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MULTIPLE NONTRIVIAL SOLUTIONS FOR NONLINEAR EIGENVALUE PROBLEMS

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ABSTRACT. In this paper we study a nonlinear eigenvalue problem driven by the p-Laplacian. Assuming for the right-hand side nonlinearity only unilateral and sign conditions near zero, we prove the existence of three nontrivial solutions, two of which have constant sign (one is strictly positive and the other is strictly negative), while the third one belongs to the order interval formed by the two opposite constant sign solutions. The approach relies on a combination of variational and minimization methods coupled with the construction of upper-lower solutions. The framework of the paper incorporates problems with concave-convex nonlinearities.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

In this paper we study the existence of multiple solutions of the following nonlinear eigenvalue problem: find $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\lambda > 0$ such that

$$\begin{cases} -\Delta_p u = f(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$ and $1 < p < +\infty$. In problem (1.1) we have the negative p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) acting as

$$\langle -\Delta_p u, v\rangle = \int_\Omega \|Du(x)\|^{p-2}(Du(x), Dv(x))_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

Our aim is to study the multiplicity of solutions for all values of the parameter λ in an interval $(0, \lambda_0)$, with $\lambda_0 > 0$. In the past this problem was examined primarily in the context of semilinear problems, i.e., for $p = 2$. In this respect we cite the work of Ambrosetti–Brezis–Cerami [1] considering a concave-convex nonlinearity $f(x,s,\lambda) = \lambda|s|^{q-2}s + |s|^{r-2}s$, with $1 < q < 2 < r < 2^*$, and the one of Jin [8] where $f(x,s,\lambda)$ is Hölder continuous with respect to $(x,s) \in \overline{\Omega} \times \mathbb{R}$ for every fixed λ. Concerning the nonlinear eigenvalue problems involving the p-Laplacian, we quote Garcia Azorero–Peral–Manfredi [6] focusing on the case where the potential

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in problem (1.1) is given by $f(x,s,\lambda)=\lambda|s|^{q-2}s+|s|^{r-2}s$ with $1<q<p<r<p^*$, where

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if } N>p,\\ +\infty & \text{otherwise.}\end{cases}$$

We also mention the work of Carl–Perera [3], treating the multiplicity of solutions of problem (1.1) without the parameter λ, namely $f(x,s)=\alpha(s^+)^{p-1}-\beta(s^-)^{p-1}+g(x,s)$, with $(\alpha,\beta)\in\mathbb{R}^2$ above the curve of the Fučik spectrum passing through (λ_2,λ_2) (λ_2 denotes the second eigenvalue of $(-\Delta_p,W_0^{1,p}(\Omega))$) and where $\lim_{s\to 0}\frac{g(x,s)}{|s|^{p-1}}=0$ uniformly in x.

Our hypotheses on the nonlinearity $f(x,s,\lambda)$ in problem (1.1) are the following:

H(f) $f:\Omega\times\mathbb{R}\times(0,\bar\lambda)\to\mathbb{R}$, with $\bar\lambda>0$, is a function such that $f(x,0,\lambda)=0$ a.e. on Ω whenever $\lambda\in(0,\bar\lambda)$, and

(i) for all $\lambda\in(0,\bar\lambda)$, $f(\cdot,\cdot,\lambda)$ is Carathéodory (that is, $f(\cdot,s,\lambda)$ is measurable for all $s\in\mathbb{R}$ and $f(x,\cdot,\lambda)$ is continuous for almost all $x\in\Omega$);

(ii) there are constants $c>0$, $r>p-1$ and functions $a(\cdot,\lambda)\in L^\infty(\Omega)_+$ ($\lambda\in(0,\bar\lambda)$) with $\|a(\cdot,\lambda)\|_\infty\to 0$ as $\lambda\downarrow 0$ such that

$$|f(x,s,\lambda)|\le a(x,\lambda)+c|s|^r \quad \text{for a.a. } x\in\Omega \text{ and all } (s,\lambda)\in\mathbb{R}\times(0,\bar\lambda);$$

(iii) for all $\lambda\in(0,\bar\lambda)$ there exists $\mu>\lambda_2$ for which we have

$$\mu<\liminf_{s\to 0}\frac{f(x,s,\lambda)}{|s|^{p-2}s} \quad \text{uniformly for a.a. } x\in\Omega;$$

(iv) there exists a constant $b>0$ such that for almost all $x\in\Omega$, all $\lambda\in(0,\bar\lambda)$ and all $s\in\mathbb{R}$ with $|s|<b$ we have $f(x,s,\lambda)s\ge 0$.

