

PLANAR FINITELY SUSLINIAN COMPACTA

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ABSTRACT. We show that a planar unshielded compact set X is *finitely Suslinian* if and only if there exists a closed set $F \subset \mathbb{S}^1$ and a *lamination* \sim of F such that F/\sim is homeomorphic to X . If X is a continuum, the analogous statement follows from Carathéodory theory and is widely used in polynomial dynamics.

1. INTRODUCTION

By a *domain* we mean a connected open subset of the plane \mathbb{C} . A planar compact set X is *unshielded* if there exists a complementary domain U with $\overline{U} \supset X$ (below we *always* assume that U contains infinity). Julia sets of polynomials (or of *expanding polymodials*, see [1, 2]) are unshielded. Below by an unshielded set we *always* mean a planar unshielded compact set.

We look at unshielded sets “from infinity.” That is, let X be a continuum, U be as above, and $\mathbb{D}_\infty = \mathbb{C} \setminus \overline{\mathbb{D}}$, where \mathbb{D} is the open unit disk in \mathbb{C} . There exists a conformal isomorphism (called a *Riemann mapping*) $\Psi : \mathbb{D}_\infty \rightarrow U$ and, by a result of Carathéodory, X is locally connected if and only if Ψ extends to a continuous function $\overline{\Psi} : \overline{\mathbb{D}_\infty} \rightarrow \overline{U}$. To state an important corollary of this result we need the following definition. Consider an equivalence relation \sim on a closed subset F of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ with the following properties [6, 9]; cf. [11] (usually $F = \mathbb{S}^1$):

- (L1) \sim is *closed*: the graph of \sim is a closed set in $F \times F$;
- (L2) \sim is *unlinked*: if $t_1 \sim t_2 \in F$ and $t_3 \sim t_4 \in F$, but $t_2 \not\sim t_3$, then the straight line segments in \mathbb{C} with endpoints t_1, t_2 and t_3, t_4 are disjoint;
- (L3) each equivalence class of \sim is totally disconnected.

Call \sim a *lamination* of F . In the situation above, with an unshielded continuum X , set $\psi = \overline{\Psi}|_{\mathbb{S}^1}$ and define an equivalence \sim_ψ on \mathbb{S}^1 by $x \sim_\psi y$ if and only if $\psi(x) = \psi(y)$. Then it is easy to see that the equivalence \sim_ψ is a lamination defined on \mathbb{S}^1 and X is homeomorphic to the quotient space \mathbb{S}^1/\sim_ψ . This yields the following theorem.

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Theorem 1.1. *Let X be an unshielded continuum. If X is locally connected, then there exists a lamination \sim of \mathbb{S}^1 such that X is homeomorphic to the quotient space \mathbb{S}^1/\sim .*

The converse to this theorem is also true. We will prove it in the next section.

The case when X is locally connected is the best case scenario for our investigation because then the structure of X is closely related to that of \mathbb{S}^1 . Our aim is to obtain a similar result without the assumption of connectivity of X . To see what unshielded compacta are similar to unshielded locally connected continua we need the following definition.

Definition 1.2. A compact metric space X is *finitely Suslinian (FS)* if for each $\varepsilon > 0$, any collection of pairwise disjoint continua in X of diameter larger than ε is finite (we call such compacta *FS-sets*).

Note that by the definition any FS-set is compact.

The next lemma shows that the notion of an unshielded FS-set is a generalization of the notion of a locally connected unshielded continuum.

Lemma 1.3 ([3, Lemma 2.7]). *An unshielded continuum Y is locally connected if and only if it is FS.*

Now we are ready to state our main result, namely Theorem 1.4, which is analogous to Theorem 1.1 and its converse; its main applications are in the field of complex dynamics and will be discussed in forthcoming papers.

Theorem 1.4. *Let X be an unshielded compact set. Then X is FS if and only if there exists a closed set $F \subset \mathbb{S}^1$ and a lamination \sim of F such that X is homeomorphic to the quotient space F/\sim .*

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2. REDUCTION

In this section we reduce Theorem 1.4 to the following theorem, which is the main technical result of the paper.

Theorem 2.1. *Any unshielded FS-set is contained in an unshielded FS-continuum.*

To prove one implication in Theorem 1.4, we have to show that if X is an unshielded FS-set, then there exist a closed set $F \subset \mathbb{S}^1$ and a lamination \sim of F such that X is homeomorphic to F/\sim . If Theorem 2.1 holds, then there exists an unshielded FS-continuum $Y \supset X$. Then by Lemma 1.3 and Theorem 1.1 there exists a lamination \sim of \mathbb{S}^1 with $Y = \mathbb{S}^1/\sim$. Let $p : \mathbb{S}^1 \rightarrow \mathbb{S}^1/\sim$ be the corresponding quotient map, and $F = p^{-1}(X)$. Then F is closed and F/\sim is homeomorphic to X . This proves the implication, provided that Theorem 2.1 holds.

