ON FINITELY INJECTIVE MODULES AND LOCALLY PURE-INJECTIVE MODULES OVER PRÜFER DOMAINS

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Abstract. Over Matlis valuation domains there exist finitely injective modules which are not direct sums of injective modules, as well as complete locally pure-injective modules which are not the completion of a direct sum of pure-injective modules. Over Prüfer domains which are either almost maximal, or h-local Matlis, finitely injective torsion modules and complete torsion-free locally pure-injective modules correspond to each other under the Matlis equivalence. Almost maximal Prüfer domains are characterized by the property that every torsion-free complete module is locally pure-injective. It is derived that semi-Dedekind domains are Dedekind.

Introduction

A result in the Ph.D. thesis of H. Bass, whose proof can be found in [3], states that a ring is left Noetherian if and only if the class of the injective modules is closed under taking direct sums. Ramanurthi and Rangaswamy [15] noted that left Noetherian rings are also characterized by the stronger condition that finitely injective modules are injective; recall that a module $M$ is finitely injective if every finite set of elements of $M$ is contained in an injective submodule; obviously, direct sums of injective modules are “trivial” examples of finitely injective modules.

Thus the following question naturally arises: if $R$ is a non-Noetherian ring, is the class of the direct sums of injective modules strictly contained in the class of finitely injective modules? In other words, does a finitely injective $R$-module exist which does not belong to the class of direct sums of injective modules? Note that such a module must be at least $\aleph_1$-generated, since a countably generated finitely injective module is necessarily a direct sum of injectives (see Proposition 1.1).

The first goal of this paper is to show that the answer to the above question is positive when $R$ is a non-Noetherian Matlis valuation domain (recall that a domain is Matlis if the projective dimension of the field of quotients $Q$ of $R$ is $\leq 1$).

The tool to obtain the required finitely injective module is a construction that was first performed by Hill [11] to exhibit $\aleph_1$-separable $p$-groups which are not direct sums of countable groups. Later on, this construction was modified by P. A. Griffith [9] to obtain flat $\aleph_1$-separable torsionless modules over non-left perfect rings,
which fail to be direct sums of countably generated modules (here $\aleph_1$-separable means that each countable subset belongs to a countably generated direct summand). B. Zimmermann-Huisgen [20] generalized Hill’s and Griffith’s constructions (and similar ones by Gruson-Jensen [10] and W. Zimmermann [18]) to prove that all $\aleph_1$-separable modules over an arbitrary associative unital ring $R$ are trivial (i.e., direct sums of countably generated modules) if and only if $R$ has pure global dimension zero, namely, every pure inclusion of left $R$-modules splits.

We will adapt the construction in [20] to the setting of divisible modules over a non-Noetherian almost maximal Matlis valuation domain $R$, obtaining $\aleph_1$-generated finitely injective torsion modules whose only injective summands are isomorphic to finite direct sums of $Q/R$. Recall that over such a domain $R$ all divisible modules are finitely injective (see [6] and [16]). Then we will extend the above result eliminating the almost maximal assumption.

Using the Matlis equivalence between $h$-divisible torsion modules and torsion-free complete modules, we obtain an example of a complete locally pure-injective torsion-free module over a non-Noetherian Matlis valuation domain, which fails to be the completion of a direct sum of pure-injective modules (see [19] for a study of locally pure-injective modules). Furthermore, we characterize almost maximal Prüfer domains $R$ by means of the property that all torsion-free complete $R$-modules are locally pure-injective. As a subproduct of this result we derive that the semidedekind domains, introduced by S. B. Lee in [13], are in fact Dedekind domains.

The latter goal of the paper is to show that, over Prüfer domains which are either almost maximal, or $h$-local Matlis, finitely injective torsion modules correspond, under the Matlis equivalence, to complete torsion-free locally pure-injective modules.

