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DECOMPOSING REPRESENTATIONS OF FINITE GROUPS ON RIEMANN-ROCH SPACES

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Abstract. If G is a finite subgroup of the automorphism group of a projective curve X and D is a divisor on X stabilized by G, then we compute a simplified formula for the trace of the natural representation of G on the Riemann-Roch space $L(D)$, under the assumption that $L(D)$ is "rational", D is nonspecial, and the characteristic is "good". We discuss the partial formulas that result if $L(D)$ is not rational.

1. Introduction

Let X be a smooth projective (irreducible) curve over an algebraically closed field k of characteristic $p \geq 0$ and let G be a finite group of automorphisms of X over k. We assume throughout this paper that either $p = 0$ or p does not divide the order of the group G. If D is a divisor of X that is stable under the action of G, then G acts on the Riemann-Roch space $L(D)$. We are interested in decomposing this representation into irreducible components.

This question was originally addressed by Hurwitz, in the case where D was the canonical divisor and G was cyclic, over $k = \mathbb{C}$. Chevalley and Weil expanded this result to any finite G [CW]. Since then further work has been done by Ellingsrud and Lønsted [EL], Kani [Ka], Nakajima [N], Köck [Ko], and Borne [B]. In the case where D is a nonspecial divisor, an equivariant Riemann-Roch formula has been given for the character of $L(D)$ (see for example [B]), giving the desired decomposition.

In this paper, we use a more concrete approach to the problem and end up with a simpler and more explicit formula for the decomposition of $L(D)$, but under slightly stronger hypotheses. First of all, we work only over "good" characteristic (as described above), while Borne's formula holds more generally. Secondly, our formula gives the complete decomposition of $L(D)$ when it is "rational," and partial information otherwise. For many groups, such as the symmetric groups S_n, or any other group all of whose characters have values in $\mathbb{Q}$, the rationality condition is automatic. For other groups the rationality of $L(D)$ may depend on D. We give several examples to illustrate different types of rationality behavior.

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Our approach is to consider quotients of X by cyclic subgroups of G and compare Riemann-Roch spaces of divisors on these quotient curves. Recall that the characters of irreducible representations over $\mathbb{C}$ form an orthonormal basis for the space of functions on the set of conjugacy classes of elements of G. It is well known that, in a similar way, the characters of irreducible representations over $\mathbb{Q}$ form an orthonormal basis for the space of functions on the set of conjugacy classes of cyclic subgroups of G ([Se], §13.1). Because we are using cyclic subgroups as our tool, they give us the coarser information of a decomposition into irreducible $\mathbb{Q}[G]$-modules rather than irreducible $\mathbb{C}[G]$-modules. If $L(D)$ does not have a $\mathbb{Q}[G]$-module structure, a naïve application of our formula will give the correct multiplicity for any character with rational values, and the average multiplicity over a Galois orbit for the others.

One motivation for seeking such a decomposition formula comes from coding theory. The construction of algebraic geometry codes uses the Riemann-Roch space $L(D)$ of a divisor on a curve defined over a finite field. Automorphisms of $L(D)$ may provide more efficient encoding and storage of information, for some algebraic geometry codes. See [JT] for more background on algebraic geometry codes and automorphisms of Riemann-Roch spaces.

In Section 2, we apply the method described above in the case where D is the pullback of a divisor on X/G, and obtain our main results: a complete decomposition of $L(D)$ if $L(D)$ has a $\mathbb{Q}[G]$-module structure, and its "Galois average" if not. In Section 3, we extend to the case that D is not necessarily a pullback. In this case we use the previously established equivariant Riemann-Roch formula (see for example [B]) which expresses $L(D)$ in terms of the equivariant degree of D and the ramification module of the cover, which does not depend on D. Our results from Section 2 then give us a simple formula for the ramification module, when it has a $\mathbb{Q}[G]$-module structure, or its "Galois average" when it does not. This simple formula for the multiplicity of a $\mathbb{Q}[G]$-module in the ramification module has also been obtained by Köck [Ko] using other methods. In Section 4, we give some examples.

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2. Decomposition of $L(D)$ when D is a pullback

We start with some definitions and notation.

Let X be a smooth projective irreducible curve over an algebraically closed field k of characteristic $p \geq 0$ and let G be a finite group of automorphisms of X over k. We assume that either $p = 0$ or p does not divide the order of the group G. This assumption will guarantee both that the cover $\pi : X \to Y = X/G$ is tamely ramified and that all $k[G]$-modules are semisimple. For any point $P \in X(k)$, let G_P be the decomposition group at P (i.e., the subgroup of G fixing P); since $\pi : X \to Y = X/G$ is tamely ramified, this group G_P will be cyclic.