In order to prove the other implication in Theorem 1.4, assume that for a set X there is a closed set $F \subset \mathbb{S}^1$ and a lamination \sim on F such that X is homeomorphic to F/\sim . We want to prove that then X is FS. Let us start with a simple lemma that follows immediately from the fact that continuous maps of metric compact spaces are uniformly continuous.

Lemma 2.2. *Any set homeomorphic to an FS-set is itself an FS-set.*

Now we prove a theorem that is essentially the converse to Theorem 1.1.

Theorem 2.3. *Let \sim be a lamination of \mathbb{S}^1 . Then any set $X \subset \mathbb{C}$ homeomorphic to the quotient space \mathbb{S}^1/\sim is locally connected and FS.*

Proof. Let L be the union of \sim -hulls, i.e., the convex hulls of \sim -classes in \mathbb{D} . Define an extension \simeq of \sim onto \mathbb{C} as follows [6]: a \simeq -class is a \sim -hull or a point of $\mathbb{C} \setminus L$. The quotient space $K = \overline{\mathbb{D}}/\simeq$ is embedded in the quotient space \mathbb{C}/\simeq . Denote the interior of K by Q . Let $p : \mathbb{C} \rightarrow \mathbb{C}/\simeq$ be the quotient map. Then $p|_{\mathbb{C} \setminus \overline{\mathbb{D}}} : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow (\mathbb{C}/\simeq) \setminus K$ and $p|_{\overline{\mathbb{D}} \setminus L} : \overline{\mathbb{D}} \setminus L \rightarrow Q$ are homeomorphisms. The set $Z = p(\mathbb{S}^1) = p(L)$ is the boundary of K in \mathbb{C}/\simeq ; clearly, Z is homeomorphic to \mathbb{S}^1/\sim . Observe that K and Z are compact, connected and locally connected because $p : \overline{\mathbb{D}} \rightarrow K$ is continuous. Note also that $p^{-1}(z)$ is a nonseparating plane continuum for each z . By a theorem of Moore [10], \mathbb{C}/\simeq is homeomorphic to the plane. By construction the set Z is unshielded in the plane \mathbb{C}/\simeq , and the fact that Z is FS follows from Lemma 1.3. Since X is homeomorphic to Z and by Lemma 2.2, X is locally connected and FS. \square

Now we can prove the second implication in Theorem 1.4. Assume that X is an unshielded compact set and there exists a closed set $F \subset \mathbb{S}^1$ and a lamination \sim of F such that X is homeomorphic to the quotient space F/\sim . If $F = \mathbb{S}^1$, then by Theorem 2.3, X is FS. Assume that $F \neq \mathbb{S}^1$. Extend the equivalence \sim onto the entire unit circle \mathbb{S}^1 by declaring that a point in $\mathbb{S}^1 \setminus F$ is equivalent only to itself. This creates a lamination \sim on \mathbb{S}^1 . By Theorem 2.3 the continuum $Y = \mathbb{S}^1/\sim$ is FS. Obviously, the quotient space F/\sim is homeomorphic to a compact subset R of Y . Since X is homeomorphic to F/\sim , it is also homeomorphic to R . A compact subset of an FS-set is clearly an FS-set, so R is an FS-set, and thus, by Lemma 2.2, X is an FS-set itself. This proves the implication and completes the reduction of Theorem 1.4 to Theorem 2.1.

3. EMBEDDING AN UNSHIELDED FS-SET INTO AN UNSHIELDED FS-CONTINUUM

In this section we prove Theorem 2.1, using an inductive construction.

Given a compact planar set A and $\varepsilon > 0$ let $B(A, \varepsilon)$ be the union of all open ε -balls centered at points of A . Open sets U and V are called *strongly disjoint* if their closures are disjoint. Given a set X we denote its boundary by $\partial X = \overline{X} \setminus \text{int}(X)$. By a *Jordan disk* we mean the bounded component of $\mathbb{C} \setminus P$, where $P \subset \mathbb{C}$ is a Jordan curve. Given a compactum $Y \subset \mathbb{C}$, the unbounded component U of $\mathbb{C} \setminus Y$ is well defined. Call the set $T(Y) = \mathbb{C} \setminus U$ the *topological hull* of Y . To get the topological hull of a compactum one adds to this compactum all bounded components of its complement (“fill-in”). We have $\text{diam}(Y) = \text{diam}(T(Y))$ and the set $T(Y)$ is also compact. If a compactum X is unshielded, then we can say more: $T(A) \cap T(B) = \emptyset$ for any two distinct components A and B of X , and X is the boundary of $T(X)$. Finally, if W is open, $K \subset W$ is a continuum and for some $\varepsilon > 0$ and for each $x \in W \setminus T(K)$ there exists an arc $A_x \subset W$ joining x to K with $\text{diam}(A_x) < \varepsilon$, then W is called an ε -arc-neighborhood of K .