1. Non-trivial finitely injective modules

It is useful to introduce some definitions. B. Zimmermann-Huisgen [20] calls a module over an arbitrary ring $\aleph_1$-separable if each countable subset belongs to a countably generated direct summand (this notion differs from that defined in [4], p. 87); she also calls an $\aleph_1$-separable module “trivial” if it is a direct sum of countably generated submodules. Inspired by this definition, we say that a finitely injective module is trivial if it is $\kappa$-injective for all $\kappa$. A finitely injective module is trivial exactly if it is $\aleph_0$-injective.

A finitely injective module $D$ is trivial exactly if it is $\kappa$-injective for all $\kappa$. Note that a separable $\aleph_0$-injective module is nothing else than a finitely injective module. The next result is similar to Proposition 2.5 in [19] (which deals with locally pure injective modules).

**Proposition 1.1.** A finitely injective module is $\aleph_1$-injective.

**Proof.** Let $M$ be a finitely injective module and $(x_n)_{n \in \omega}$ a sequence of elements in $M$. We argue by induction. Let $E_1 \oplus \cdots \oplus E_n \oplus N = M$, where $E_1 \oplus \cdots \oplus E_n$ is a finite direct sum of injective submodules of $M$ containing $x_1, \ldots, x_n$. Let $x_{n+1} = y + z$, with $y \in E_1 \oplus \cdots \oplus E_n$ and $z \in N$. Since $N$ is finitely injective, $z$ belongs to an injective submodule $E_{n+1}$ of $N$, thus $E_1 \oplus \cdots \oplus E_{n+1}$ contains $x_1, \ldots, x_{n+1}$. There follows that $\bigoplus_{n \in \omega} E_n$ contains $(x_n)_{n \in \omega}$. □
An immediate consequence of Proposition 1.1 is that countably generated finitely injective modules are trivial, i.e., direct sums of injective modules.

Most of this section is devoted to proving the following.

**Theorem 1.2.** Let $R$ be a non-Noetherian Matlis almost maximal valuation domain. Then there exists a non-trivial separable $\aleph_1 \oplus$ injective module $D$.

**Proof.** Let $0 < R r_1 < R r_2 < \cdots < R r_n < \cdots$ be a strictly increasing sequence of principal ideals of $R$, and let $J = \bigcup_{n \in \omega} R r_n$. For each ordinal $\sigma \leq \aleph_1$ set

$$S_{\sigma} = \bigoplus_{\rho < \sigma} K_{\rho}, \quad P_{\sigma} = \prod_{\rho < \sigma} K_{\rho},$$

where $K_{\rho} = K = Q/R$ for all $\rho$.

First we construct a countably generated submodule $T_{\omega}$ of $P_{\omega}$ containing $S_{\omega}$, which is isomorphic to $\bigoplus_{\aleph_0} K$ such that $S_{\omega}$ is not a direct summand of $T_{\omega}$.

Let $x = (x_n)_{n \in \omega} \in P_{\omega}$ be the element defined by setting, for each $n \in \omega$, $x_n = r_n^{-1} + R$. Then $\text{Ann}_R(x) = R r_1$, hence $Rx \cong R/R r_1$, and an easy computation shows that $\text{Ann}_R(x + S_{\omega}) = J$. Let us set

$$T_{\omega} = S_{\omega} + E_{\omega},$$

where $E_{\omega}$ denotes the injective envelope of the cyclic module $Rx$ inside the injective module $P_{\omega}$. $E_{\omega}$ is countably generated, since $E_{\omega} = E(Rx) \cong Q/R r_1 \cong K$. Therefore, $T_{\omega}$ is countably generated. Moreover, since $E_{\omega}$ is uniserial and $\text{Ann}_R(x + S_{\omega}) = J$, we have

$$T_{\omega}/S_{\omega} \cong E_{\omega}/(E_{\omega} \cap S_{\omega}) = E(Rx)/Jx \cong Q/J.$$