Let $\langle G \rangle$ denote the set of conjugacy classes of cyclic subgroups of G. For each class in $\langle G \rangle$ we choose a representative cyclic subgroup H_ℓ, $\ell = 1, \ldots, M$, and partially order them so that $|H_i| \leq |H_j|$ if $i \leq j$. For each branch point of the cover $\pi : X \to Y$, the inertia groups at the ramification points above that branch point will be cyclic and conjugate to each other. For each $\ell > 1$, let R_ℓ denote the number of branch points in Y above which the inertia groups are conjugate to H_ℓ. (R_1 may be set to 0; it does not play a role in the formula.)

Let $G^*_{\mathbb{Q}}$ denote the set of isomorphism classes of irreducible $\mathbb{Q}[G]$-modules. By results in ([Se], §13.1, §12.4), this set has the same number of elements, M, as $\langle G\rangle$. For each class in $G^*_{\mathbb{Q}}$, choose a representative irreducible $\mathbb{Q}[G]$-module V_j, $j = 1, \dots, M$, and denote its character by χ_j. The character table of G over $\mathbb{Q}$ is a square matrix with rows labelled by $G^*_{\mathbb{Q}}$ and columns labelled by $\langle G\rangle$. The rows are linearly independent (as $\mathbb{Q}$-class functions), so in fact the character table is an invertible matrix.

Let μ be the least common multiple of the orders of the elements of G, and let F be a finite abelian extension of $\mathbb{Q}$ containing the μth roots of unity. Then every irreducible $F[G]$-module is absolutely irreducible (irreducible over $\mathbb{C}$), so that the character table of G over F is the same as the character table for G over $\mathbb{C}$ ([Se], p. 94). Let V_j be an irreducible $\mathbb{Q}[G]$-module; then $V_j \otimes_{\mathbb{Q}[G]} F[G]$ decomposes into irreducible $F[G]$-modules. The Galois group of F over $\mathbb{Q}$ permutes the components transitively, so each must have the same multiplicity (the Schur index of the representation V_j) and the same dimension. We write

$$V_j \otimes_{\mathbb{Q}[G]} F[G] \simeq m_j \bigoplus_{r=1}^{d_j} W_{jr}, \tag{1}$$

where m_j is the Schur index and the W_{jr}'s are irreducible $F[G]$-modules. Note that $\dim_{\mathbb{Q}} V_j = \dim_F(V_j \otimes_{\mathbb{Q}[G]} F[G]) = m_j d_j \dim_F W_{jr}$ for all r. Let χ_{jr} denote the character of W_{jr}.

Now we will use the technique described in the introduction to prove the following theorem. The proof is similar to the proof of Theorem 2.3 in [Ks]. Recall that a divisor D is *nonspecial* if $L(K - D) = \{0\}$.

Theorem 1. *Let $D = \pi^*(D_0)$ be a nonspecial divisor on X which is a pullback of a divisor D_0 on $Y = X/G$, and assume that the (Brauer) character of $L(D)$ is the character of a $\mathbb{Q}[G]$-module $L(D)_{\mathbb{Q}}$. Then for each irreducible $\mathbb{Q}[G]$-module V_j, its multiplicity in $L(D)_{\mathbb{Q}}$ is given by*

$$n_j = \frac{1}{m_j^2 d_j}\left(\dim(V_j)(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^{M}(\dim(V_j) - \dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right). \tag{2}$$

Proof. We consider the quotients X/H_ℓ of X by cyclic subgroups H_ℓ. The morphism $\pi : X \to Y$ factors through these quotients, so on each X/H_ℓ there is a pull-back divisor D_ℓ of D_0. Let π_ℓ denote the covering $\pi_\ell : X \to X/H_\ell$.

First, note that our assumption that D is nonspecial means that for any quotient X/H_ℓ, the pullback D_ℓ of D_0 to X/H_ℓ is also nonspecial. This is because

$$K_X - D = \pi_\ell^*(K_{X/H_\ell}) + \mathrm{Ram}(X/H_\ell) - \pi_\ell^*(D_\ell) = \pi_\ell^*(K_{X/H_\ell} - D_\ell) + \mathrm{Ram}(X/H_\ell),$$

where $\mathrm{Ram}(X/H_\ell)$ is the ramification divisor of the covering π_ℓ. Any nontrivial element of $L(K_{X/H_\ell} - D_\ell)$ would pull back to X to give a nontrivial element of $L(K_X - D - \mathrm{Ram}(X/H_\ell))$. Since $\mathrm{Ram}(X/H_\ell)$ is effective, this would also give a nontrivial element of $L(K_X - D)$, contradicting our assumption that D is nonspecial.