Lemma 3.1. *Let $X \subset \mathbb{C}$ be an unshielded FS-set and let $\eta > 0$. Then for every component K of X there exists a Jordan disk U such that $K \subset T(K) \subset U \subset B(T(K), \eta)$ and ∂U is disjoint from X .*

Proof. Set $X' = T(X)$. Fix a component K of X . By [5], there exists a sequence $\varepsilon_n \searrow 0$ such that for each n , each component of the boundary of $B(X', \varepsilon_n)$ is a singleton or an arc or a Jordan curve. One of those closed curves has to be the boundary of a Jordan disk U_n containing K .

Set $K' = \bigcap_n \overline{U_n}$. Clearly, $T(K) \subset K'$. To prove the reverse inclusion, observe that the boundaries of $B(X', \varepsilon_n)$ and $B(X', \varepsilon_{n+1})$ are disjoint, so $U_{n+1} \subset U_n$. Therefore K' is a continuum as an intersection of the descending sequence of continua. Suppose that there is a point $x \in K' \setminus X'$. Then it is contained in the unbounded component of $\mathbb{C} \setminus X$, so it can be connected with ∞ by an arc A disjoint from X' . The distance between A and X' is positive, so if n is sufficiently large, then A is disjoint from ∂U_n . Thus, $x \notin \overline{U_n}$, a contradiction. This proves that $K' \subset X'$. Thus, K' is a connected subset of X' containing $T(K)$. Since $T(K)$ is a connected component of X' , we get $K' = T(K)$. Therefore, if n is sufficiently large, then $K \subset U_n \subset B(T(K), \eta)$.

By the definition of ∂U_n , it is disjoint from X' . This completes the proof. \square

Lemma 3.2. *Let $X \subset \mathbb{C}$ be an unshielded FS-set, Y a closed subset of X such that every component of X intersecting Y is contained in Y , and let $\varepsilon \geq \eta > 0$. Assume that every component of Y has diameter smaller than ε . Then there exist strongly disjoint Jordan disks U_1, \dots, U_n such that $Y \subset \bigcup_{i=1}^n U_i \subset B(T(Y), \eta)$ and for $i = 1, \dots, n$ the diameter of U_i is less than 3ε , the boundary of U_i is disjoint from X and $U_i \cap Y \neq \emptyset$.*

Proof. By Lemma 3.1 for every component K of Y there exists a Jordan disk U such that $K \subset U \subset B(T(K), \eta)$ and ∂U is disjoint from X . This gives us an open cover of Y , so we can choose from it a finite subcover V_1, \dots, V_k . We can modify each V_i by moving slightly its boundary in $\mathbb{C} \setminus T(X)$, so that the resulting disks V'_i still satisfy $K_i \subset V'_i \subset B(T(K_i), \eta)$ for the appropriate components K_i of Y , and so that $\partial V'_i$ is disjoint from X and $\partial V'_i \cap \partial V'_j$ is finite if $i \neq j$. Let $U'_i, i = 1, \dots, n$, be the bounded components of $\mathbb{C} \setminus \bigcup_{j=1}^k \partial V'_j$ whose intersection with Y is nonempty. Then all U'_i are Jordan disks. Their boundaries are contained in $\bigcup_{j=1}^k \partial V'_j$, so they are disjoint from X . Each U'_i is contained in some $V'_j \subset B(T(K_i), \eta)$, so $U'_i \subset B(T(Y), \eta)$ and $\text{diam}(U'_i) < \varepsilon + 2\eta \leq 3\varepsilon$. Now we can replace each U'_i by a slightly smaller Jordan disk U_i such that $\overline{U_i} \subset U'_i, U_i \cap X = U'_i \cap X$ and $\partial U_i \cap X = \emptyset$. Then the disks U_1, \dots, U_n satisfy the assertions of the lemma. \square

Now we can prove a stronger version of Lemma 3.1.