It follows that $\text{p.d.} T_{\omega}/S_{\omega} = 2$ (see [8] VI, Exercise 3.3]; as usual, $\text{p.d.} M$ denotes the projective dimension of the module $M$). On the other hand, $\text{p.d.} T_{\omega} = 1$; in fact, every finitely generated submodule of $T_{\omega}$ is contained in a submodule of the form $K_1 \oplus \cdots \oplus K_n + E_{\omega}$, which is isomorphic to $K_1 \oplus \cdots \oplus K_n \oplus K$, since $\text{Ann}_R(x + K_1 \oplus \cdots \oplus K_n) = R r_n + 1$. There it follows that $T_{\omega}$ is coherent, hence is isomorphic to $\bigoplus_{\aleph_0} K$. (N.B.: It is easily seen that the module $T_{\omega}$ is isomorphic to the module $T$ constructed in [1] Lemma 3.4). However, it is important to construct $T_{\omega}$ as above, in order to adapt Griffith’s technique.)

Let us now fix a ladder system on the set $\lim(\aleph_1)$ of the limit ordinals in $\aleph_1$ (see [4] p. 40]; that is, for each limit ordinal $\lambda < \aleph_1$, we fix an increasing sequence $J(\lambda)$ of ordinals: $\sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots < \lambda$, whose supremum is $\lambda$. We denote by $S_{J(\lambda)}$ and $P_{J(\lambda)}$, respectively, the two modules $\bigoplus_{\rho \in J(\lambda)} K_{\rho}$ and $\prod_{\rho \in J(\lambda)} K_{\rho}$.

Let us define an element $x^\lambda = (x^\lambda_\sigma)_{\sigma < \lambda} \in P_{\lambda}$ in the following way:

$$x^\lambda_\sigma = \begin{cases} r_n^{-1} + R & \text{if } \sigma = \sigma_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

Let us set

$$T_{\lambda} = S_{J(\lambda)} + E_{\lambda},$$

where $E_{\lambda}$ denotes the injective envelope of the cyclic module $Rx^\lambda$ inside the injective module $P_{J(\lambda)}$ (we look at $x^\lambda$ as an element of $P_{J(\lambda)}$ in the obvious way). The module $T_{\lambda}$ is a “clone” (B. Zimmermann-Huisgen calls the similar module in her construction a “carbon copy” [30]) of the module $T_{\omega}$ defined above. In particular, $T_{\lambda}/S_{J(\lambda)} \cong Q/J$; hence, p.d.$T_{\lambda}/S_{J(\lambda)} = 2$. 


We are now in a position to define the desired finitely injective module $D$. Let us set

$$D = S_{\aleph_1} + \sum_{\lambda \in \lim(\aleph_1)} E_\lambda.$$  

We will prove that $D$ is separable $\aleph_1$-injective and is not a direct sum of injective modules. Actually, we will prove more: the only injective summands of $D$ are isomorphic to $(Q/R)^n$ $(n \in \mathbb{N})$. Note that $D$ is the union of an increasing chain of submodules: $D = \bigcup_{\sigma < \aleph_1} D_\sigma$, where, for each $\sigma < \aleph_1$,

$$D_\sigma = S_\sigma + \sum_{\lambda \in \lim(\sigma)} E_\lambda.$$  

Each module $D_\sigma$ is divisible, countably generated and coherent; hence, $D_\sigma \cong \bigoplus_{\aleph_0} K$ for all $\sigma < \aleph_1$. We need the following technical lemmas.

**Lemma 1.3.** If $\lambda \in \lim(\aleph_1)$, then $D \cap P_\lambda = D_\lambda + E_\lambda$.

**Proof.** The inclusion $D \cap P_\lambda \geq D_\lambda + E_\lambda$ is obvious. Conversely, let $x = \sum_{1 \leq i \leq n} k_{\rho_i} + \sum_{1 \leq j \leq m} e_{\lambda_j} \in D \cap P_\lambda$, where $k_{\rho_i} \in K_{\rho_i}$, $e_{\lambda_j} \in E_{\lambda_j}$, and $\lambda_1 < \cdots < \lambda_m$ are limit ordinals. Since $J(\lambda)$ has finite intersection with $J(\lambda_j)$ for each $j$, $x \in P_\lambda$ implies that $\lambda_m \leq \lambda$. Then $\sum_{1 \leq j \leq m} e_{\lambda_j} \in P_\lambda$ and consequently also $\sum_{1 \leq i \leq n} k_{\rho_i} \in P_\lambda$. Therefore, $\rho_i < \lambda$ for all $i$; hence, $\sum_{1 \leq i \leq n} k_{\rho_i} + \sum_{1 \leq j \leq m} e_{\lambda_j} \in D_\lambda$ and $e_{\lambda_m} \in E_\lambda$, so that $x \in D_\lambda + E_\lambda$ as desired. \hfill $\square$