In order to formulate our main result, we recall that in the Banach space $C_0^1(\overline{\Omega})=\{u\in C^1(\overline{\Omega})\,:\,u(x)=0,\ \forall x\in\partial\Omega\}$ the interior of the positive cone $C_0^1(\overline{\Omega})_+=\{u\in C_0^1(\overline{\Omega})\,:\,u(x)\ge 0,\ \forall x\in\Omega\}$ is given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+)=\{u\in C_0^1(\overline{\Omega})\,:\,u(x)>0,\ \forall x\in\Omega,\ \text{and}\ \frac{\partial u}{\partial n}(x)<0,\ \forall x\in\partial\Omega\},$$

where $n=n(x)$ is the outer unit normal at $x\in\partial\Omega$. We now state our main result on the existence of multiple nontrivial solutions to problem (1.1).

Theorem 1.1. *Under hypotheses* H(f)*, there is a number $\lambda_0\in(0,\bar\lambda)$ such that for all $\lambda\in(0,\lambda_0)$ problem* (1.1) *has at least three distinct nontrivial solutions $u_0{=}u_0(\lambda)\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_0{=}v_0(\lambda)\in-\operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $w_0{=}w_0(\lambda)\in C_0^1(\overline{\Omega})$ satisfying $-b<v_0(x)\le w_0(x)\le u_0(x)<b$ for all $x\in\Omega$ and with negative energies, that is,*

$$\frac{1}{p}\|Du\|_p^p-\int_\Omega\int_0^{u(x)}f(x,s,\lambda)\,ds\,dx<0 \quad \textit{for } u\in\{u_0,v_0,w_0\}.$$

The next example illustrates that large classes of nonlinear eigenvalue problems, among which those with concave-convex potentials, fit in our setting. For the sake of simplicity we drop the x dependence for the potential f in the right-hand side of equation (1.1).

Example 1.2. The situation treated in [6], including the concave-convex case of [1], is covered by our hypotheses. More generally, without assuming a subcritical growth, the function $f:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ given by

$$f(s,\lambda)=c_1\lambda\,|s|^{q-2}s+c_2|s|^{r-2}s \quad \text{for all } (s,\lambda)\in\mathbb{R}\times(0,+\infty),$$

with $c_1, c_2 > 0$ and $1 < q < p < r < +\infty$, satisfies hypotheses H(f). Indeed, (i), (iii), (iv) (with any $b > 0$) are obviously verified, while (ii) holds since by Young inequality we have

$$|f(s,\lambda)| \leq c_1 \frac{r-q}{r-1} \lambda^{\frac{r-1}{r-q}} + \left(c_1 \frac{q-1}{r-1} + c_2\right) |s|^{r-1} \quad \text{for all } (s,\lambda) \in \mathbb{R} \times (0,+\infty).$$

Further examples of functions $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ verifying assumptions H(f) are

$$f(s,\lambda) = \begin{cases} \lambda(|s|^{q-2}s + |s|^{r-2}s) & \text{if } |s| \leq 1, \\ 2\lambda|s|^{\ell-2}s + s|s|^{t-3}\ln|s| & \text{if } |s| > 1, \end{cases}$$

where $1 < \min\{q,r\} < p$, $\max\{\ell, p\} < t$, and

$$f(s,\lambda) = \begin{cases} \lambda(\max\{s^{q-1}, s^{r-1}\} + \ln(1+s^{t-1})) & \text{if } s \geq 0, \\ \lambda(|s|^{q-2}s + \arctan s) & \text{if } s < 0, \end{cases}$$

where $1 < q < r$, $q < p$, $1 < t$. Notice that in the last example the function f is not odd with respect to s. In all these cases, Theorem 1.1 can be applied to the corresponding problems (1.1).

Commenting on our hypotheses, it is seen from H(f) that we do not assume symmetries, subcritical growth or asymptotic conditions at infinity. We only impose that the function $f(\cdot,\cdot,\lambda)$ is Carathéodory, satisfies an arbitrary polynomial growth, a unilateral condition near zero related to the second eigenvalue λ_2 of $(-\Delta_p, W_0^{1,p}(\Omega))$ and a sign condition near zero. We remark that the sign condition (in assumption H(f)(iv)) cannot be derived from the unilateral condition (in H(f)(iii)) because the constant b in H(f)(iv) must be independent on λ.