Now we decompose $L(D)_{\mathbb{Q}}$ as

$$L(D)_{\mathbb{Q}} \simeq \bigoplus_{j=1}^{M} n_j V_j. \tag{3}$$

For each H_ℓ in $\langle G\rangle$, consider the dimension of the piece of this module fixed by H_ℓ. Since the elements of $L(D)$ fixed by H_ℓ are exactly the elements of $L(D_\ell)$, $\dim_{\mathbb{Q}} L(D)_{\mathbb{Q}}^{H_\ell} = \dim_k L(D)^{H_\ell} = \dim_k L(D_\ell)$ and we get an equation for each ℓ:

$$\dim_k L(D_\ell) = \sum_{j=1}^{M} n_j \dim_{\mathbb{Q}}(V_j^{H_\ell}), \qquad 1 \le \ell \le M. \tag{4}$$

This gives us a system of M equations in the M unknowns n_j. We need to show that the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is invertible, so this system has a unique solution, and that the above equation is the claimed solution.

First let us consider the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$. Each matrix entry is equal to the multiplicity of the trivial representation $\mathbf{1}$ of H_ℓ in the restricted representation of H_ℓ on V_j. This is the inner product of characters $\langle \mathrm{Res}_{H_\ell}^G \chi_j, \mathbf{1}\rangle$, which is defined as

$$\dim V_j^{H_\ell} = \frac{1}{|H_\ell|} \sum_{a\in H_\ell} \chi_j(a). \tag{5}$$

Thus each column of the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is a sum of columns of the character table of G over $\mathbb{Q}$. Each element a in H_ℓ generates either all of H_ℓ or a cyclic subgroup of lower order, hence earlier in the list $\langle G\rangle$. Thus if we write our matrix in terms of the basis of columns of this character table, we get a lower triangular matrix. Since at least one element of H_ℓ generates all of H_ℓ, this lower triangular matrix has nonzero entries on the diagonal. This implies that our matrix is also invertible.

Now it remains to verify that our equation is the correct solution to (4).

Note that

$$\dim L(D_\ell) = \frac{|G|}{|H_\ell|}\deg(D_0) + 1 - g(X/H_\ell), \tag{6}$$

for $1 \le \ell \le M$, by the Riemann-Roch theorem and the hypothesis that D_ℓ is nonspecial.

We will now substitute (2) into (4) and verify that the result agrees with (6), for each $1 \le \ell \le M$. The argument is similar to that in [Ks].

Plugging (2) into (4) gives

$$\begin{aligned}\sum_{j=1}^{M} n_j \dim(V_j^{H_\ell}) &= (\deg(D_0) + 1 - g_Y)\sum_{j=1}^{M} \frac{1}{m_j^2 d_j}\dim(V_j^{H_\ell})\dim(V_j)\\ &\quad - \sum_{i=1}^{M}\left(\sum_{j=1}^{M}\frac{1}{m_j^2 d_j}[\dim(V_j^{H_\ell})\dim(V_j) - \dim(V_j^{H_\ell})\dim(V_j^{H_i})]\frac{R_i}{2}\right).\end{aligned}$$

Note that

$$\dim(V_j^{H_\ell}) = \langle \mathrm{Res}_{H_\ell}^G \chi_j, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \chi_{jr}, \mathrm{Ind}_{H_\ell}^G \mathbf{1}\rangle, \tag{7}$$

using (1) and Frobenius reciprocity. This gives us

$$\begin{aligned}\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim V_j^{H_\ell} \dim V_j &= \sum_{j=1}^{M} \sum_{r=1}^{d_j} \dim W_{jr} \langle \operatorname{Res}_{H_\ell}^{G} \chi_{jr}, \mathbf{1} \rangle \\ &= \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e) \chi_{jr}(a).\end{aligned} \tag{8}$$

The last part of this is the sum over all irreducible F-characters of G, so the last expression is in fact the inner product of two columns of the character table for G over F. This inner product will be zero unless $a = e$, so the sum becomes

$$\frac{1}{|H_\ell|} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e)^2 = \frac{|G|}{|H_\ell|}. \tag{9}$$