**Lemma 1.4.** If $\lambda \in \lim(\aleph_1)$, then

(1) $D_\lambda = (D_\lambda \cap \prod_{\rho \in \lambda \setminus J(\lambda)} K_{\rho}) \oplus S_{J(\lambda)}$;

(2) $D_{\lambda+1} = (D_\lambda \cap \prod_{\rho \in \lambda \setminus J(\lambda)} K_{\rho}) \oplus T_\lambda \oplus K_\lambda$;

(3) $D_{\lambda+1} = D_\lambda \cap \prod_{\rho \leq \lambda} K_{\rho}$.

**Proof.** (1) Using the notation in the proof of Lemma [1.3] let $x = \sum_{1 \leq i \leq n} k_{\rho_i} + \sum_{1 \leq j \leq m} e_{\lambda_j} \in D_\lambda$, where $\rho_1 < \cdots < \rho_m < \lambda$, and $\lambda_1 < \cdots < \lambda_m < \lambda$ are limit ordinals. Since $J(\lambda)$ has finite intersection with $J(\lambda_j)$ for each $j$, each $e_{\lambda_j}$ has only finitely many components in $S_{J(\lambda)}$. Let $y$ denote the element of $S_{J(\lambda)}$ which is the sum of all of these components for $1 \leq j \leq m$, and of all the elements $k_{\rho_i}$, such that $\rho_i \in J(\lambda)$. Then $x - y \in D_\lambda \cap \prod_{\rho \in \lambda \setminus J(\lambda)} K_{\rho}$, so we are done.

(2) $D_{\lambda+1} = (D_\lambda + E_\lambda) \oplus K_\lambda = ((D_\lambda \cap \prod_{\rho \in \lambda \setminus J(\lambda)} K_{\rho}) \oplus S_{J(\lambda)}) \oplus E_\lambda \oplus K_\lambda = (D_\lambda \cap \prod_{\rho \in \lambda \setminus J(\lambda)} K_{\rho}) \oplus T_\lambda \oplus K_\lambda$.

(3) Using Lemma [1.3] and the equality $P_{\lambda+1} = P_\lambda \oplus K_\lambda$, we get $D_{\lambda+1} \leq D \cap (P_\lambda \oplus K_\lambda) = (D \cap P_\lambda) \oplus K_\lambda = (D_\lambda + E_\lambda) \oplus K_\lambda \leq D_{\lambda+1}$; hence, $D_{\lambda+1} = D \cap (P_\lambda \oplus K_\lambda) = D \cap \prod_{\rho \leq \lambda} K_{\rho}$. \hfill $\square$

**Lemma 1.5.** If $\lambda \in \lim(\aleph_1)$, then

$$D = D_{\lambda+1} \oplus \left( D \cap \prod_{\lambda < \rho \leq \aleph_1} K_{\rho} \right).$$

**Proof.** By Lemma [1.4] (3), we must prove that $D = (D \cap \prod_{\rho \leq \lambda} K_{\rho}) \oplus (D \cap \prod_{\lambda < \rho \leq \aleph_1} K_{\rho})$. Let $x = \sum_{1 \leq i \leq n} k_{\rho_i} + \sum_{1 \leq j \leq m} e_{\lambda_j} \in D$, where $\rho_1 < \cdots < \rho_k < \lambda < \rho_{k+1} < \cdots < \rho_m$, and $\lambda_1 < \cdots < \lambda_h < \lambda < \lambda_{h+1} < \cdots < \lambda_m$ are limit ordinals.
Let \( y \) denote the element of \( \prod_{\rho \leq \lambda} K_{\rho} \) which is the sum of the finitely many components in \( \sum_{\rho \leq \lambda} K_{\rho} \) of the elements \( e_{\lambda_j} \) for \( h+1 \leq j \leq m \). Then