Our main result stated as Theorem 1.1 provides two opposite constant sign solutions uniformly bounded by the constant b and a third nontrivial solution belonging to their order interval. We note that the result cannot be repeatedly used to yield new solutions by taking the constant b in H(f)(iv) smaller than the uniform norm of the already found solutions because they depend on λ. Theorem 1.1 complements the works in [1], [3], [6], [8] (see also Example 1.2). Actually, it is an extension of Theorem 1 in [8] from semilinear problems to those driven by the p-Laplacian, except for a slightly different sign condition near zero. Specifically, in addition to the extension from $p = 2$ to any $1 < p < +\infty$, we drop the requirement on $f(\cdot,\cdot,\lambda)$ to be Hölder continuous on $\overline{\Omega} \times \mathbb{R}$, provide additional information in the conclusion and use an argument different from the one in [8] based on the pseudo-gradient vector field.

Concerning the method of proof, it is worth pointing out that here we cannot consider the usual Euler functional associated to problem (1.1), that is,

$$\varphi(u,\lambda) = \frac{1}{p}\,\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x,s,\lambda)\,ds\,dx$$

whenever $u \in W_0^{1,p}(\Omega)$ and $\lambda \in (0,\bar{\lambda})$, due to the general growth condition in H(f)(ii), so the common variational method seeking the weak solutions of (1.1) as the critical points of $\varphi(\cdot,\lambda)$ is not applicable. Our approach combines variational, truncation and minimization techniques with construction of upper-lower solutions. First, in Proposition 3.1, we find an upper bound λ_0 of λ such that for any $\lambda \in (0,\lambda_0)$ there exist two solutions $u_0 = u_0(\lambda)$, $v_0 = v_0(\lambda)$ of problem (1.1) having opposite constant sign. These solutions arise as global minimizers for some functionals constructed by means of a positive upper solution and a negative lower solution. Then, through suitable variational techniques we obtain another nontrivial solution $w_0 = w_0(\lambda)$ of (1.1) belonging to the order interval determined by v_0, u_0 with $w_0 \neq u_0$ and $w_0 \neq v_0$. The fact that w_0 is nontrivial is deduced by applying the second deformation lemma.

The rest of the paper is organized as follows. Section 2 recalls an essential mathematical background needed in the sequel. Section 3 sets forth the construction of two solutions of opposite constant sign. Section 4 contains the proof of Theorem 1.1.

2. Mathematical background

In this section we briefly recall basic mathematical tools which will be used in the analysis of problem (1.1). Let us start with some properties regarding the spectrum of the negative p-Laplacian with Dirichlet boundary conditions on Ω. We consider the nonlinear eigenvalue problem of $-\Delta_p$ on $W_0^{1,p}(\Omega)$: find $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\lambda > 0$ such that

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The first eigenvalue for (2.1), denoted by λ_1, is variationally characterized as

$$\lambda_1 = \min \left\{ \frac{\|Du\|_p^p}{\|u\|_p^p} \ : \ u \in W_0^{1,p}(\Omega), \ u \neq 0 \right\},$$

and the minimum is realized at an eigenfunction u_1 which is taken to satisfy $\|u_1\|_p = 1$. By using the strong maximum principle (see [12]), we may assume that $u_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Taking into account that λ_1 is isolated, the second eigenvalue λ_2 of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ satisfies

$$\lambda_2 = \inf\{\lambda \ : \ \lambda \text{ is an eigenvalue of } (-\Delta_p, W_0^{1,p}(\Omega)), \ \lambda > \lambda_1\} > \lambda_1$$

(see Anane–Tsouli [2]). A variational expression for λ_2 was given by Cuesta–de Figueiredo–Gossez [4] as follows:

$$\lambda_2 = \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([-1,1])} \|Du\|_p^p, \tag{2.2}$$

where

$$\Gamma_0 = \{\gamma \in C([-1,1], S) \ : \ \gamma(-1) = -u_1, \ \gamma(1) = u_1\},$$

$S = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$ and $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) \ : \ \|u\|_p = 1\}$.

Given $\varphi \in C^1(X)$ on a Banach space X and $c \in \mathbb{R}$, we set $\varphi^c = \{u \in X : \varphi(u) \leq c\}$ and $K_c(\varphi) = \{u \in X \ : \ \varphi'(u) = 0, \ \varphi(u) = c\}$. We recall that φ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (the PS_c-condition for short) if any sequence $\{u_n\} \subset X$ verifying $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$ has a strongly convergent subsequence. We refer to [10] for the necessary mathematical background concerning the critical point theory. For later use we also mention the second deformation lemma (see, e.g., [7, p. 366]).

