ON IGUSA ZETA FUNCTIONS OF MONOMIAL IDEALS

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ABSTRACT. We show that the real parts of the poles of the Igusa zeta function of a monomial ideal can be computed from the torus-invariant divisors on the normalized blow-up of the affine space along the ideal. Moreover, we show that every such number is a root of the Bernstein-Sato polynomial associated to the monomial ideal.

1. INTRODUCTION

If f is a nonconstant polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ and p is a fixed prime, then the Igusa zeta function of f is defined by

$$(1) \qquad Z_f(s) = \int_{(\mathbb{Z}_p)^n} |f(y)|_p^s \, |dy|,$$

for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers, endowed with the discrete valuation ord_p and with the p-adic absolute value defined by

$$|a|_p = \left(\frac{1}{p}\right)^{\mathrm{ord}_p(a)}.$$

The measure μ on $(\mathbb{Z}_p)^n$ that is used in the above integral is the Haar measure characterized by

$$\mu\left(\prod_{i=1}^{n} p^{m_i} \mathbb{Z}_p\right) = \left(\frac{1}{p}\right)^{\sum_i m_i}.$$

As defined, Z_f is a holomorphic function on the half plane $\{s \mid \mathrm{Re}(s) > 0\}$, and one can show that it admits a meromorphic extension to $\mathbb{C}$. In fact, Z_f is a rational function of $\left(\frac{1}{p}\right)^s$. A proof of rationality that also gives information on the real parts of the possible poles of Z_f proceeds as follows. Let $\pi\colon Y \to X = \mathbb{A}^n$ be an embedded resolution of singularities of f defined over $\mathbb{Q}$. This means that π is proper and birational, Y is smooth, and the union of $\pi^*(\mathrm{div}(f))$ and of the exceptional locus of π is a divisor with simple normal crossings. For every prime divisor E in this union, denote by $a_E(f)$ the order of E in $\pi^*(\mathrm{div}(f))$ and by k_E the order of E in the relative canonical class $K_{Y/X}$ (this is the divisor locally defined by $\det(\mathrm{Jac}(\pi))$). Using the Change of Variable formula for p-adic integrals to express

Z_f as an integral over $Y(\mathbb{Z}_p)$, Igusa obtained the rationality of Z_f as a function of $\left(\frac{1}{p}\right)^s$ and the fact that if s is a pole of Z_f, then $\mathrm{Re}(s) = -\frac{k_E + 1}{a_E(f)}$ for some divisor E as above. Our main reference for Igusa zeta functions is Igusa's book [Ig] (see also Denef's Bourbaki report [De]).

While every divisor on a log resolution of f gives a candidate for the real part of a pole of Z_f, examples show that most of these numbers do not come actually from poles of Z_f. In fact an outstanding open problem in the field is the following conjecture of Igusa, relating the poles to one of the basic invariants of the singularities of f, its Bernstein-Sato polynomial.

Conjecture 1.1. *Let f be a nonconstant polynomial in $\mathbb{Z}[X_1, \ldots, X_n]$. For almost all primes p the following holds: if s is a pole of Z_f, then the real part of s is a root of the Bernstein-Sato polynomial b_f of f.*

We recall that the Bernstein-Sato polynomial of f is a polynomial in one variable introduced independently in [Be] and [SS]. It is a subtle but very fundamental invariant of the singularities of f. We do not give its definition as we will not need it, but we mention that its roots are related to the eigenvalues of the monodromy of the hypersurface $f^{-1}(0)$. There is, in fact, a weaker version of the above conjecture that is stated in terms of these eigenvalues and that is known as the Monodromy Conjecture (see [De] for more on these conjectures and also [Ve] for some recent work in this direction).

Our goal in this note is to prove the analogue of Conjecture 1.1 when we replace f by a monomial ideal. Though less studied, Igusa zeta functions for nonnecessarily principal ideals in $\mathbb{Z}[x_1, \ldots, x_n]$ can be defined in a very similar way with (1). More precisely, if I is a nonzero proper ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ and if we put for $y \in (\mathbb{Z}_p)^n$

$$\mathrm{ord}_p\, I(y) = \min\{\mathrm{ord}_p(f(y)) | f \in I\},$$

then we have

$$(2) \qquad Z_I(s) = \int_{(\mathbb{Z}_p)^n} \left(\frac{1}{p}\right)^{\mathrm{ord}_p\, I(y)} |dy|.$$