\[
x = \left( \sum_{1 \leq k \leq h} k_{\rho_k} + \sum_{1 \leq j \leq h} e_{\lambda_j} + y \right) + \left( \sum_{k < i \leq n} k_{\rho_i} + \sum_{h < j \leq m} e_{\lambda_j} - y \right),
\]

where the first summand belongs to \( D \cap \prod_{\rho \leq \lambda} K_{\rho} \), and the second summand belongs to \( D \cap \prod_{\lambda < \rho < \aleph_1} K_{\rho} \). Consequently, \( D \subseteq D_{\lambda+1} \oplus (D \cap \prod_{\lambda < \rho < \aleph_1} K_{\rho}) \). The converse inclusion is trivial. \( \square \)

We now continue the proof of Theorem 1.2. By Lemma 1.5, \( D_{\lambda+1} \) is a direct summand of \( D \) for each limit ordinal \( \lambda < \aleph_1 \); there follows that \( D \) is not an injective module, as \( D_{\lambda+1} \cong \bigoplus_{\aleph_0} K \) is not injective. \( D \) is separable \( \aleph_1 \)-injective, since a countable subset of \( D \) is contained in \( D_{\lambda+1} \) for some limit ordinal \( \lambda < \aleph_1 \), and \( D_{\lambda+1} \) is a direct sum of injective modules which is a summand in \( D \). Finally, we must show that \( D \) is not a direct sum of injective modules.

It remains to show that \( D \) is not a direct sum of injective modules of some different kind. Assume, by way of contradiction, that \( D \) has a direct summand \( E \) which is the injective hull of a module \( B \cong \bigoplus_{\aleph_0} K \). Then there exists a countable ordinal \( \lambda \) such that \( B \subseteq D_{\lambda+1} \). As \( B \) is essential in \( E \), we get from Lemma 1.5 that \( E \cap (D \cap \prod_{\lambda < \rho < \aleph_1} K_{\rho}) = 0 \). This implies that \( E \) embeds as a direct summand in \( D_{\lambda+1} \), which is impossible, since \( E \) is not countably generated. \( \square \)

We now extend Theorem 1.2 to arbitrary non-Noetherian Matlis valuation domains. We need the following technical lemma; recall that, if \( S \) denotes a maximal immediate extension of \( R \), then \( E(Q/R) \cong QS/S \) (see [5]), which is an injective \( S \)-module, and, as \( S \) is a flat overring of \( R \), injective \( S \)-modules are also injective as \( R \)-modules (see [12], p. 62, 3.6A or [8], IX, Exercise 1.3)

\textbf{Lemma 1.6.} Let \( S \) be a maximal immediate extension of the valuation domain \( R \), and let \( E \) be the injective hull as an \( R \)-module of a module of the form \( B = \bigoplus_{\aleph_0} QS/S \) for some \( R \). Then \( E \) is an injective \( S \)-module.

\textbf{Proof.} Let \( A \) be the injective hull of \( B \) as an \( S \)-module. Then \( A \) is an injective \( R \)-module which decomposes as an \( R \)-module in the form \( A = E \oplus E' \). If \( E' \neq 0 \), then \( E' \) contains a direct \( R \)-summand \( C \cong QS/S \). But this is impossible, since \( B \oplus C \) is also a direct decomposition of \( S \)-modules, while \( B \) is an essential \( S \)-submodule of \( A \). \( \square \)

\textbf{Theorem 1.7.} Let \( R \) be a non-Noetherian Matlis valuation domain. Then there exists a non-trivial finitely injective module \( D \).