Lemma 2.1. *If $\varphi \in C^1(X)$, $-\infty < a < b \leq +\infty$, φ satisfies the PS_c-condition for every $c \in [a, b)$, φ has no critical values in the open interval (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a continuous* *mapping* $\eta : [0,1] \times (\varphi^b \setminus K_b(\varphi)) \to \varphi^b$ *such that*

$$\varphi(\eta(t,u)) \le \varphi(\eta(s,u)) \;\; \text{for all } t,s \in [0,1],\; s \le t \;\; \text{and all } u \in \varphi^b \setminus K_b(\varphi)\,,$$
$$\eta(1,\varphi^b \setminus K_b(\varphi)) \subset \varphi^a\,,\; \eta(0,\cdot) = \mathrm{id}|_{\varphi^b \setminus K_b(\varphi)}\,,$$
$$\eta(t,u) = u \;\; \text{for all } (t,u) \in [0,1] \times \varphi^a.$$

3. Constant sign solutions through local minima

We start by generating two solutions of problem (1.1) having constant sign.

Proposition 3.1. *Assume* H(f)(i), (ii), (iv) *and a weaker form of* H(f)(iii)*: for all* $\lambda \in (0,\bar\lambda)$ *there exists* $\mu > \lambda_1$ *for which we have*

$$\mu < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \quad \text{uniformly for a.a. } x \in \Omega.$$

Then there is a number $\lambda_0 \in (0,\bar\lambda)$ *such that if* $\lambda \in (0,\lambda_0)$ *problem* (1.1) *has a solution* $u_0 = u_0(\lambda) \in \operatorname{int}(C_0^1(\overline\Omega)_+)$ *and a solution* $v_0 = v_0(\lambda) \in -\operatorname{int}(C_0^1(\overline\Omega)_+)$ *satisfying* $\|u_0\|_\infty < b$, $\|v_0\|_\infty < b$ *and with negative energies.*

Proof. It is well known that there exists $e \in \operatorname{int}(C_0^1(\overline\Omega)_+)$ satisfying $-\Delta_p\, e = 1$.

We claim that there is $\lambda_0 \in (0,\bar\lambda)$ with the property that if $\lambda \in (0,\lambda_0)$ we can choose $\xi_0 = \xi_0(\lambda) \in (0, \frac{b}{\|e\|_\infty})$ such that

$$c(\xi_0\|e\|_\infty)^r + \|a(\cdot,\lambda)\|_\infty < \xi_0^{p-1}, \tag{3.1}$$

with $c > 0$ and $a(\cdot,\lambda)$ as in H(f)(ii). Arguing by contradiction, suppose that there is a sequence $\lambda_n \downarrow 0$ as $n \to \infty$ such that

$$c(\xi\|e\|_\infty)^r + \|a(\cdot,\lambda_n)\|_\infty \ge \xi^{p-1} \;\; \text{for all } n \in \mathbb{N} \text{ and } \xi \in (0, \frac{b}{\|e\|_\infty}).$$

Letting $n \to \infty$ we get $c\|e\|_\infty^r \xi^{r-p+1} \ge 1$ for all $\xi \in (0, \frac{b}{\|e\|_\infty})$ because, by hypothesis H(f)(ii), we have $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \downarrow 0$. Since $r > p-1$, we arrive at a contradiction as $\xi \downarrow 0$. Therefore (3.1) holds true.

Fix $\lambda \in (0,\lambda_0)$ and set $\overline{u} = \xi_0 e \in \operatorname{int}(C_0^1(\overline\Omega)_+)$, with $\xi_0 \in (0, \frac{b}{\|e\|_\infty})$ given by (3.1). Define the truncation function $\hat\tau_+ : \mathbb{R} \to \mathbb{R}_+$ by

$$\hat\tau_+(s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 < s < \|\overline{u}\|_\infty, \\ \|\overline{u}\|_\infty & \text{if } s \ge \|\overline{u}\|_\infty. \end{cases}$$