We would like to do a similar simplification of

$$\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j^{H_i}) \tag{10}$$

using (7) twice. The induced representation $\operatorname{Ind}_{H_i}^{G} \mathbf{1}$ corresponds to the permutation action of G on the cosets of H_i, and thus has a $\mathbb{Q}[G]$-module structure as well as an $F[G]$-module structure. It can be decomposed into irreducible $F[G]$-modules, such that for each j the multiplicities $\langle \chi_{jr}, \operatorname{Ind}_{H_i}^{G} \mathbf{1} \rangle$ of W_{jr} do not depend on r. Using that fact, Frobenius reciprocity, and the definition of the Schur inner product, we have

$$\begin{aligned}&\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j^{H_i}) \\ &\qquad= \sum_{j=1}^{M} \frac{1}{d_j} \sum_{r=1}^{d_j} \langle \operatorname{Res}_{H_\ell}^{G} \chi_{jr}, \mathbf{1} \rangle \sum_{s=1}^{d_j} \langle \chi_{js}, \operatorname{Ind}_{H_i}^{G} \mathbf{1} \rangle \\ &\qquad= \sum_{j=1}^{M} \sum_{r=1}^{d_j} \langle \operatorname{Res}_{H_\ell}^{G} \chi_{jr}, \mathbf{1} \rangle \langle \chi_{jr}, \operatorname{Ind}_{H_i}^{G} \mathbf{1} \rangle \\ &\qquad= \sum_{j=1}^{M} \sum_{r=1}^{d_j} \langle \operatorname{Res}_{H_\ell}^{G} \chi_{jr}, \mathbf{1} \rangle \langle \operatorname{Res}_{H_i}^{G} \chi_{jr}, \mathbf{1} \rangle \\ &\qquad= \frac{1}{|H_\ell|} \frac{1}{|H_i|} \sum_{a \in H_\ell} \sum_{b \in H_i} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(a) \chi_{jr}(b).\end{aligned} \tag{11}$$

Again, this last sum is an inner product of columns of the character table of G over k, so will be zero unless a and b are in the same conjugacy class. Let $\mathcal{C}_G(a)$ denote the conjugacy class of a in G. We end up with

$$\begin{aligned}\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j^{H_i}) &= \frac{1}{|H_\ell||H_i|} \sum_{a \in H_\ell} \#(H_i \cap \mathcal{C}_G(a)) \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(a)^2 \\ &= |H_\ell \backslash G / H_i|,\end{aligned} \tag{12}$$

the number of double cosets. Therefore, we get

$$\begin{aligned}\sum_{j=1}^{M} n_j \dim V_j^{H_\ell} &= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} - \sum_{i=1}^{M}(\frac{|G|}{|H_\ell|} - |H_i\backslash G/H_\ell|)\frac{R_i}{2} \\ &= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} + 1 + \frac{|G|}{|H_\ell|}(g_Y-1) - g_{X/H_\ell} \\ &= \deg(D_0)\frac{|G|}{|H_\ell|} + 1 - g_{X/H_\ell},\end{aligned}$$

where the last equalities come from applying the Hurwitz formula to the cover $X/H_\ell \to Y$ (see [Ks] for details). This is (6), as desired. □

Theorem 2. *Let $D = \pi^*(D_0)$ be a nonspecial divisor on X which is a pullback of a divisor D_0 on $Y = X/G$ and assume that the (Brauer) character of $L(D)$ is the character of a $\mathbb{Q}[G]$-module. Then for each absolutely irreducible character of G, the multiplicity of the corresponding module W in $L(D)$ is given by*

$$n = \dim(W)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(W) - \dim(W^{H_\ell}))\frac{R_\ell}{2}. \tag{13}$$

Proof. We use the decomposition (1) to compute the multiplicity of each W_{jr} in $L(D)_{\mathbb{Q}} \otimes F$. By our definition of F, each absolutely irreducible character is the character of one of the W_{jr}'s, and the character of $L(D)$ is the same as the character of $L(D)_{\mathbb{Q}} \otimes F$, so this will give us the correct answer.