The above-mentioned results in the case of one polynomial extend in a straightforward way to the case of an arbitrary ideal. Note that in order to prove rationality, we need to consider a log resolution of I: this is a morphism $\pi \colon Y \to \mathbb{A}^n$ as before, such that $\pi^{-1}(V(I))$ is a Cartier divisor and its union with the exceptional locus of π is a divisor with simple normal crossings. If E is a prime divisor on Y contained in this union, then $a_E(I)$ is by definition the coefficient of E in $\pi^{-1}(V(I))$. As in the case of a principal ideal, one can show that given a log resolution π, for every pole s of Z_I there is a divisor E on Y such that $\mathrm{Re}(s) = -\frac{k_E + 1}{a_E(I)}$.

On the other hand, the definition of the Bernstein-Sato polynomial has been extended in [BMS3] from the case of one polynomial to that of an arbitrary ideal. This is again a polynomial in one variable, and therefore the analogue of Conjecture 1.1 makes sense in this case. We will prove the monomial case, i.e. when I is generated by monomials.

Theorem 1.2. *If I is a nonzero proper monomial ideal of $\mathbb{Z}[x_1, \ldots, x_n]$, then for every prime p and every pole s of Z_I, the real part of s is a root of the Bernstein-Sato polynomial of I.*

The key ingredient in the proof of the above theorem is a result on the poles of Igusa-type zeta functions associated to cones. Suppose that $N \simeq \mathbb{Z}^n$ is a lattice and that σ is a pointed, rational, polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by σ° the relative interior of the cone σ. If $\sigma^{\vee}$ is the dual cone in $M_{\mathbb{R}}$, where $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and if ℓ_1, ℓ_2 are elements in $\sigma^{\vee} \cap M$ such that $\sigma \cap \{v \mid \ell_2(v) = 0\} = \{0\}$, then we put

$$(3) \qquad Z_{\sigma,\ell_1,\ell_2}(s) := \sum_{v \in \sigma^{\circ} \cap N} \left(\frac{1}{p}\right)^{\ell_1(v)s + \ell_2(v)}.$$

It is easy to see (and it will follow from our computations) that this is well-defined and holomorphic in $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$. We prove the following

Theorem 1.3. *For every σ, ℓ_1 and ℓ_2 as above, Z_{σ,ℓ_1,ℓ_2} is a rational function of $\left(\frac{1}{p}\right)^s$, and therefore can be meromorphically extended to $\mathbb{C}$. Moreover, for every pole s of Z_{σ,ℓ_1,ℓ_2} there is a primitive generator v of a ray of σ such that $\operatorname{Re}(s) = -\frac{\ell_2(v)}{\ell_1(v)}$.*

Given a monomial ideal I, we give in the next section a formula for Z_I in terms of suitable zeta functions for the cones in the normal fan to the Newton polyhedron of I (we refer for the relevant definition and for the precise formula to that section). Let us just mention that this fan defines the toric variety that is the normalized blowing-up of $\mathbb{A}^n$ along the ideal I. Using this formula and Theorem 1.3, we will show in §3 that the real part of every pole of Z_I corresponds to a torus-invariant divisor in the normalized blow-up of $\mathbb{A}^n$ along I (despite the fact that the normalized blowing-up is *not* a log resolution of I).

Theorem 1.4. *Let I be a nonzero proper monomial ideal of $\mathbb{Z}[x_1, \ldots, x_n]$. For every pole s of Z_I, there is a torus-invariant divisor E on the normalized blowing-up of $\mathbb{A}^n$ along I such that*

$$\operatorname{Re}(s) = -\frac{k_E + 1}{a_E(I)}.$$

On the other hand, explicit descriptions of the roots of the Bernstein-Sato polynomial of a monomial ideal have been obtained in [BMS1] and [BMS2]. We use the description in [BMS2] and Theorem 1.4 to prove Theorem 1.2 in the last section.

We mention that a description for the Igusa zeta function of a monomial ideal has also been obtained by Zúñiga-Galindo in [Zu]. Moreover, similar results appear in the work of Denef and Hoornert [DH], in which one describes the poles of the Igusa zeta functions for nondegenerate hypersurfaces with respect to their Newton polyhedron. It is shown in *loc. cit.* that for such f the real part of essentially any pole corresponds to a facet of the Newton polyhedron of f, as above. Moreover, Loeser [Lo] showed that under some mild extra assumptions these numbers are roots of the Bernstein-Sato polynomial of f, thus proving Conjecture 1.1 for such nondegenerate hypersurfaces. On the other hand, note that the relations between the respective Igusa zeta functions and between the Bernstein-Sato polynomials of f and of the corresponding monomial ideal are not clear in general.