We introduce the functional $\hat\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ as follows:

$$\hat\varphi_+(u) = \frac{1}{p}\,\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x,\hat\tau_+(s),\lambda)\,ds\,dx \;\; \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\hat\varphi_+$ is continuously differentiable, coercive and weakly lower semicontinuous. So there exists $u_0 = u_0(\lambda) \in W_0^{1,p}(\Omega)$ such that

$$\hat\varphi_+(u_0) = \inf\{\hat\varphi_+(u) \;:\; u \in W_0^{1,p}(\Omega)\}.$$

It turns out that $\hat\varphi_+'(u_0) = 0$, which is equivalent to

$$-\Delta_p\, u_0 = f(\cdot,\hat\tau_+(u_0(\cdot)),\lambda) \;\; \text{in } W^{-1,p'}(\Omega). \tag{3.2}$$

By H(f)(ii) it follows that $-\Delta_p u_0 \in L^\infty(\Omega)$. From the nonlinear regularity theory (see [9], [11], [7, p. 115-116]), we deduce $u_0 \in C_0^1(\overline{\Omega})$. Using $-u_0^- = -\max\{-u_0, 0\} \in W_0^{1,p}(\Omega)$ as a test function in (3.2) we obtain

$$\|Du_0^-\|_p^p = \langle -\Delta_p u_0, -u_0^- \rangle = -\int_\Omega f(x, \hat{\tau}_+(u_0(x)), \lambda) u_0^-(x)\, dx.$$

Notice $\hat{\tau}_+(u_0(x)) \le \|\overline{u}\|_\infty = \xi_0 \|e\|_\infty < b$ and

$$\int_\Omega f(x, \hat{\tau}_+(u_0(x)), \lambda) u_0^-(x)\, dx = 0.$$

Hence we get $Du_0^- = 0$, so $u_0 \ge 0$.

By hypothesis, we can find $\mu = \mu(\lambda) > \lambda_1$ and $\delta = \delta(\lambda) > 0$ such that

$$\mu < \frac{f(x, s, \lambda)}{|s|^{p-2}s} \quad \text{for a.a. } x \in \Omega \text{ and all } 0 < |s| \le \delta. \tag{3.3}$$

Choose $t > 0$ with $t\|u_1\|_\infty < \min\{\delta, \|\overline{u}\|_\infty\}$. Then by (3.3) we have

$$\hat{\varphi}_+(tu_1) \le \frac{t^p}{p}\,(\lambda_1 - \mu) < 0,$$

which ensures that $\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0)$, thus $u_0 \ne 0$. Furthermore, because $f(x, \hat{\tau}_+(u_0(x)), \lambda) \ge 0$ a.e. on Ω (cf. H(f)(iv)), from (3.2) we derive $-\Delta_p u_0 \ge 0$ for a.e. on Ω. The nonlinear strong maximum principle yields

$$u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{3.4}$$

In addition, denoting $s^+ = \max\{s, 0\}$ for $s \in \mathbb{R}$, we note from (3.2), H(f)(ii) and (3.1) that

$$\begin{aligned}\langle -\Delta_p u_0 + \Delta_p \overline{u}, (u_0 - \overline{u})^+ \rangle &= \int_\Omega (f(x, \hat{\tau}_+(u_0(x)), \lambda) - \xi_0^{p-1})(u_0 - \overline{u})^+(x)\, dx \\ &\le \int_\Omega (\|a(\cdot, \lambda)\|_\infty + c\|\overline{u}\|_\infty^r - \xi_0^{p-1})(u_0 - \overline{u})^+\, dx \le 0,\end{aligned}$$

which shows $\|D(u_0 - \overline{u})^+\|_p^p \le 0$, and thus $(u_0 - \overline{u})^+ = 0$. Consequently, we have $0 < u_0(x) \le \overline{u}(x) < b$ for all $x \in \Omega$ and from (3.2) we deduce that u_0 is a solution of problem (1.1). The fact that u_0 is with negative energy follows from $\hat{\varphi}_+(u_0) < 0$.

Now set $\underline{u} = -\xi_0 e \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, with $\xi_0 \in (0, \frac{b}{\|e\|_\infty})$ given in (3.1), and introduce the truncation function $\hat{\tau}_- : \mathbb{R} \to \mathbb{R}_-$ as

$$\hat{\tau}_-(s) = \begin{cases} -\|\underline{u}\|_\infty & \text{if } s \le -\|\underline{u}\|_\infty, \\ s & \text{if } -\|\underline{u}\|_\infty < s < 0, \\ 0 & \text{if } s \ge 0. \end{cases}$$

Then we define the functional $\hat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\hat{\varphi}_-(u) = \frac{1}{p}\,\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \hat{\tau}_-(s), \lambda)\, ds\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Arguing with $\hat{\varphi}_-$ in a similar way to the one for the functional $\hat{\varphi}_+$, we find a solution $v_0 = v_0(\lambda) \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ to problem (1.1) such that $-b < \underline{u}(x) \le v_0(x)$ on Ω and having negative energy. This completes the proof. $\square$

Remark 3.2. The functions $\overline{u} = \xi_0 e$ and $\underline{u} = -\xi_0 e$ constructed in the proof of Proposition 3.1 are an upper solution and a lower solution, respectively, for problem (1.1). In Proposition 3.1 the nonlinearity f does not have a subcritical growth in the infinity. This is handled by employing a suitable truncation function argument used in [5].

