambda \oplus K_\lambda) = (D \cap P_\lambda) \oplus K_\lambda = (D_\lambda + E_\lambda) \oplus K_\lambda \leq D_{\lambda+1}$; hence, $D_{\lambda+1} = D \cap (P_\lambda \oplus K_\lambda) = D \cap \prod_{\rho\leq\lambda} K_\rho$. $\square$

Lemma 1.5. *If $\lambda \in \lim(\aleph_1)$, then*

$$D = D_{\lambda+1} \oplus \Big(D \cap \prod_{\lambda<\rho<\aleph_1} K_\rho\Big).$$

Proof. By Lemma 1.4 (3), we must prove that $D = (D \cap \prod_{\rho\leq\lambda} K_\rho) \oplus (D \cap \prod_{\lambda<\rho<\aleph_1} K_\rho)$. Let $x = \sum_{1\leq i\leq n} k_{\rho_i} + \sum_{1\leq j\leq m} e_{\lambda_j} \in D$, where $\rho_1 < \cdots < \rho_k \leq \lambda < \rho_{k+1} < \cdots < \rho_m$, and $\lambda_1 < \cdots < \lambda_h \leq \lambda < \lambda_{h+1} < \cdots < \lambda_m$ are limit ordinals.

Let y denote the element of $\prod_{\rho\leq\lambda} K_\rho$ which is the sum of the finitely many components in $\sum_{\rho\leq\lambda} K_\rho$ of the elements e_{λ_j} for $h+1\leq j\leq m$. Then

$$x=\Bigg(\sum_{1\leq i\leq k} k_{\rho_i}+\sum_{1\leq j\leq h} e_{\lambda_j}+y\Bigg)+\Bigg(\sum_{k<i\leq n} k_{\rho_i}+\sum_{h<j\leq m} e_{\lambda_j}-y\Bigg),$$

where the first summand belongs to $D\cap\prod_{\rho\leq\lambda} K_\rho$, and the second summand belongs to $D\cap\prod_{\lambda<\rho<\aleph_1} K_\rho$. Consequently, $D\subseteq D_{\lambda+1}\oplus(D\cap\prod_{\lambda<\rho<\aleph_1} K_\rho)$. The converse inclusion is trivial. $\square$

We now continue the proof of Theorem 1.2. By Lemma 1.5, $D_{\lambda+1}$ is a direct summand of D for each limit ordinal $\lambda<\aleph_1$; there it follows that D is not an injective module, as $D_{\lambda+1}\cong\bigoplus_{\aleph_0} K$ is not injective. D is separable $\aleph_1$-$\oplus$ injective, since a countable subset of D is contained in $D_{\lambda+1}$ for some limit ordinal $\lambda<\aleph_1$, and $D_{\lambda+1}$ is a direct sum of injective modules which is a summand in D. Finally, we must show that D is not a direct sum of injective modules. First we observe that D is not isomorphic to $\bigoplus_{\aleph_1} K$; in fact, from Lemma 1.4 we derive that $D_{\lambda+1}/D_\lambda\cong(T_\lambda/S_{J(\lambda)})\oplus K$; hence, $\operatorname{p.d.} D_{\lambda+1}/D_\lambda=2$ for all limit ordinals λ; a well-known result by Eklof (see [7, p. 75]) implies that $\operatorname{p.d.} D=2$, while $\operatorname{p.d.}\bigoplus_{\aleph_1} K=1$.

It remains to show that D is not a direct sum of injective modules of some different kind. Assume, by way of contradiction, that D has a direct summand E which is the injective hull of a module $B\cong\bigoplus_{\aleph_0} K$. Then there exists a countable ordinal λ such that $B\leq D_{\lambda+1}$. As B is essential in E, we get from Lemma 1.5 that $E\cap(D\cap\prod_{\lambda<\rho<\aleph_1} K_\rho)=0$. This implies that E embeds as a direct summand in $D_{\lambda+1}$, which is impossible, since E is not countably generated. $\square$

We now extend Theorem 1.2 to arbitrary non-Noetherian Matlis valuation domains. We need the following technical lemma; recall that, if S denotes a maximal immediate extension of R, then $E(Q/R)\cong QS/S$ (see [5]), which is an injective S-module, and, as S is a flat overring of R, injective S-modules are also injective as R-modules (see [12, p. 62, 3.6A] or [8, IX, Exercise 1.3]).