2. Igusa zeta functions of monomial ideals

Let I be a nonzero proper ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by monomials. If $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we denote by x^u the corresponding monomial $x_1^{u_1} \ldots x_n^{u_n}$.

The Newton polyhedron P_I of I is the convex hull of those u in $\mathbb{N}^n$ such that x^u is in I.

If $N = \mathbb{Z}^n$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, we think of P_I as lying in $M_\mathbb{R}$, where $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between M and N. If we consider in $N_\mathbb{R}$ the cone generated by the elements of the standard basis $e_1, \ldots, e_n$, then the corresponding toric variety is the affine space $\mathbb{A}^n$ and the subscheme $V(I)$ is invariant under the torus action (we refer to [Fu] for the basic notions on toric varieties). Hence the normalized blowing-up of $\mathbb{A}^n$ along I is again a toric variety, and therefore it corresponds to a fan subdividing the above cone. This is the normal fan to the polyhedron P_I, that we will denote by Δ_I. It is defined as follows: to each face Q of P_I one associates the cone

$$\sigma_Q := \{v \in N_\mathbb{R} | \langle u, v \rangle \leq \langle u', v \rangle \text{ for every } u \in Q \text{ and } u' \in P_I\}.$$

The fan Δ_I consists of the cones σ_Q, when Q varies over the faces of P_I. Note that $\dim(\sigma_Q) = n - \dim(Q)$, so the rays of Δ_I correspond to the facets of P_I, and the maximal cones of Δ_I correspond to the vertices of P_I.

Let p be a fixed prime. We now proceed to the computation of Z_I. For every $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we consider the set $C_a = \prod_{i=1}^n (p^{a_i}\mathbb{Z}_p \smallsetminus p^{a_i+1}\mathbb{Z}_p)$. Since each $p^{a_i}\mathbb{Z}_p \smallsetminus p^{a_i+1}\mathbb{Z}_p$ is a disjoint union of $(p-1)$ translates of $p^{a_i+1}\mathbb{Z}_p$, we see that

$$\mu(C_a) = (p-1)^n \left(\frac{1}{p}\right)^{n+\sum_i a_i}.$$

We denote by e the vector $(1, \ldots, 1)$, so $\langle e, a \rangle = \sum_i a_i$.

The function $\mathrm{ord}_p I$ is constant on C_a with value

$$\nu(a) := \min\{\langle u, a \rangle \mid x^u \in I\} = \min\{\langle u, a \rangle | u \in P_I\}.$$

Since the sets C_a are disjoint and the complement of their union has measure zero, we deduce

$$(4) \qquad Z_I(s) = \sum_{a \in \mathbb{N}^n} \left(1 - \frac{1}{p}\right)^n \cdot \left(\frac{1}{p}\right)^{\langle e, a \rangle + s\nu(a)}.$$

Note that ν is a linear function on each of the cones in Δ_I. Indeed, if w is a vertex of P_I, then $\nu(a) = \langle w, a \rangle$ whenever a is in σ_w.

If σ is a cone in Δ_I, choose a vertex w of P_I such that σ is contained in σ_w and put $\ell_\sigma := w$. By letting the a in (4) vary inside the relative interior of each cone in Δ_I, and using the definition in the Introduction, we get the following

Proposition 2.1. *With the above notation, we have*

$$(5) \qquad Z_I(s) = \left(1 - \frac{1}{p}\right)^n \cdot \sum_{\sigma \in \Delta_I} Z_{\sigma, \ell_\sigma, e}(s).$$

3. IGUSA ZETA FUNCTIONS FOR CONES

Our goal in this section is to discuss Igusa-type zeta functions associated to cones and prove Theorem 1.3. Let N be a lattice, M its dual, and σ a pointed, rational polyhedral cone in $N_\mathbb{R}$. We consider ℓ_1 and ℓ_2 in $\sigma^\vee \cap M$, where $\sigma^\vee$ is the dual cone of σ, such that $\sigma \cap \{v \mid \ell_2(v) = 0\} = \{0\}$. We want to study the function $Z_{\sigma, \ell_1, \ell_2}$ and its poles.