Lemma 3.3. *Let $X \subset \mathbb{C}$ be an unshielded FS-set and let $\varepsilon > 0$. Then for every component K of X there exists a Jordan disk U such that U is an ε -arc-neighborhood of $T(K)$ and ∂U is disjoint from X .*

Proof. By Lemma 1.3 each component K is locally connected, so the Riemann mapping $\Psi : \mathbb{D}_\infty \rightarrow \mathbb{C} \setminus T(K)$ extends to a continuous map $\overline{\Psi} : \overline{\mathbb{D}_\infty} \rightarrow \overline{\mathbb{C} \setminus T(K)}$. Restricted to $D_R \cap \mathbb{D}_\infty$, where D_R is the disk of radius $R > 1$ centered at the origin, this map is uniformly continuous. Therefore for a given $\delta > 0$ there exists $\xi > 0$ such that the diameter of the ray piece $\{\overline{\Psi}(z) : z = tz_0 \in \mathbb{C}, 1 \leq t \leq 1 + \xi\}$ is smaller than δ for any z_0 of modulus 1. Then the set $V_\delta = T(K) \cup \Psi(\{z \in \mathbb{C} : 1 < |z| < 1 + \xi\})$ is a Jordan disk with the boundary $\Psi(\{z \in \mathbb{C} : |z| = 1 + \xi\})$ and is a δ -arc-neighborhood of $T(K)$.

Since X is an FS-set, we may choose $\delta < \varepsilon/7$ so small that every component of X that meets ∂V_δ has diameter less than $\varepsilon/7$. Denote the union of all components of X that meets ∂V_δ by X_K . Then X_K is a closed subset of X and each component of X_K has diameter less than $\varepsilon/7$. Let $\eta > 0$ be smaller than the distance between X_K and K . By Lemma 3.2, there exist strongly disjoint Jordan disks U_1, \dots, U_n such that $X_K \subset \bigcup_{i=1}^n U_i \subset B(T(X_K), \eta)$ and for $i = 1, \dots, n$ the diameter of U_i is less than $3\varepsilon/7$ and the boundary of U_i is disjoint from X . For each i , since $U_i \subset B(T(X_K), \eta)$, we have $\overline{U_i} \cap K = \emptyset$. Let U be the component of $\mathbb{C} \setminus (\bigcup_{i=1}^n \partial U_i \cup \partial V_\delta)$ that contains K . By Theorem 4, p. 512 of [8], U is a Jordan disk. We claim that U is the required ε -arc-neighborhood of $T(K)$. To see this choose $x \in U \setminus T(K)$. Since $x \in V_\delta$, there exists an arc $A \subset V_\delta$ of diameter less than $\varepsilon/7$ joining x to K . If A meets a $\overline{U_i}$, we can modify it by using a piece of ∂U_i . Using the triangle inequality it is easy to see that doing this for all $\overline{U_i}$ that A meets results in a new arc A' of diameter less than $\varepsilon/7 + 6\varepsilon/7 = \varepsilon$ as required. By the construction, ∂U is disjoint from X . \square

Lemma 3.4. *Let X be an unshielded FS-set contained in a Jordan disk U and let K_0, \dots, K_m be some components of X . Then for each $\varepsilon > 0$ there exists a finite cover $\{U_0, \dots, U_n\}$ ($n \geq m$) of X by strongly disjoint Jordan disks and components K_{m+1}, \dots, K_n of X such that for each $i \in \{0, \dots, n\}$, $K_i \subset U_i \subset \overline{U_i} \subset U$ and U_i is an ε -arc-neighborhood of $T(K_i)$. Moreover, for each component K of X such that $K \notin \{K_0, \dots, K_n\}$, $\text{diam}(K) < \varepsilon/2$.*

Proof. Let X, U and K_0, \dots, K_m be as specified above. Fix $\varepsilon > 0$. Since X is FS, there exist $k \geq m$ and components K_{m+1}, \dots, K_k such that $\text{diam}(K) < \varepsilon/3$ for any component K of X which is not one of K_0, \dots, K_k . By Lemma 3.3 there exist strongly disjoint Jordan disks U_0, \dots, U_k such that U_i is an ε -arc-neighborhood of $T(K_i)$, $\overline{U_i} \subset U$ and ∂U_i is disjoint from X for $i = 0, \dots, k$.

