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A CHARACTERIZATION OF MODULES LOCALLY OF FINITE INJECTIVE DIMENSION

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Abstract. In this note, we characterize finite modules locally of finite injective dimension over commutative Noetherian rings in terms of vanishing of Ext modules.

1. Introduction

Let R be a commutative Noetherian ring. Goto [2] proved the following theorem.

Theorem 1.1 (Goto)**.** *The following are equivalent:*

(1) *R is Gorenstein;*
(2) *For every finite R-module M, there exists an integer n such that $\mathrm{Ext}_R^i(M,R) = 0$ for all $i > n$.*

We should note that this theorem remains valid even in the case where the ring R has infinite Krull dimension.

The purpose of this note is to give a characterization of finite modules locally of finite injective dimension. Our theorem is the following.

Theorem 1.2. *The following are equivalent for a finite R-module N:*

(1) $\mathrm{id}_{R_\mathfrak{p}} N_\mathfrak{p} < \infty$ *for every* $\mathfrak{p} \in \operatorname{Spec} R$*;*
(2) *For every finite R-module M, there exists an integer n such that $\mathrm{Ext}_R^i(M,N) = 0$ for all $i > n$.*

This theorem is a generalization of Goto's. In fact, applying our theorem to $N = R$, we immediately obtain Goto's theorem.

2. Proof of the theorem

We denote by $\mathrm{CM}(R)$ the *Cohen-Macaulay locus* of R, that is, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is Cohen-Macaulay. The following lemma can be shown in a similar way to the proof of [3, Theorem 24.5].

Lemma 2.1. *Let $\mathfrak{p}$ be a prime ideal of R such that both $R_\mathfrak{p}$ and $R/\mathfrak{p}$ are Cohen-Macaulay rings. Then there exists an element $f \in R - \mathfrak{p}$ such that $D(f) \cap V(\mathfrak{p}) \subseteq \mathrm{CM}(R)$.*

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It is known that the Cohen-Macaulay locus of a homomorphic image of a Cohen-Macaulay ring is an open subset; see [3, Exercises 24.2]. To prove our theorem, we need to generalize this fact. For an ideal I of R, let $\mathrm{CM}_R(R/I)$ denote the set of prime ideals $\mathfrak{p} \in V(I)$ such that the local ring $(R/I)_{\mathfrak{p}}$ is Cohen-Macaulay.

Lemma 2.2. *Let I and J be ideals of R. Suppose that $V(J)$ is contained in $\mathrm{CM}(R)$. Then the following hold:*

(1) *For any prime ideal $\mathfrak{p} \in \mathrm{CM}_R(R/I) \cap V(J)$, there exists an element $f \in R - \mathfrak{p}$ such that $D(f) \cap V(\mathfrak{p}) \subseteq \mathrm{CM}_R(R/I)$.*

(2) *There exists an ideal K of R such that*

$$\mathrm{CM}_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J).$$

In other words, $\mathrm{CM}_R(R/I) \cap V(J)$ is an open subset of $V(I) \cap V(J)$ in the relative topology induced by the Zariski topology of $\operatorname{Spec} R$.

Proof. (1) Let $\mathfrak{p} \in \mathrm{CM}_R(R/I) \cap V(J)$. Then by the assumption that $V(J)$ is contained in $\mathrm{CM}(R)$, the ring $R_{\mathfrak{p}}$ is a Cohen-Macaulay local ring. Making a similar argument to the proof of [3, Theorem 24.5], we can assume without loss of generality that there is an R-regular sequence $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ in $\mathfrak{p}$ with $\mathfrak{p}^r \subseteq \boldsymbol{x}R$ for some $r > 0$ and that $\overline{\mathfrak{p}}^i/\overline{\mathfrak{p}}^{i+1}$ is a free $\overline{R}/\overline{\mathfrak{p}}$-module for all $i > 0$, where $\overline{R} = R/\boldsymbol{x}R$ and $\overline{\mathfrak{p}} = \mathfrak{p}/\boldsymbol{x}R$.