The definition of Z_{σ,ℓ_1,ℓ_2} was motivated by the formula in Proposition 2.1, but sometimes it is more natural to consider a version of this function in which we sum over all the integer points in σ:

$$(6) \qquad \overline{Z}_{\sigma,\ell_1,\ell_2}(s) := \sum_{v \in \sigma \cap N} \left(\frac{1}{p}\right)^{\ell_1(v)s+\ell_2(v)}.$$

Again, this is well-defined if $\operatorname{Re}(s) > 0$ and we have $\overline{Z}_{\sigma,\ell_1,\ell_2} = \sum_\tau Z_{\tau,\ell_1,\ell_2}$, where the sum is over all faces τ of σ. Moreover, it follows from this formula that Z_{σ,ℓ_1,ℓ_2} can be computed in terms of the functions $\overline{Z}_{\tau,\ell_1,\ell_2}$, where τ varies over the faces of σ.

We start with the following

Lemma 3.1. *Let $v_1, \ldots, v_r$ in N be linearly independent over $\mathbb{Q}$. If w is in N and ℓ_1, ℓ_2 are elements in M, with ℓ_1 nonnegative and ℓ_2 positive on all the v_i, then we put*

$$A(s) := \sum_{v \in S} \left(\frac{1}{p}\right)^{\ell_1(v)s+\ell_2(v)},$$

where $S = \{w + a_1 v_1 + \ldots + a_r v_r \mid a = (a_i) \in \mathbb{N}^r\}$. The function A is well-defined and holomorphic for $\operatorname{Re}(s) > 0$ and it is a rational function in $\left(\frac{1}{p}\right)^s$, so it has a meromorphic continuation to $\mathbb{C}$. Moreover, if s is a pole of A, then there is i such that $\operatorname{Re}(s) = -\frac{\ell_2(v_i)}{\ell_1(v_i)}$.

Proof. If $\operatorname{Re}(s) > -\frac{\ell_2(v_i)}{\ell_1(v_i)}$ for all i such that $\ell_1(v_i)$ is nonzero, then we have

$$A(s) = \left(\frac{1}{p}\right)^{\ell_1(w)s+\ell_2(w)} \cdot \prod_{i=1}^n \sum_{a_i \in \mathbb{N}} \left(\frac{1}{p}\right)^{a_i(\ell_1(v_i)s+\ell_2(v_i))}$$

$$= \left(\frac{1}{p}\right)^{\ell_1(w)s+\ell_2(w)} \cdot \prod_{i=1}^n \frac{1}{1 - \left(\frac{1}{p}\right)^{\ell_1(v_i)s+\ell_2(v_i)}}.$$

The assertions in the lemma are direct consequences of this formula. $\square$

We can now give the proof of our result on Igusa-type zeta functions associated to cones.

Proof of Theorem 1.3. Arguing by induction on the dimension of σ, we may assume that the theorem holds for all cones of smaller dimension than $\dim(\sigma)$ (the case when $\dim(\sigma)$ is zero being trivial). In this case, we see that proving the assertions in the theorem for Z_{σ,ℓ_1,ℓ_2} is equivalent with proving them for $\overline{Z}_{\sigma,\ell_1,\ell_2}$.

We first show that it is enough to prove the theorem when σ is a simplicial cone. Indeed, it is well-known that one can always find a fan Γ refining the cone σ such that every cone in Γ is simplicial and the one-dimensional cones in Γ are precisely the rays of σ. Since

$$\overline{Z}_{\sigma,\ell_1,\ell_2} = \sum_{\tau \in \Gamma} Z_{\tau,\ell_1,\ell_2},$$

and since each ray of a cone in Γ is a ray of σ, we see that it is enough to prove the theorem for each (maximal) cone in Γ.

Therefore we may assume that σ is simplicial and our goal is to show that $\overline{Z}_{\sigma,\ell_1,\ell_2}$ satisfies the assertions in the theorem. Let $v_1,\ldots,v_r$ be the primitive generators of the rays of σ. Since σ is simplicial, the v_i are linearly independent over $\mathbb{Q}$. The semigroup $\sigma \cap N$ is finitely generated, so we may choose generators $w_1,\ldots,w_s$. The v_i span σ over $\mathbb{Q}$, hence we can find a positive integer m such that every mw_j is in the semigroup generated by the v_i. It follows that after replacing $\{w_1,\ldots,w_s\}$ by $\{q_1w_1 + \ldots + q_sw_s \mid 0 \le q_j \le m-1\}$, we may assume that

$$(7) \qquad \sigma \cap N = \bigcup_{j=1}^{s}(w_j + S),$$

where S is the semigroup generated by the v_i.