Set $Y = X \setminus \bigcup_{i=0}^k U_i$. Then Y is a closed subset of X and its distance from $(X \cap \bigcup_{i=0}^k U_i) \cup (\mathbb{C} \setminus U)$ is positive. Therefore by Lemma 3.2 there exist Jordan disks U_{k+1}, \dots, U_n of diameter less than ε such that $Y \subset \bigcup_{i=k+1}^n U_i$, $\overline{U_i} \subset U$, ∂U_i is disjoint from X and $U_i \cap Y \neq \emptyset$ for $i = k+1, \dots, n$ and all Jordan disks $U_0, \dots, U_k, U_{k+1}, \dots, U_n$ are strongly disjoint. To complete the proof, we choose components K_i of X , $i = k+1, \dots, n$, with $K_i \subset U_i$. \square

Given a Jordan disk U and two points $a, b \in U$ there exists a unique arc $S(a, b)$ of shortest arc length joining a, b in \overline{U} [4]. We may, and from now on do, assume that all Jordan disks have a piecewise linear boundary. Under this assumption there exists a unique shortest arc $S(a, b) \subset \overline{U}$ between any two points $a, b \in \overline{U}$ and the length $|S(a, b)|$ of $S(a, b)$ is finite. We call $S(a, b)$ the *s-arc* (between a and b in U). By [4] the family of s-arcs is continuous in the following sense: if $a_i \rightarrow a, b_i \rightarrow b$ and all these points belong to U , then $S(a_i, b_i)$ converge to $S(a, b)$ in the Hausdorff topology and $|S(a_i, b_i)| \rightarrow |S(a, b)|$. Hence, given two disjoint compact subsets A, B of U there is an arc of shortest length between a point of A and a point of B : just take a sequence of pairs of points a_i, b_i such that $|S(a_i, b_i)|$ converges to the infimum of the lengths over all arcs connecting a point of A with a point of B , and use the continuity. Thus we can talk of s-arcs between compacta; in particular, given a point $x \in U$ and a Jordan disk $V \subset \overline{V} \subset U$ we can find an arc (contained in \overline{U}) of shortest length from x to \overline{V} .

Similarly, one can consider arcs of smallest diameter between two points in U . Given two points $a, b \in U$, call an arc with endpoints a and b a *d-arc* (between a and

b in U) if its diameter is the smallest possible among all arcs connecting a and b in \overline{U} . By [7], if $a, b \in X \subset \overline{U}$ and X is a continuum, then $\text{diam}(X) \geq \text{diam}(S(a, b))$. Hence the s-arc between a and b is a d-arc between a and b . The case of arcs between sets is analogous to the above. Given two compacta $A, B \subset U$ call an arc with the endpoints in A, B a *d-arc (between A and B)* if it has the smallest diameter among all such arcs. To show that d-arcs between A and B exist, choose sequences $a_i \in A, b_i \in B$ so that $a_i \rightarrow a \in A, b_i \rightarrow b \in B$ and $\text{diam}(S(a_i, b_i))$ converges to the infimum of the diameters of arcs connecting points of A and B . Then by [4] the arcs $S(a_i, b_i)$ converge to the arc $S(a, b)$ in the Hausdorff topology, and it is easy to see that $\text{diam}(S(a_i, b_i)) \rightarrow \text{diam}(S(a, b))$. Thus, $S(a, b)$ is a d-arc between A and B . An upper bound on the diameter of an s-arc between a point in U and a Jordan disk in U is established below.

Lemma 3.5. *Let U be a Jordan disk, $V \subset \overline{V} \subset U$ be a Jordan disk, $x \in U$ be a point, δ be the diameter of a d-arc between x and \overline{V} , and let $S(x, y)$ be an s-arc between x and \overline{V} . Then $\text{diam}(S(x, y)) \leq 9\delta$.*

Proof. Choose a d-arc $S(x, z)$ between x and \overline{V} of diameter δ , such that $S(x, z) \cap \overline{V} = \{z\}$. We may assume that $x \notin \overline{V}$ and $z \neq y$. Observe that since $S(x, y)$ is an s-arc between x and \overline{V} , we have $S(x, y) \cap \overline{V} = \{y\}$. Fix $\varepsilon > \delta$; we will show that $\text{diam}(S(x, y)) < 9\varepsilon$. Draw the circle P centered at x of radius ε . If $S(x, y) \subset T(P)$, then $\text{diam}(S(x, y)) \leq 2\varepsilon < 9\varepsilon$ as desired. Thus we may assume that $S(x, y)$ is not a subset of $T(P)$.

By the uniqueness of shortest arcs, there is a point q such that $S(x, y) \cap S(x, z) = S(x, q)$. Moreover, $\overline{S(x, y) \setminus S(x, q)} = S(q, y)$ and $\overline{S(x, z) \setminus S(x, q)} = S(q, z)$. Since $S(x, z) \subset T(P)$, we have $q \in T(P)$. Choose an arc $R \subset \partial V$ connecting y and z and consider the Jordan curve $A = S(q, y) \cup R \cup S(q, z)$.