We have only to prove that the residue ring $R/\mathfrak{p}$ is Cohen-Macaulay. In fact, if $R/\mathfrak{p}$ is Cohen-Macaulay, then so are $(R/I)/(\mathfrak{p}/I)$ and $(R/I)_{\mathfrak{p}/I}$ since $\mathfrak{p}$ is in $\mathrm{CM}_R(R/I)$. Hence Lemma 2.1 implies that there is an element $f \in R - \mathfrak{p}$ such that $D(\overline{f}) \cap V(\mathfrak{p}/I)$ is contained in $\mathrm{CM}(R/I)$, where $\overline{f}$ denotes the residue class of f in R/I. We easily see that $D(f) \cap V(\mathfrak{p})$ is contained in $\mathrm{CM}_R(R/I)$.

Let us show that $R/\mathfrak{p}$ is a Cohen-Macaulay ring. It is easy to see from [3, Exercises 24.1] that $R/\mathfrak{p} = \overline{R}/\overline{\mathfrak{p}}$ is Cohen-Macaulay if and only if so is $\overline{R}$. Take a prime ideal $\mathfrak{q} \in V(\boldsymbol{x}) = V(\mathfrak{p})$. Then we have $\mathfrak{q} \supseteq \mathfrak{p} \supseteq J$; hence $\mathfrak{q} \in V(J) \subseteq \mathrm{CM}(R)$. Therefore $R_{\mathfrak{q}}$ is a Cohen-Macaulay local ring, and so is $\overline{R}_{\mathfrak{q}}$, because $\boldsymbol{x}$ is an $R_{\mathfrak{q}}$-regular sequence. This shows that $\overline{R}$ is a Cohen-Macaulay ring. Thus we conclude that the residue ring $R/\mathfrak{p}$ is Cohen-Macaulay, as desired.

(2) Set $U = \left\{ \mathfrak{p}/I+J \,\middle|\, \mathfrak{p} \in \mathrm{CM}_R(R/I) \cap V(J) \right\}$. This is a subset of $\operatorname{Spec} R/I+J$. Note that this subset is stable under generalization. Let $P \in U$. Then there is a prime ideal $\mathfrak{p} \in \mathrm{CM}_R(R/I) \cap V(J)$ such that $P = \mathfrak{p}/I+J$. By the assertion (1) of the lemma, the set $D(f) \cap V(\mathfrak{p})$ is contained in $\mathrm{CM}_R(R/I)$ for some $f \in R - \mathfrak{p}$. Denote by $\overline{f}$ the residue class of f in R/I. It is easy to see that $P \in D(\overline{f}) \cap V(P) \subseteq U$. Thus U contains a nonempty open subset of $V(P)$. By virtue of the topological Nagata criterion [3, Theorem 24.2], U is an open subset of $\operatorname{Spec} R/I+J$; we have $U = D(K/I+J)$ for some ideal K of R containing $I+J$. Then it is easily checked that $\mathrm{CM}_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J)$. □

Now, we can prove our theorem.

Proof of Theorem 1.2. (2) ⇒ (1): Let $\mathfrak{p}$ be a prime ideal of R. Then there is an integer n such that $\mathrm{Ext}^i_R(R/\mathfrak{p}, N) = 0$ for all $i > n$. Hence we have $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), N_{\mathfrak{p}}) = 0$ for all $i > n$. Therefore by [1, Theorem 3.1.14] we obtain $\mathrm{id}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq n < \infty$.

(1) ⇒ (2): First of all, note that (2) is equivalent to the statement that for each ideal I of R there is an integer n such that $\mathrm{Ext}^i_R(R/I, N) = 0$ for all $i > n$. (This can easily be proved by induction on the number of generators of the R-module

M.) Suppose that there exists an ideal I of R such that for any integer n there is an integer $i > n$ such that $\operatorname{Ext}_R^i(R/I, N) \neq 0$. We want to derive a contradiction. Since R is Noetherian, one can choose I to be a maximal one among such ideals. Making a similar argument to the proof of Theorem 1.1, we see that the ideal I is prime and that for any element $f \in R - I$, the map

$$\operatorname{Ext}_R^i(R/I, N) \xrightarrow{f} \operatorname{Ext}_R^i(R/I, N)$$

is an isomorphism for $i \gg 0$.

Claim 1. *One has* $I \in \operatorname{Supp}_R N \subseteq \operatorname{CM}(R)$.