\textbf{Proof.} Let \( S \) be a maximal immediate extension of \( R \). Let \( D \) be a finitely injective \( S \)-module constructed as in Theorem 1.2 with \( S \) instead of \( R \); this makes sense, since \( S \) is a non-Noetherian Matlis maximal valuation domain. Clearly finitely injective \( S \)-modules are also finitely injective as \( R \)-modules; hence, it is enough to show that \( D \) is not a direct sum of injective \( R \)-modules. We will show more: the only injective summands of \( D \) as an \( R \)-module are isomorphic to finite direct sums.
of copies of $E(Q/R)$. Let us assume, by way of contradiction, that $D$ decomposes as an $R$-module in the following way:

$$D = E\left(\bigoplus_{\mathbb{N}_0} Q/R\right) \oplus D' \cong E\left(\bigoplus_{\mathbb{N}_0} QS/S\right) \oplus D'$$

(note that an indecomposable injective $R$-summand of $D$ is necessarily isomorphic to $E(Q/R)$, since $D$ is a coherent $R$-module). To reach the contradiction it is enough to prove that $E(\bigoplus_{\mathbb{N}_0} QS/S)$ is the injective hull of $\bigoplus_{\mathbb{N}_0} QS/S$ as an $S$-module, and this is ensured by Lemma 1.6.

## 2. Finitely injective modules and Matlis equivalence

A classical result by Matlis establishes an equivalence between the category of $h$-divisible torsion modules over an arbitrary domain $R$ on one side, and the category of torsion-free complete $R$-modules on the other side; it is understood that the completion is made with respect to the $R$-topology. The two inverse correspondences send the $h$-divisible torsion $R$-module $D$ to the torsion-free complete $R$-module $\text{Hom}_R(K,D)$ ($K = Q/R$), and the torsion-free complete $R$-module $M$ to the $h$-divisible torsion $R$-module $K \otimes_R M$ (see [8, VIII.2]).

As finitely injective torsion modules are $h$-divisible (see [16, Corollary 1.3]), it is natural to ask which torsion-free complete modules correspond to them in the Matlis equivalence. Recall that, by a Warfield result [17] (see [8, VIII.2.9]), injective torsion modules and torsion-free Warfield cotorsion modules correspond to each other in the Matlis equivalence.

We will investigate the above question over Pr"ufer domains. Note that, over such a domain $R$, as torsion-free $R$-modules are flat, RD-injective and pure-injective modules coincide, as well as Enochs cotorsion and Warfield cotorsion modules, and torsion-free cotorsion modules are pure-injective (see [8, XIII]). Notice also that, in view of the Warfield’s result quoted above, it is quite obvious that direct sums of injective torsion $R$-modules and completions of direct sums of torsion-free pure-injective $R$-modules correspond to each other under the Matlis equivalence.

Therefore, natural candidates as modules corresponding to the finitely injective torsion $R$-modules are the complete $l$-pure-injective modules. Recall that a module $M$ over an arbitrary ring is said to be locally pure-injective ($l$-pure-injective for short), if every element $x \in M$ belongs to a pure-injective summand of $M$. These modules have been investigated recently by W. Zimmermann [19].

The first simple result is the following

**Proposition 2.1.** Let $M$ be a torsion-free (complete) $l$-pure-injective module over a domain $R$. Then $M \otimes_R K$ is finitely injective.

**Proof.** It is enough to prove that every element of the form $m \otimes k$ ($0 \neq m \in M, k \in K$) belongs to an injective summand of $M \otimes_R K$. By hypothesis, $M = X \oplus Y$, where $X$ is pure-injective and contains $m$. Then $M \otimes_R K = (X \otimes_R K) \oplus (Y \otimes_R K)$, where $X \otimes_R K$ contains $m \otimes k$ and is injective, by the Warfield result quoted above. □

We give a simple direct proof of the converse of Proposition 2.1 for almost maximal Pr"ufer domains. Recall that, by [16], every $h$-divisible module over such a domain is finitely injective.