Clearly, $A \subset \overline{U}$ and $\text{int}(T(A)) \subset U$ is a Jordan disk. Let W be the component of $\text{int}(T(P)) \cap \text{int}(T(A))$ containing q (and hence $S(q, z)$) in its boundary (by Theorem 4, p. 512 of [8], all components of $\text{int}(T(P)) \cap \text{int}(T(A))$ are Jordan disks). Note that $\partial W \subset \overline{A} \subset \overline{U}$. Let $B \subset \partial W \setminus \{z\}$ be the minimal subarc of ∂W that connects q to ∂V . Then $B \subset S(q, y) \cup P$. Since $S(q, y)$ is an s-arc between q and V , then $|B| \geq |S(q, y)|$. Hence the parts of $S(q, y)$ that are not in B have total length at most $2\pi\varepsilon$. In particular, all parts of $S(x, y)$ outside $T(P)$ have total length at most $2\pi\varepsilon$. Therefore by the triangle inequality $\text{diam}(S(x, y)) \leq 2\varepsilon + 2\pi\varepsilon < 9\varepsilon$. Since ε can be chosen arbitrarily close to δ , we get $\text{diam}(S(x, y)) \leq 9\delta$. \square

We use Lemma 3.5 in the proof of the next lemma.

Lemma 3.6. *Let U be a Jordan disk containing the closures of strongly disjoint Jordan disks V_0, V_1, \dots, V_i and let V_{i+1}, \dots, V_n be points on ∂U . Choose $\varepsilon > 0$ so that U is an ε -arc-neighborhood of $\overline{V_0}$ and $\text{diam}(V_k) < \varepsilon, k = 1, \dots, n$. Then there exist disjoint trees $T_1, \dots, T_m, m < \infty$, such that $\text{diam}(T_j) < 20\varepsilon$, exactly one point of each T_j is in $\overline{V_0}$, and $(\bigcup_{j=1}^m T_j) \cap \overline{V_i}$ is a singleton for each $i = 1, \dots, n$.*

Proof. By the remarks preceding Lemma 3.5, there is an s-arc $A'_i = S(a_i, b_i)$ between $\overline{V_i}$ and $\overline{V_0}$ ($a_i \in \overline{V_i}, b_i \in \overline{V_0}$). If $x \in A'_i \cap A'_j$, then $S(x, b_i)$ is the arc in A'_i from x to b_i , $S(x, b_j)$ is the arc of A'_j from x to b_j , and these arcs have the same length. Hence the arc, say, $S(a_i, x) \cup S(x, b_j)$ is still an s-arc between a_i and $\overline{V_0}$ (the original s-arc is A'_i). Endow A'_i with the order $a_i < b_i$ and use interval notation for subarcs of A'_i . Inductively define s-arcs $A_i, 1 \leq i \leq n$, between a_i and

$\overline{V_0}$ so that each $G_k = \bigcup_{j=1}^k A_j, 1 \leq k \leq n$, is the union of disjoint trees, each of which intersects $\overline{V_0}$ at one point. We can do this as follows. First we set $A_1 = A'_1$. Suppose A_1, \dots, A_k have been defined; choose the smallest point x_{k+1} on A'_{k+1} which belongs to $\overline{V_0} \cup \bigcup_{j=1}^k A_j$, denote by B the unique arc in G_k joining x_{k+1} to $\overline{V_0}$ (such an arc is unique by induction), and set $A_{k+1} = [a_{k+1}, x_{k+1}] \cup B$. It is now easy to verify the inductive assumptions, which completes the construction.

Let C be a component of $G_n, C \cap \overline{V_0} = \{z\}$. By the construction, C is the union of s -arcs between a point in U and $\overline{V_0}$. By Lemma 3.5 each such arc has diameter at most 9ε , so $\text{diam}(C) \leq 18\varepsilon$. Since some arcs A_i and $\overline{V_j}, j \neq i$, may intersect, to complete the proof we modify G_n . Choose strongly disjoint Jordan disks $V'_i \supset \overline{V_i}$ of diameters less than ε . For each i choose a Jordan disk $D_i \subset V_i$ such that $\partial D_i \cap \partial V_i = \overline{D_i} \cap G_n = \{a_i\}$ and also a Jordan disk $D'_i \subset \overline{D'_i} \subset V'_i$ containing V_i and such that $\partial D'_i \cap \partial V_i = \{a_i\}$. Then we can use an isotopy in $\overline{V'_i}$ which expands D_i onto D'_i and is the identity on $\partial V'_i \cup \{a_i\}$. Repeating this for all i , we construct the desired union of trees. Since each change we made is confined to a set of diameter less than ε , the diameter of each tree does not change by more than 2ε . □