4. Proof of Theorem 1.1

With the positive number λ_0 given by Proposition 3.1, fix $\lambda \in (0, \lambda_0)$. Let $u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $v_0 \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the two solutions obtained in Proposition 3.1. We introduce on $\Omega \times \mathbb{R}$ the truncation functions

$$\tau_+(x,s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } 0 < s < u_0(x), \\ u_0(x) & \text{if } s \ge u_0(x), \end{cases} \qquad \tau_-(x,s) = \begin{cases} v_0(x) & \text{if } s \le v_0(x), \\ s & \text{if } v_0(x) < s < 0, \\ 0 & \text{if } s \ge 0, \end{cases}$$

$$\tau_0(x,s) = \begin{cases} v_0(x) & \text{if } s \le v_0(x), \\ s & \text{if } v_0(x) < s < u_0(x), \\ u_0(x) & \text{if } s \ge u_0(x). \end{cases}$$

For all $u \in W_0^{1,p}(\Omega)$ let us now define

$$\varphi_+(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_+(x,s), \lambda)\, ds\, dx\,,$$
$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_-(x,s), \lambda)\, ds\, dx\,,$$
$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \int_0^{u(x)} f(x, \tau_0(x,s), \lambda)\, ds\, dx\,,$$

and notice that $\varphi_+\,, \varphi_-\,, \varphi_0 \in C^1(W_0^{1,p}(\Omega))$.

It readily follows that u_0 is a critical point of φ_+ because u_0 is a solution of problem (1.1). Let us show that

$$v \text{ is a critical point of } \varphi_+ \implies 0 \le v(x) \le u_0(x) \ \text{ for a.a. } x \in \Omega. \tag{4.1}$$

To this end, let v be a critical point of φ_+. We note that

$$\langle \Delta_p u_0 - \Delta_p v, (v-u_0)^+\rangle = \int_\Omega (f(x, \tau_+(x, v(x)), \lambda) - f(x, u_0(x), \lambda))(v-u_0)^+\, dx = 0,$$

which implies $v \le u_0$. Similarly, we check that $v \ge 0$, which leads to (4.1). On the basis of (4.1) we see that every critical point v of φ_+ is a solution of (1.1).

We also observe that the function φ_+ is coercive, weakly lower semicontinuous, so φ_+ has a global minimizer $z_0 \in W_0^{1,p}(\Omega)$. Furthermore, using (3.3), it turns out

$$\varphi_+(z_0) = \inf_{W_0^{1,p}(\Omega)} \varphi_+ < 0,$$

thus $z_0 \ne 0$. If $z_0 \ne u_0$, then by (4.1) the desired conclusion follows with $w_0 = z_0$. So we may assume that $z_0 = u_0$, and it is the unique global minimizer of φ_+ (see (4.1)). On the other hand, according to (3.4) we find a neighborhood V of u_0 in the space $C_0^1(\overline{\Omega})$ such that $V \subset C_0^1(\overline{\Omega})_+$, thus $\varphi_0 = \varphi_+$ on V. This guarantees that u_0 is a local minimizer of φ_0 on $C_0^1(\overline{\Omega})$. Then it follows that u_0 is a local minimizer of φ_0 in the space $W_0^{1,p}(\Omega)$ (see [7, p. 655-656]). By the same reasoning as above for

φ_+, we may admit that v_0 is the unique global minimizer of φ_-. Proceeding as in the case of u_0, we establish that v_0 is a local minimizer of φ_0 on $W_0^{1,p}(\Omega)$.