Lemma 1.6. *Let S be a maximal immediate extension of the valuation domain R, and let E be the injective hull as an R-module of a module of the form $B=\bigoplus_{\aleph} QS/S$ for some $\aleph$. Then E is an injective S-module.*

Proof. Let A be the injective hull of B as an S-module. Then A is an injective R-module which decomposes as an R-module in the form $A=E\oplus E'$. If $E'\neq 0$, then E' contains a direct R-summand $C\cong QS/S$. But this is impossible, since $B\oplus C$ is also a direct decomposition of S-modules, while B is an essential S-submodule of A. $\square$

Theorem 1.7. *Let R be a non-Noetherian Matlis valuation domain. Then there exists a non-trivial finitely injective module D.*

Proof. Let S be a maximal immediate extension of R. Let D be a finitely injective S-module constructed as in Theorem 1.2 with S instead of R; this makes sense, since S is a non-Noetherian Matlis maximal valuation domain. Clearly finitely injective S-modules are also finitely injective as R-modules; hence, it is enough to show that D is not a direct sum of injective R-modules. We will show more: the only injective summands of D as an R-module are isomorphic to finite direct sums of copies of $E(Q/R)$. Let us assume, by way of contradiction, that D decomposes as an R-module in the following way:

$$D = E\left(\bigoplus_{\aleph_0} Q/R\right) \oplus D' \cong E\left(\bigoplus_{\aleph_0} QS/S\right) \oplus D'$$

(note that an indecomposable injective R-summand of D is necessarily isomorphic to $E(Q/R)$, since D is a coherent R-module). To reach the contradiction it is enough to prove that $E(\bigoplus_{\aleph_0} QS/S)$ is the injective hull of $\bigoplus_{\aleph_0} QS/S$ as an S-module, and this is ensured by Lemma 1.6. □

2. Finitely injective modules and Matlis equivalence

A classical result by Matlis establishes an equivalence between the category of h-divisible torsion modules over an arbitrary domain R on one side, and the category of torsion-free complete R-modules on the other side; it is understood that the completion is made with respect to the R-topology. The two inverse correspondences send the h-divisible torsion R-module D to the torsion-free complete R-module $\mathrm{Hom}_R(K, D)$ $(K = Q/R)$, and the torsion-free complete R-module M to the h-divisible torsion R-module $K \otimes_R M$ (see [8, VIII.2]).

As finitely injective torsion modules are h-divisible (see [16, Corollary 1.3]), it is natural to ask which torsion-free complete modules correspond to them in the Matlis equivalence. Recall that, by a Warfield result [17] (see [8, VIII.2.9]), injective torsion modules and torsion-free Warfield cotorsion modules correspond to each other in the Matlis equivalence.

We will investigate the above question over Prüfer domains. Note that, over such a domain R, as torsion-free R-modules are flat, RD-injective and pure-injective modules coincide, as well as Enochs cotorsion and Warfield cotorsion modules, and torsion-free cotorsion modules are pure-injective (see [8, XIII]). Notice also that, in view of the Warfield's result quoted above, it is quite obvious that direct sums of injective torsion R-modules and completions of direct sums of torsion-free pure-injective R-modules correspond to each other under the Matlis equivalence.

Therefore, natural candidates as modules corresponding to the finitely injective torsion R-modules are the complete l-pure-injective modules. Recall that a module M over an arbitrary ring is said to be *locally pure-injective* (l-pure-injective for short), if every element $x \in M$ belongs to a pure-injective summand of M. These modules have been investigated recently by W. Zimmermann [19].