Proof of Claim. Our assumption (1) implies that for any $\mathfrak{p} \in \operatorname{Supp}_R N$, the nonzero finite $R_\mathfrak{p}$-module $N_\mathfrak{p}$ has finite injective dimension. Hence $R_\mathfrak{p}$ is a Cohen-Macaulay local ring; see [1, Corollary 9.6.2 and Remarks 9.6.4]. Thus $\operatorname{Supp}_R N$ is contained in $\operatorname{CM}(R)$. On the other hand, assume that I is not in $\operatorname{Supp}_R N$. Then there exists an element $f \in \operatorname{Ann}_R N - I$, and the map $\operatorname{Ext}_R^i(R/I, N) \xrightarrow{f} \operatorname{Ext}_R^i(R/I, N)$ is an isomorphism for $i \gg 0$. Since $fN = 0$, this map is the zero map, and we get $\operatorname{Ext}_R^i(R/I, N) = 0$ for $i \gg 0$. It follows from this contradiction that I belongs to $\operatorname{Supp}_R N$. □

Noting that $\operatorname{Supp}_R N = V(\operatorname{Ann}_R N)$, we see from Claim 1 and Lemma 2.2(2) that there is an ideal K of R such that $\operatorname{CM}_R(R/I) \cap \operatorname{Supp}_R N = D(K) \cap V(I) \cap \operatorname{Supp}_R N$. The localization $(R/I)_I = \kappa(I)$ is a field, hence a Cohen-Macaulay ring. It is seen from Claim 1 again that $I \in \operatorname{CM}_R(R/I) \cap \operatorname{Supp}_R N \subseteq D(K)$. Thus there is an element $f \in K - I$.

Claim 2. *For any prime ideal* $\mathfrak{p} \in D(f)$ *and any integer* $i > \operatorname{ht} I$*, one has* $\operatorname{Ext}_{R_\mathfrak{p}}^i(R_\mathfrak{p}/IR_\mathfrak{p}, N_\mathfrak{p}) = 0$.

Proof of Claim. We may assume that $\mathfrak{p}$ belongs to both $V(I)$ and $\operatorname{Supp}_R N$, because otherwise the module $\operatorname{Ext}_{R_\mathfrak{p}}^i(R_\mathfrak{p}/IR_\mathfrak{p}, N_\mathfrak{p})$ automatically vanishes. Hence Claim 1 implies that $\mathfrak{p}$ belongs to $\operatorname{CM}(R)$, namely the local ring $R_\mathfrak{p}$ is Cohen-Macaulay. Added to it, since $D(f)$ is contained in $D(K)$, we have $\mathfrak{p} \in D(K) \cap V(I) \cap \operatorname{Supp}_R N \subseteq \operatorname{CM}_R(R/I)$, and therefore $R_\mathfrak{p}/IR_\mathfrak{p}$ is Cohen-Macaulay. Thus we get the following equalities:

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p}/IR_\mathfrak{p} = \dim R_\mathfrak{p} - \dim R_\mathfrak{p}/IR_\mathfrak{p} = \operatorname{ht} IR_\mathfrak{p} = \operatorname{ht} I.$$

Since $N_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module of finite injective dimension by assumption, it follows from the result of Ischebeck [1, Exercises 3.1.24] that $\operatorname{Ext}_{R_\mathfrak{p}}^i(R_\mathfrak{p}/IR_\mathfrak{p}, N_\mathfrak{p}) = 0$ for every $i > \operatorname{ht} I$. □

Claim 2 means that $(\operatorname{Ext}_{R_f}^i(R_f/IR_f, N_f))_\mathfrak{P} = 0$ for every $\mathfrak{P} \in \operatorname{Spec} R_f$ and every $i > \operatorname{ht} I$. Therefore, $\operatorname{Ext}_{R_f}^i(R_f/IR_f, N_f) = 0$ for $i > \operatorname{ht} I$. The R-module $\operatorname{Ext}_R^i(R/I, N)$ is isomorphic to $\operatorname{Ext}_{R_f}^i(R_f/IR_f, N_f)$ for $i \gg 0$, and thus $\operatorname{Ext}_R^i(R/I, N) = 0$ for $i \gg 0$. This contradiction completes the proof of our theorem. □

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