**Proposition 2.2.** Let $R$ be an almost maximal Pr"ufer domain, and let $D$ be a torsion $h$-divisible $R$-module. Then $\text{Hom}_R(K,D)$ is $l$-pure-injective.
Proof. Let \( 0 \neq \phi : K \to D \) be a homomorphism. Since all quotients of \( Q \) are injective \([2]\) (see also \([8, IX.4.5]\)), we get that \( D = \phi(K) \oplus A \), with \( \phi(K) \) injective. Therefore, \( \text{Hom}_R(K, D) = \text{Hom}_R(K, \phi(K)) \oplus \text{Hom}_R(K, A) \), where \( \text{Hom}_R(K, \phi(K)) \) is pure-injective by the Warfield’s result quoted above, and clearly
\[
\phi \in \text{Hom}_R(K, \phi(K)). \quad \square
\]

There is another very good reason why the statement of Proposition \(2.2\) is true: every complete torsion-free module over an almost maximal Prüfer domain is \( l \)-pure-injective. Actually, with the help of Theorem \(2.4\) in \([16]\), we can prove the following.

**Theorem 2.3.** All torsion-free complete modules over a domain \( R \) are \( l \)-pure-injective if and only if \( R \) is an almost maximal Prüfer domain.

**Proof.** The sufficiency is an obvious consequence of the following facts:

(i) As almost maximal Prüfer domains \( R \) are \( h \)-local, a torsion-free complete \( R \)-module \( M \) (i.e., reduced weakly cotorsion) is isomorphic to \( \prod_P M_P \), where \( P \) ranges over \( \text{Max}(R) \) and \( M_P = \text{Hom}_R(\hat{R}_P, M) \) is a complete module over the completion \( \hat{R}_P \) of the localization \( R_P \) (see \([14, Corollary 8.6]\));

(ii) \( \hat{R}_P \) is a maximal valuation domain, and every torsion-free module over such a domain is \( l \)-pure-injective both as an \( \hat{R}_P \)-module and as an \( R \)-module (see \([8, XIV.3]\));

(iii) the class of \( l \)-pure-injective modules is closed under direct products \([18, Prop. 2.4 (2)]\).

We will prove the necessity by showing that every \( h \)-divisible torsion \( R \)-module is finitely injective and then applying Theorem \(2.4\) in \([10]\). Let \( D \) be a torsion \( h \)-divisible \( R \)-module. Then \( \text{Hom}_R(K, D) \) is \( l \)-pure-injective, by hypothesis; hence, \( D \cong K \otimes_R \text{Hom}_R(K, D) \) is finitely injective, by Proposition \(2.1\).

An immediate consequence is the following.

**Corollary 2.4.** Let \( R \) be an almost maximal Prüfer domain. Then the two classes of torsion finitely injective \( R \)-modules and of torsion-free complete \( l \)-pure-injective \( R \)-modules correspond to each other under the Matlis equivalence. \( \square \)

Another interesting consequence is that semi-Dedekind domains, introduced and investigated by S. B. Lee in \([13]\), are Dedekind. Recall that a domain \( R \) is semi-Dedekind if all \( h \)-divisible \( R \)-modules are pure-injective. Lee proved that, if \( R \) is semi-Dedekind, then every torsion-free complete \( R \)-module is pure-injective. As pure-injective modules are trivially \( l \)-pure-injective, and since semi-Dedekind Prüfer domains are necessarily Dedekind domains, from Theorem \(2.3\) we derive the following.

**Corollary 2.5.** A semi-Dedekind domain is a Dedekind domain. \( \square \)

Recall that an almost maximal Prüfer domain is necessarily \( h \)-local. In the remaining part of this paper we will extend the result in Corollary \(2.4\) to \( h \)-local Prüfer domains, but with the additional assumption that they are Matlis.

We will need the next technical result.