The first simple result is the following

Proposition 2.1. *Let M be a torsion-free (complete) l-pure-injective module over a domain R. Then $M \otimes_R K$ is finitely injective.*

Proof. It is enough to prove that every element of the form $m \otimes k$ $(0 \neq m \in M, k \in K)$ belongs to an injective summand of $M \otimes_R K$. By hypothesis, $M = X \oplus Y$, where X is pure-injective and contains m. Then $M \otimes_R K = (X \otimes_R K) \oplus (Y \otimes_R K)$, where $X \otimes_R K$ contains $m \otimes k$ and is injective, by the Warfield result quoted above. □

We give a simple direct proof of the converse of Proposition 2.1 for almost maximal Prüfer domains. Recall that, by [16], every h-divisible module over such a domain is finitely injective.

Proposition 2.2. *Let R be an almost maximal Prüfer domain, and let D be a torsion h-divisible R-module. Then* $\mathrm{Hom}_R(K, D)$ *is l-pure-injective.*

Proof. Let $0 \neq \phi : K \to D$ be a homomorphism. Since all quotients of Q are injective [2] (see also [8, IX.4.5]), we get that $D = \phi(K) \oplus A$, with $\phi(K)$ injective. Therefore, $\mathrm{Hom}_R(K, D) = \mathrm{Hom}_R(K, \phi(K)) \oplus \mathrm{Hom}_R(K, A)$, where $\mathrm{Hom}_R(K, \phi(K))$ is pure-injective by the Warfield's result quoted above, and clearly

$$\phi \in \mathrm{Hom}_R(K, \phi(K)). \qquad \square$$

There is another very good reason why the statement of Proposition 2.2 is true: every complete torsion-free module over an almost maximal Prüfer domain is l-pure-injective. Actually, with the help of Theorem 2.4 in [16], we can prove the following.

Theorem 2.3. *All torsion-free complete modules over a domain R are l-pure-injective if and only if R is an almost maximal Prüfer domain.*

Proof. The sufficiency is an obvious consequence of the following facts:

(i) As almost maximal Prüfer domains R are h-local, a torsion-free complete R-module M (i.e., reduced weakly cotorsion) is isomorphic to $\prod_P M^P$, where P ranges over $\mathrm{Max}(R)$ and $M^P = \mathrm{Hom}_R(\tilde{R}_P, M)$ is a complete module over the completion $\tilde{R}_P$ of the localization R_P (see [14, Corollary 8.6]);

(ii) $\tilde{R}_P$ is a maximal valuation domain, and every torsion-free module over such a domain is l-pure-injective both as an $\tilde{R}_P$-module and as an R-module (see [8, XIV.3]);

(iii) the class of l-pure-injective modules is closed under direct products [18, Prop. 2.4 (2)].

We will prove the necessity by showing that every h-divisible torsion R-module is finitely injective and then applying Theorem 2.4 in [16]. Let D be a torsion h-divisible R-module. Then $\mathrm{Hom}_R(K, D)$ is l-pure-injective, by hypothesis; hence, $D \cong K \otimes_R \mathrm{Hom}_R(K, D)$ is finitely injective, by Proposition 2.1. $\square$

An immediate consequence is the following.

Corollary 2.4. *Let R be an almost maximal Prüfer domain. Then the two classes of torsion finitely injective R-modules and of torsion-free complete l-pure-injective R-modules correspond to each other under the Matlis equivalence.* $\square$

Another interesting consequence is that semi-Dedekind domains, introduced and investigated by S. B. Lee in [13], are Dedekind. Recall that a domain R is *semi-Dedekind* if all h-divisible R-modules are pure-injective. Lee proved that, if R is semi-Dedekind, then every torsion-free complete R-module is pure-injective. As pure-injective modules are trivially l-pure-injective, and since semi-Dedekind Prüfer domains are necessarily Dedekind domains, from Theorem 2.3 we derive the following.

Corollary 2.5. *A semi-Dedekind domain is a Dedekind domain.* $\square$

Recall that an almost maximal Prüfer domain is necessarily h-local. In the remaining part of this paper we will extend the result in Corollary 2.4 to h-local Prüfer domains, but with the additional assumption that they are Matlis.

We will need the next technical result.