If $I \subseteq \{1,\ldots,s\}$, let us put

$$A_I(s) := \sum_v \left(\frac{1}{p}\right)^{\ell_1(v)s+\ell_2(v)},$$

where the sum is over v in $\bigcap_{j \in I}(w_j+S)$. We claim that $\bigcap_{j \in I}(w_j+S)$ is either empty or it is equal to $w + S$ for a suitable w in N. Indeed, by an obvious induction on $|I|$ it is enough to show this when I has two elements, say j and k. The intersection of $w_j + S$ and $w_k + S$ is nonempty if and only if $w_j - w_k$ lies in the lattice generated by the v_i. If this is the case, let us write $w_j - w_k = \sum_{i=1}^{r} a_i v_i$ for suitable integers $a_1,\ldots,a_r$. If we put $w = w_j + \sum_{i=1}^{r} \max\{0,-a_i\}v_i$, then it is easy to check that $(w_j + S) \cap (w_k + S) = w + S$, which proves our claim.

It follows from our claim and Lemma 3.1 that each A_I is a rational function of $\left(\frac{1}{p}\right)^s$. Moreover, if s is a pole of A_I, then there is i such that $\mathrm{Re}(s) = -\frac{\ell_2(v_i)}{\ell_1(v_i)}$. On the other hand, it follows from (7) that

$$(8) \qquad \overline{Z}_{\sigma,\ell_1,\ell_2} = \sum_I (-1)^{|I|-1} A_I(s),$$

where the sum is over all nonempty subsets I of $\{1,\ldots,s\}$. Therefore $\overline{Z}_{\sigma,\ell_1,\ell_2}$ satisfies the assertions of the theorem, which completes the proof. $\qquad\square$

Putting together Theorem 1.3 and the description of the Igusa zeta function of a monomial ideal from the previous section, we can relate the poles of this zeta function with the torus-invariant divisors in the blowing-up along the ideal.

Proof of Theorem 1.4. It follows from Proposition 2.1 and Theorem 1.3 that if s is a pole of Z_I, then there is a primitive generator v of a ray of the normal fan Δ_I to the Newton polyhedron P_I such that

$$\mathrm{Re}(s) = -\frac{\langle e, v\rangle}{\langle w, v\rangle}.$$

Here w is a vertex of P_I such that v is contained in the maximal cone σ_w of Δ_I corresponding to w.

On the other hand, recall that the torus-invariant divisors on the toric variety defined by Δ_I correspond precisely to the rays of Δ_I. Moreover, if E is the divisor corresponding to the ray through v, then $k_E = \langle e, v\rangle - 1$. Since we also have

$$a_E(I) = \min\{\langle u, v\rangle \mid x^u \in I\} = \langle w, v\rangle,$$

as v lies in σ_w, we deduce the assertion in the theorem. $\qquad\square$

4. Poles and roots of the Bernstein-Sato polynomial

We now show that the real part of any pole of Z_I is a root of the Bernstein-Sato polynomial b_I associated to I. In fact, we prove the following stronger statement that together with Theorem 1.4 implies Theorem 1.2.

Proposition 4.1. *If I is a nonzero proper monomial ideal and if E is a prime divisor in the normalized blowing-up of the affine space along I such that $a_E(I)$ is nonzero, then $-\frac{k_E+1}{a_E(I)}$ is a root of the Bernstein-Sato polynomial b_I.*

Proof. The divisor E corresponds to a ray in the normal fan Δ_I to P_I. Let v be a primitive generator of this ray. If w is a vertex of P_I such that the corresponding maximal cone σ_w of Δ_I contains v, then we have seen in the proof of Theorem 1.4 that $k_E + 1 = \langle e, v \rangle$ and $a_E(I) = \langle w, v \rangle$. Note that since $\langle w, v \rangle \neq 0$, the facet Q of P_I corresponding to v is not contained in a coordinate hyperplane: if, for example, Q is contained in the hyperplane $(x_i = 0)$, then $v = e_i$, and since w lies in Q we get $\langle w, v \rangle = 0$, a contradiction.