To prove Theorem 2.1 we apply Lemma 3.6; the inductive construction is close to [2]. For brevity and to follow the terminology of [2], below we call Jordan disks *bubbles*. Choose a bubble $U_0 \supset X, R > \text{diam}(U_0)$, and a component K_0 of X . Set $\delta_n = 4^{-n}$. By Lemma 3.4 we can find a finite cover $\{U_0^1, \dots, U_{m(1)}^1\}$ of X by strongly disjoint bubbles and components $K_i^1 \subset U_i^1$ of X satisfying the conditions of Lemma 3.4 with $\varepsilon = \delta_1$ and $K_0 = K_0^1$. Thus, after the first step we have $m(1) + 1$ components $K_i^1, 0 \leq i \leq m(1)$ of X with strongly disjoint δ_1 -arc-neighborhoods of their topological hulls: the bubbles $U_i^1, 0 \leq i \leq m(1)$. By Lemma 3.6 the sets $\overline{U_i^1}, 1 \leq i \leq m(1)$ can be connected to $\overline{U_0^1}$ by means of pairwise disjoint trees of diameters less than R . On sets $\partial U_i^1, 0 \leq i \leq m(1)$ we mark finitely many points at which those trees intersect them (there will be at most $m(1)$ marked points on ∂U_0^1 and one marked point on each $\partial U_i^1, 1 \leq i \leq m(1)$). Finally, by Lemma 3.4 all other components of X (not belonging to $\{K_0^1, \dots, K_{m(1)}^1\}$) are of diameter less than $\delta_1/2$. Let us denote the union of all connecting trees and all sets $U_i^1, 0 \leq i \leq m(1)$ by T_1 . The set T_1 can be thought of as a tree in which some points are replaced by bubbles. Observe that by Lemma 3.6 the connecting trees do not penetrate the bubbles.

Apply the same construction to the pairs $(U_i^1, K_i^1), 0 \leq i \leq m(1)$ (K_i^1 and U_i^1 replace $K_0 = K_0^1$ and U_0 , and the constant is δ_2 , not δ_1). The difference is that now we have bubbles and finitely many points on the boundary of each U_i^1 . Still, Lemma 3.6 is applicable. Thus, inside each bubble of the first generation we construct bubbles of the the second generation each of which is a δ_2 -arc-neighborhood of the appropriate component of X . Moreover, components $K_0^1, \dots, K_{m(1)}^1$ chosen on the previous step remain on the list of chosen components of the second generation, and some bubbles of the second generation are δ_2 -arc-neighborhoods of the topological hulls of those components. The trees with bubbles, constructed inside the bubbles of the first generation, replace the bubbles of the first generation and are then added to already existing trees. To sum it all up, once the construction is applied inside each U_i^1 , we complete the second step which gives rise to the set

$T_2 \subset T_1$ which is also a tree with bubbles. Now apply the same arguments inside each bubble of the second generation with the new constant δ_3 , etc. In the end we get the infinite intersection $Z = \bigcap T_n$. Then Z is a nonseparating continuum. Put $Y = \partial Z$. Then Y is an unshielded continuum. Let us now establish some properties of the construction.

Property A. *We have $X \subset Z$.*

Proof. It follows from the fact that $X \subset T_n$ for any n . □

If $K \subset X$ is a component of X , denote by U_K^n the bubble of the n -th generation containing K .

Property B. *If K is a component of X , then $\bigcap_n \overline{U_K^n} = \bigcap_n U_K^n = T(K)$.*

Proof. By the construction, $\bigcap_n \overline{U_K^n} = \bigcap_n U_K^n$. Set $B = \bigcap_n \overline{U_K^n}$. Then by the construction $T(K) \subset B$. On the other hand, no point outside $T(K)$ belongs to B because for any $\varepsilon > 0$ from some time on the sets U_K^n are contained in ε -balls around K . Hence $B = T(K)$. □

Property C. *We have $X \subset Y$.*

Proof. Let K be any component of X and let U_K^n be the bubble of the n -th generation containing K . By Property B, $\bigcap_n \overline{U_K^n} = \bigcap_n U_K^n = T(K)$. Since X is unshielded, $K = \lim \partial U_K^n$. It follows from the construction that $\partial U_K^n \subset \mathbb{C} \setminus Z$ for every n . Hence $K \subset \partial Z = Y$. □