Lemma 2.6. *Let R be a valuation domain and $\{M_i\}_{i\in I}$ a family of pure-injective torsion-free R-modules. Then the completion of $\bigoplus_{i\in I} M_i$ in the R-topology is l-pure-injective.*

Proof. Let $M = \widetilde{\bigoplus_{i\in I} M_i}$ be the completion of $\bigoplus_{i\in I} M_i$ and let S be a maximal immediate extension of R. An element $x \in M$ is the limit of a Cauchy net $\{x_r\}_{0\neq r\in R}$ of elements in $\bigoplus_{i\in I} M_i$, which is an S-module. If $\alpha \in S$, then αx is the limit of the net $\{\alpha x_r\}_{0\neq r\in R}$, hence $\alpha x \in M$; therefore, M is an S-module too. Pick a non-zero element $m \in M$; if xJ is the pure R-submodule of M generated by x $(J \leq Q)$, then xJS is a pure-injective summand of M containing x; hence, M is l-pure-injective. □

We can now prove the converse of Proposition 2.1 for modules over Matlis valuation domains.

Proposition 2.7. *Let R be a Matlis valuation domain, and let D be a finitely injective torsion R-module. Then $\mathrm{Hom}_R(K,D)$ is l-pure-injective.*

Proof. Let $0 \neq \phi \in \mathrm{Hom}_R(K,D)$. Since K is $\aleph_0$-generated, by Proposition 1.1 we have that $\phi(K) \leq \bigoplus_{n\in\omega} E_n$, where the E_n are injective modules. From the exact sequence

$$0 \to \bigoplus_{n\in\omega} E_n \to D \to X \to 0,$$

where $X = D/\bigoplus_{n\in\omega} E_n$, we obtain the exact sequence

$$0 \to \mathrm{Hom}_R\left(K, \bigoplus_{n\in\omega} E_n\right) \to \mathrm{Hom}_R(K,D) \to \mathrm{Hom}_R(K,X) \to \mathrm{Ext}^1_R\left(K, \bigoplus_{n\in\omega} E_n\right),$$

where the last term vanishes, since $\mathrm{p.d.}\, K = 1$ and $\bigoplus_{n\in\omega} E_n$ is h-divisible (see [7, VII.2.5]). Now $\mathrm{Hom}_R(K, \bigoplus_{n\in\omega} E_n)$ is the completion of a direct sum of torsion-free pure-injective modules by the Warfield result in [17]; hence, it is l-pure-injective by Lemma 2.6. Furthermore, it is pure in $\mathrm{Hom}_R(K,D)$, since $\mathrm{Hom}_R(K,X)$ is torsion-free. Clearly $\phi \in \mathrm{Hom}_R(K, \bigoplus_{n\in\omega} E_n)$, so it belongs to a pure-injective summand of it, which is as well a direct summand of $\mathrm{Hom}_R(K,D)$, being pure in it. Therefore, we can conclude that $\mathrm{Hom}_R(K,D)$ is l-pure-injective. □

From Theorem 1.7 and Proposition 2.7 we immediately obtain the following.

Corollary 2.8. *Let R be a Matlis non-Noetherian valuation domain. There exists a complete l-pure-injective torsion-free R-module which fails to be the completion of a direct sum of pure-injective modules.* □

We can also extend Corollary 2.4 to h-local Matlis Prüfer domains.

Corollary 2.9. *Let R be an h-local Matlis Prüfer domain. Then the two classes of torsion finitely injective R-modules and of torsion-free complete l-pure-injective R-modules correspond to each other under the Matlis equivalence.*

Proof. By h-locality, we can reduce to the local case, namely, to Matlis valuation domains, so the proof follows from Propositions 2.1 and 2.7. □

We close the paper with two open questions.

Question 1. Given an arbitrary non-Noetherian ring R, does a non-trivial finitely injective R-module exist?

Question 2. Given an arbitrary integral domain R, which torsion-free complete R-modules correspond to the torsion finitely injective R-modules under the Matlis equivalence, and which torsion finitely injective R-modules correspond to the complete torsion-free l-pure-injective R-modules?

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Dipartimento di Matematica Pura e Applicata, Università di Padova, via Trieste 63, I-35121 Padova, Italy

E-mail address: salce@math.unipd.it