In order to show that $(k_E + 1)/a_E(I)$ is a root of the Bernstein-Sato polynomial b_I associated to I, we use the description of the roots of b_I from [BMS2] (in fact, the ones that we need for the theorem are "the most straightforward" of the roots of b_I). Since Q is a facet of P_I that is not contained in a coordinate hyperplane, there is a unique linear function L_Q on $M_\mathbb{R}$ such that $Q = P_I \cap L_Q^{-1}(1)$. With this notation, it is shown in [BMS2] (see Remark 4.6) that $-L_Q(e)$ is a root of b_I.

On the other hand, since the ray through v corresponds to the facet Q and since w is in Q, we have
$$Q = \{u \in P_I \mid \langle u, v \rangle = \langle w, v \rangle\}.$$
Therefore L_Q is given by $\frac{1}{\langle w,v \rangle} \cdot v$, and since $-L_Q(e)$ is a root of b_I, we see that $(k_E + 1)/a_E(I)$ is, indeed, a root of b_I. $\qquad\square$

Remark 4.2. We do not know whether the analogue of Proposition 4.1 holds for a nonnecessarily monomial ideal I. Note that if $I = (f)$ is principal, then the assertion is trivial: the divisor E is one of the irreducible components of the divisor H defined by f, $k_E = 0$, and $a_E(I)$ is the multiplicity of E in H. The fact that $-\frac{1}{a_E(f)}$ is a root of b_f then follows by restricting to an open subset where E is smooth and $H = a_E(f) \cdot E$.

Remark 4.3. The arguments in the previous two sections can also be used to analyze the orders of the possible poles of the Igusa zeta function Z_I. Indeed, it follows from Proposition 2.1 and from the proof of Theorem 1.3 that if s is a pole of order r of Z_I, then $r \leq n$ and there are r invariant divisors $E_1, \ldots, E_r$ on the normalized blowing-up along I such that $E_1 \cap \ldots \cap E_r \neq \emptyset$ and $\mathrm{Re}(s) = -(k_{E_i} + 1)/a_{E_i}(I)$ for every i. We would like to deduce that in this case $\mathrm{Re}(s)$ is a root of order r of b_I, but unfortunately, we do not understand well enough the multiplicities of the roots of b_I.

Remark 4.4. While Proposition 2.1 gives in principle a formula for the Igusa zeta function of a monomial ideal, and Theorem 1.4 gives an estimate on the denominator of this function (written as a rational function of $1/p^s$), getting a general explicit formula for the denominator is rather difficult. A *Maple* code for computing p-adic and motivic zeta functions of monomial ideals via resolution of singularities is available, upon request, from the first author.

Remark 4.5. Using motivic integration, Denef and Loeser defined in [DL] a motivic analogue of the Igusa zeta function. For concreteness, we preferred to work with p-adic integrals. However, as the reader familiar with this topic will certainly notice, all the above results have analogues in the motivic setting, "replacing p by $\mathbb{L}$". For example, if σ, ℓ_1 and ℓ_2 are as in Theorem 1.3, then the series

$$(9) \qquad \sum_{v \in \sigma \cap N} \mathbb{L}^{-(\ell_1(v)s + \ell_2(v))}$$

can be written as a sum of fractions with numerator in $K[\mathbb{L}^{-s}]$ and denominator of the form

$$(10) \qquad \prod_{i=1}^{r} \left(1 - \mathbb{L}^{-(\ell_1(v_i)s + \ell_2(v_i))} \right),$$

where $r \le \dim(\sigma)$ and $v_1, \ldots, v_r$ are primitive generators of the rays of σ. Here K is the ring obtained from the Grothendieck ring of varieties over a base field k by inverting $\mathbb{L} = [\mathbb{A}_k^1]$. Similarly, if I is a monomial ideal, then the motivic zeta function of I

$$(11) \qquad \int_{(\mathbb{A}^n)_\infty} \mathbb{L}^{-s \cdot \mathrm{ord}_t I}$$

can be written as a sum of fractions with numerator in $K[\mathbb{L}^{-s}]$ and denominator of the form

$$\prod_{i=1}^{r} \left(1 - \mathbb{L}^{-(a_{E_i}(I)s + k_{E_i} + 1)} \right),$$

where $r \le n$ and $E_1, \ldots, E_r$ are divisors on the normalized blowing-up of $\mathbb{A}^n$ along I such that $E_1 \cap \ldots \cap E_r$ is nonempty.

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