Call a pair (U, K) which appears in the construction (on the step n) a *building pair (of generation n)*. Recall that $T(X)$ denotes the union of the topological hulls of all components of X and set $A = Y \setminus T(X)$. Represent A as a countable union of *open* (homeomorphic to $(0, 1)$) arcs as follows. Let (U, K) be a building pair of generation n , (V, K) and (W, L) be building pairs of generation $n + 1$, $W \subset U$. Then there is a unique arc $I \subset T_{n+1}$ connecting W and V . Outside W this arc is extended on each step so that its endpoint approaches K . Finitely many points on ∂U (coming from the previous step) are connected to V similarly. These sets, connecting to K either bubbles of the next generation (contained in U), or the finitely many points chosen on ∂U , are called *partial connectors (of generation n)*. On the other hand, a similar process takes place inside W where the partial connector towards L is being constructed. The entire inductively constructed set connecting L and K is said to be a *connector*. Clearly, there are countably many connectors, and A is the union of all of them. By the construction, a connector is the union of two partial connectors (above, one partial connector extends towards K , and the other one towards L). Hence the claim that all connectors are arcs follows from Property D below.

Property D. *Let (U, K) be a building pair of generation n , (W, L) be a building pair of generation $n + 1$, $W \subset U$ and $K \neq L$. Then the following estimates hold.*

- (1) $\text{diam}(W) \leq 4^{-n}$.
- (2) *Let I be a partial connector between W and K , or between one of the finitely many points chosen on ∂U , and K . Then \overline{I} is a closed arc, $\text{diam}(I) < \frac{80}{3}4^{-n}$, and if I is a partial connector between W and K , then $\text{diam}(I \cup W) < \frac{83}{3}4^{-n}$.*

Proof. (1) By the construction $\text{diam}(L) < \delta_n/2$. By Lemma 3.4 and the construction, for each point $x \in W$ there is a point $y \in L$ such that $|x - y| \leq \delta_{n+1}$. Hence $\text{diam}(W) \leq \delta_n/2 + 2\delta_{n+1} = 4^{-n}$.

(2) By the construction and Lemma 3.6 the partial connector I between W and K (or between an appropriate point on ∂U and K) is the concatenation of disjoint (except endpoints where two consecutive arcs meet) arcs of diameters at most $20\delta_n, 20\delta_{n+1}, \dots$. Hence $\text{diam}(I) \leq 20(\delta_n + \delta_{n+1} + \dots) = \frac{80}{3}4^{-n}$. Since the endpoint of I belongs to K , it does not belong to any of the arcs that we concatenate. Therefore \bar{I} is a closed arc. Together with (1) this implies that $\text{diam}(I \cup W) < \frac{83}{3}4^{-n}$ and completes the proof. \square

Property E. *If $C \subset Y$ is a continuum, then its intersection with a component K of X is a continuum too.*

Proof. Let C_n be the intersection of C with the bubble of n -th generation containing K . Then, by the definition of T_n , the set C_n is connected. The sequence of continua $(C_n)_{n=1}^\infty$ is descending, so its intersection $C \cap T(K)$ is a continuum. Since $C \subset Y$ and $Y \cap T(K) = K$, we have $C \cap T(K) = C \cap K$. \square

To prove Theorem 2.1 we need to show that Y is FS. Suppose that for some $\varepsilon > 0$ there are infinitely many disjoint continua $C_n \subset Y$ with diameter greater than ε . Choose $m > 1$ so that $\delta_m = 4^{-m} < \frac{3\varepsilon}{332}$. Consider T_m . As we know T_m is a tree with finitely many bubbles; hence only finitely many C_n 's are not contained in one of those bubbles. Thus we may assume that for a building pair (U, K) of generation m all C_n 's are contained in U .

Let us estimate the diameters of the C_n 's. Since the C_n 's are disjoint, their intersections with K are disjoint too. By Property E each intersection $K \cap C_n$ is a continuum, and since K is FS we see that $\text{diam}(C_n \cap K) \rightarrow 0$. Hence we may assume that no C_n is contained in K and that $\text{diam}(C_n \cap K) < \varepsilon/2$. Now, it follows from the construction that the set $(Y \cap U) \setminus K$ is the union of sets, each of which is contained in a set of the form listed in Property D (2). That is, it is either contained in the union of a bubble $V \subset U$ of generation greater than m and a partial connector between K and V , or it is contained in a partial connector between one of the finitely many chosen points on ∂U and K . By Property D every set in the list is of diameter at most $\frac{83}{3}4^{-m}$. By the triangle inequality, this implies that $\text{diam}(C_n) \leq \frac{166}{3}4^{-m} + \varepsilon/2 < \varepsilon$ (the last inequality follows from the fact that $4^{-m} < \frac{3\varepsilon}{332}$), a contradiction. Hence, Y is finitely Suslinian.

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