As before in (4.1) we verify that every critical point of φ_0 belongs to the set $\{u \in W_0^{1,p}(\Omega) : v_0(x) \le u(x) \le u_0(x) \text{ a.e. } x \in \Omega\}$. This implies that every critical point of φ_0 is a solution to problem (1.1). The functional φ_0 is coercive, weakly lower semicontinuous, so it is bounded from below. In addition, one has $\inf_{W_0^{1,p}(\Omega)} \varphi_0 < 0$. Thus φ_0 has a global minimizer $y_0 \in W_0^{1,p}(\Omega)$ with $y_0 \neq 0$. If $y_0 \neq u_0$ and $y_0 \neq v_0$, then the conclusion in the theorem is obtained by taking $w_0 = y_0$. It remains to consider the cases $y_0 = u_0$ or $y_0 = v_0$. To make a choice, suppose $y_0 = u_0$. We may also admit that v_0 is a strict local minimizer of φ_0 since on the contrary we are done. Consequently, we can find $\rho \in (0, \|u_0 - v_0\|)$ such that

$$\varphi_0(u_0) \le \varphi_0(v_0) < \inf\{\varphi_0(u) : u \in \partial B_\rho(v_0)\}, \tag{4.2}$$

where $\partial B_\rho(v_0) = \{u \in W_0^{1,p}(\Omega) : \|u - v_0\| = \rho\}$. Relation (4.2) in conjunction with the PS$_c$-condition for φ_0 at any $c \in \mathbb{R}$ (due to its coercivity) enables us to apply the mountain pass theorem to φ_0 (see, e.g., [10]) and get $w_0 \in W_0^{1,p}(\Omega)$ satisfying $\varphi_0'(w_0) = 0$ and

$$\inf\{\varphi_0(u) : u \in \partial B_\rho(v_0)\} \le \varphi_0(w_0) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \varphi_0(\gamma(t)), \tag{4.3}$$

where

$$\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(\Omega)) : \gamma(-1) = v_0,\ \gamma(1) = u_0\}.$$

We infer from (4.2) and (4.3) that $w_0 \neq u_0$ and $w_0 \neq v_0$. Next let us prove that

$$\varphi_0(w_0) < 0. \tag{4.4}$$

Towards this, in view of (4.3), it suffices to produce a path $\hat{\gamma} \in \Gamma$ such that

$$\varphi_0(\hat{\gamma}(t)) < 0 \quad \text{for all } t \in [-1,1]. \tag{4.5}$$

Let $S = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$ and $S_C = S \cap C_0^1(\overline{\Omega})$ be endowed with the topologies induced by $W_0^{1,p}(\Omega)$ and $C_0^1(\overline{\Omega})$, respectively. We set

$$\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -u_1,\ \gamma(1) = u_1\}.$$

In view of assumption H(f)(iii), fix numbers $\mu > \lambda_2$ and $\delta > 0$ satisfying (3.3), and let $\rho_0 \in (0, \mu - \lambda_2)$. By (2.2) there exists $\gamma \in \Gamma_0$ such that

$$\max_{t \in [-1,1]} \|D\gamma(t)\|_p^p < \lambda_2 + \frac{\rho_0}{2}.$$

Choose some r with $0 < r \le (\lambda_2 + \rho_0)^{\frac{1}{p}} - (\lambda_2 + \frac{\rho_0}{2})^{\frac{1}{p}}$. The density of S_C in S implies that $\Gamma_{0,C}$ is dense in Γ_0, so there is $\gamma_0 \in \Gamma_{0,C}$ satisfying

$$\max_{t \in [-1,1]} \|D\gamma(t) - D\gamma_0(t)\|_p < r.$$

This ensures

$$\max_{t \in [-1,1]} \|D\gamma_0(t)\|_p^p < \lambda_2 + \rho_0. \tag{4.6}$$

Thanks to the boundedness of $\gamma_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ there exists $\varepsilon > 0$ such that

$$\varepsilon |u(x)| \le \delta \quad \text{for all } x \in \Omega \text{ and all } u \in \gamma_0([-1,1]). \tag{4.7}$$

Since $u_0, -v_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, for every $u \in \gamma_0([-1,1])$, we can find arbitrarily large positive numbers h_u and j_u satisfying $h_u u_0 - u, -j_u v_0 + u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Then there exists a neighborhood V_u of u in $C_0^1(\overline{\Omega})$ such that

$$h_u u_0 - v \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{and} \quad - j_u v_0 + v \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{for all } v \in V_u \,.$$