**Lemma 2.6.** Let \( R \) be a valuation domain and \( \{ M_i \}_{i \in I} \) a family of pure-injective torsion-free \( R \)-modules. Then the completion of \( \bigoplus_{i \in I} M_i \) in the \( R \)-topology is \( l \)-pure-injective.
Proof. Let \( M = \bigoplus_{i \in I} M_i \) be the completion of \( \bigoplus_{i \in I} M_i \) and let \( S \) be a maximal immediate extension of \( R \). An element \( x \in M \) is the limit of a Cauchy net \( \{ x_r \}_{0 \neq r \in R} \) of elements in \( \bigoplus_{i \in I} M_i \), which is an \( S \)-module. If \( \alpha \in S \), then \( \alpha x \) is the limit of the net \( \{ \alpha x_r \}_{0 \neq r \in R} \), hence \( \alpha x \in M \); therefore, \( M \) is an \( S \)-module too. Pick a non-zero element \( m \in M \); if \( xJ \) is the pure \( R \)-submodule of \( M \) generated by \( x \) (\( J \leq Q \)), then \( xJ \) is a pure-injective summand of \( M \) containing \( x \); hence, \( M \) is \( l \)-pure-injective.

We can now prove the converse of Proposition 2.1 for modules over Matlis valuation domains.

**Proposition 2.7.** Let \( R \) be a Matlis valuation domain, and let \( D \) be a finitely injective torsion \( R \)-module. Then \( \text{Hom}_R(K, D) \) is \( l \)-pure-injective.

**Proof.** Let \( 0 \neq \phi \in \text{Hom}_R(K, D) \). Since \( K \) is \( S_0 \)-generated, by Proposition 1.1 we have that \( \phi(K) \leq \bigoplus_{n \in \omega} E_n \), where the \( E_n \) are injective modules. From the exact sequence

\[
0 \to \bigoplus_{n \in \omega} E_n \to D \to X \to 0,
\]

where \( X = D/\bigoplus_{n \in \omega} E_n \), we obtain the exact sequence

\[
0 \to \text{Hom}_R\left(K, \bigoplus_{n \in \omega} E_n\right) \to \text{Hom}_R(K, D) \to \text{Hom}_R(K, X) \to \text{Ext}^1_R\left(K, \bigoplus_{n \in \omega} E_n\right),
\]

where the last term vanishes, since \( \text{p.d. } K = 1 \) and \( \bigoplus_{n \in \omega} E_n \) is \( h \)-divisible (see [7 VII 2.5]). Now \( \text{Hom}_R(K, \bigoplus_{n \in \omega} E_n) \) is the completion of a direct sum of torsion-free pure-injective modules by the Warfield result in [7]; hence, it is \( l \)-pure-injective by Lemma 2.6. Furthermore, it is pure in \( \text{Hom}_R(K, D) \), since \( \text{Hom}_R(K, X) \) is torsion-free. Clearly \( \phi \in \text{Hom}_R(K, \bigoplus_{n \in \omega} E_n) \), so it belongs to a pure-injective summand of it, which is as well a direct summand of \( \text{Hom}_R(K, D) \), being pure in it. Therefore, we can conclude that \( \text{Hom}_R(K, D) \) is \( l \)-pure-injective.

From Theorem 1.7 and Proposition 2.7 we immediately obtain the following.

**Corollary 2.8.** Let \( R \) be a Matlis non-Noetherian valuation domain. There exists a complete \( l \)-pure-injective torsion-free \( R \)-module which fails to be the completion of a direct sum of pure-injective modules.

We can also extend Corollary 2.4 to \( h \)-local Matlis Prüfer domains.

**Corollary 2.9.** Let \( R \) be an \( h \)-local Matlis Prüfer domain. Then the two classes of torsion finitely injective \( R \)-modules and of torsion-free complete \( l \)-pure-injective \( R \)-modules correspond to each other under the Matlis equivalence.

**Proof.** By \( h \)-locality, we can reduce to the local case, namely, to Matlis valuation domains, so the proof follows from Propositions 2.1 and 2.7.

We close the paper with two open questions.

**Question 1.** Given an arbitrary non-Noetherian ring \( R \), does a non-trivial finitely injective \( R \)-module exist?

**Question 2.** Given an arbitrary integral domain \( R \), which torsion-free complete \( R \)-modules correspond to the torsion finitely injective \( R \)-modules under the Matlis equivalence, and which torsion finitely injective \( R \)-modules correspond to the complete torsion-free \( l \)-pure-injective \( R \)-modules?
References


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