Using the compactness of $\gamma_0([-1,1])$ in $C_0^1(\overline{\Omega})$ a number $\zeta > 0$ can be determined to fulfill

$$v_0(x) \leq \varepsilon u(x) \leq u_0(x) \quad \text{for all } x \in \Omega \text{ and all } u \in \gamma_0([-1,1]) \tag{4.8}$$

provided $\varepsilon \in (0, \zeta)$. By (4.6), (4.8), (4.7), (3.3) and the fact that $\gamma_0([-1,1]) \subset \partial B_1^{L^p(\Omega)}$ we obtain for the continuous path $\varepsilon\gamma_0$ joining $-\varepsilon u_1$, εu_1 that

$$\begin{aligned}\varphi_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}\,\|D\gamma_0(t)\|_p^p - \int_\Omega \int_0^{\varepsilon\gamma_0(t)(x)} f(x, \tau_0(x,s), \lambda)\, ds\, dx \\ &\leq \frac{\varepsilon^p}{p}\,(\lambda_2 + \rho_0 - \mu) < 0 \quad \text{for all } t \in [-1,1].\end{aligned}$$

Let us denote $c_+ = c_+(\lambda) = \varphi_+(\varepsilon u_1)$ and $m_+ = m_+(\lambda) = \varphi_+(u_0)$. Since u_0 is the unique global minimizer of φ_+, we have that $m_+ < c_+$. Taking into account (4.1) and that $c_+ = \varphi_0(\varepsilon u_1) < 0$, we may assume that $K_{c_+}(\varphi_+) = \emptyset$ and φ_+ has no critical values in the open interval (m_+, c_+) (for, otherwise, a third solution of (1.1) is found with the properties in the statement of the theorem). Moreover, $\varphi_+^{m_+} = \{u_0\}$ and φ_+ satisfies the PS_c-condition at any level $c \in \mathbb{R}$ because it is coercive. Therefore all the assumptions in Lemma 2.1 are verified with $a = m_+$ and $b = c_+$ for the functional φ_+. Lemma 2.1 yields a homotopy $\eta \in C([0,1] \times \varphi_+^{c_+}, \varphi_+^{c_+})$ such that $\eta(0,u) = u$ and $\eta(1,u) = u_0$ for all $u \in \varphi_+^{c_+}$, as well as $\varphi_+(\eta(t,u)) \leq \varphi_+(u)$ whenever $t \in [0,1]$ and $u \in \varphi_+^{c_+}$. Defining $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ by $\gamma_+(t) = \eta(t, \varepsilon u_1)$ for all $t \in [0,1]$, it is seen that γ_+ is a continuous path in $W_0^{1,p}(\Omega)$ joining εu_1 and u_0. We claim that

$$\varphi_0(\gamma_+(t)) \leq \varphi_+(\gamma_+(t)) \quad \text{for all } t \in [0,1]. \tag{4.9}$$

If $\gamma_+(t)(x) < 0$, then, in view of H(f)(iv), we have for all $s \in [\gamma_+(t)(x), 0]$ that $f(x, \tau_0(x,s), \lambda) \leq 0 = f(x, \tau_+(x,s), \lambda)$. If $\gamma_+(t)(x) \geq 0$, then for all $s \in [0, \gamma_+(t)(x)]$ there holds $f(x, \tau_0(x,s), \lambda) = f(x, \tau_+(x,s), \lambda)$, so the claim follows from the expressions of φ_0 and φ_+. On the basis of (4.9) and the properties of the deformation η we get

$$\varphi_0(\gamma_+(t)) \leq \varphi_+(\eta(t, \varepsilon u_1)) \leq \varphi_+(\varepsilon u_1) < 0 \quad \text{for all } t \in [0,1].$$

Similarly, applying Lemma 2.1 for the functional φ_-, we construct a continuous path $\gamma_- : [0,1] \to W_0^{1,p}(\Omega)$ joining v_0 and $-\varepsilon u_1$ such that $\varphi_0(\gamma_-(t)) < 0$ for all $t \in [0,1]$.

Piecing together the paths γ_-, $\varepsilon\gamma_0$ and γ_+, we produce a continuous curve $\hat{\gamma} \in \Gamma$ joining v_0 and u_0 that satisfies (4.5). Hence (4.4) holds, which implies $w_0 \neq 0$. Recalling that the critical points of φ_0 are in the order interval $\{u \in W_0^{1,p}(\Omega) : v_0(x) \leq u(x) \leq u_0(x) \text{ a.e. } x \in \Omega\}$ we derive that w_0 is a nontrivial solution of (1.1) distinct from v_0 and u_0, with $v_0(x) \leq w_0(x) \leq u_0(x)$ for a.a. $x \in \Omega$ and negative energy (see (4.4)). By the nonlinear regularity theory we have that $w_0 \in C_0^1(\overline{\Omega})$ which completes the proof.

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