The multiplicity of W_{jr} in V_j is m_j, and $\dim V_j = m_j d_j \dim W_{jr}$. Equation (7) and the fact that $\operatorname{Ind}_{H_\ell}^G \mathbf{1}$ has a $\mathbb{Q}[G]$-module structure imply that $\dim W_{jr}^{H_\ell}$ is the same for each r, so $\dim V_j^{H_\ell} = m_j d_j \dim W_{jr}^{H_\ell}$. Thus we can factor $m_j d_j$ out from the inside of (2) and multiply the whole thing by m_j to get formula (13). □

Remark 1. If $L(D)$ does not have a $\mathbb{Q}[G]$-module structure, formula (13) instead computes an average of multiplicities. If the character of $L(D)$ is not the character of a $\mathbb{Q}[G]$-module, it will still be the character of an F-module $L(D)_F$, and $L(D)_F$ will decompose into irreducibles W_{jr}. However in this case for a given j, the multiplicities of the W_{jr}'s may not be all the same. We can replace (3) with

$$L(D)_F \simeq \bigoplus_{j=1}^{M}\bigoplus_{r=1}^{d_j} n_{jr} W_{jr}$$

and the system of equations (4) with

$$\dim L(D_\ell) = \sum_{j=1}^{M}\sum_{r=1}^{d_j} n_{jr}\dim(W_{jr}^{H_\ell}) = \sum_{j=1}^{M}\frac{\sum_{r=1}^{d_j} n_{jr}}{m_j d_j}\dim(V_j^{H_\ell}), \qquad 1 \le \ell \le M.$$

This is now the same system of equations, but with our old unknowns replaced by $\frac{1}{m_j d_j}\sum_{r=1}^{d_j} n_{jr}$. We get the same solution,

$$\frac{1}{m_j d_j}\sum_{r=1}^{d_j} n_{jr} = \frac{1}{m_j^2 d_j}\left(\dim(V_j)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(V_j) - \dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right).$$

Recall from the proof of Theorem 2 that $\dim W_{jr}^{H_\ell}$ does not depend on r, so this can be simplified to

$$\frac{1}{d_j}\sum_{r=1}^{d_j} n_{jr} = \dim(W_{jr})(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(W_{jr}) - \dim(W_{jr}^{H_\ell}))\frac{R_\ell}{2}. \tag{14}$$

We see that our formula has computed the average of the multiplicities over a Galois orbit.

3. The ramification module

Now we wish to extend our results to the case where D is not necessarily the pullback of a divisor on $Y = X/G$. For this we need to build on work previously done on this problem by Nakajima, Borne, Ellingsrud and Lønsted, Köck, Kani, and others. We refer to [B] for references. We start with two definitions: the ramification module of the cover $X \to X/G$ and the equivariant degree of a divisor.

For any point $P \in X(k)$, the inertia group G_P acts on the cotangent space of $X(k)$ at P by a k-character ψ_P. This character is the *ramification character* of X at P. Because the ramification is tame, ψ_P will be a primitive character of the cyclic group G_P. The ramification module is defined by

$$\Gamma_G = \sum_{P\in X(k)_{ram}} \mathrm{Ind}_{G_P}^G(\sum_{\ell=1}^{e_P-1} \ell\psi_P^\ell),$$

where $e_P = |G_P|$. It was shown by Kani [Ka] and Nakajima [N] that there is a unique G-module $\tilde\Gamma_G$ such that

$$\Gamma_G = |G|\tilde\Gamma_G.$$

In this paper we are only concerned with $\tilde\Gamma_G$, so we abuse terminology and call $\tilde\Gamma_G$ the *ramification module*.

Now consider a G-invariant divisor D on $X(k)$. If $D = \frac{1}{e_P}\sum_{g\in G} g(P)$, then we call D a *reduced orbit*. The reduced orbits generate the group of G-invariant divisors $\mathrm{Div}(X)^G$.

Definition 3. The *equivariant degree* is a map from $\mathrm{Div}(X)^G$ to the Grothendieck group $R_k(G) = \mathbb{Z}[G_k^*]$ of virtual k-characters of G,

$$\deg_{eq} : \mathrm{Div}(X)^G \to R_k(G),$$

defined by the following conditions:

(1) $\deg_{eq}$ is additive on G-invariant divisors of disjoint support;
(2) if $D = r\frac{1}{e_P}\sum_{g\in G} g(P)$ is a multiple of a reduced orbit, then

$$\deg_{eq}(D) = \begin{cases} \mathrm{Ind}_{G_P}^G(\sum_{\ell=1}^r \psi_P^{-\ell}), & \text{if } r>0,\\ -\mathrm{Ind}{G_P}^G(\sum_{\ell=0}^{-(r+1)} \psi_P^{\ell}), & \text{if } r<0,\\ 0, & r=0,\end{cases}$$

where ψ_P is the ramification character of X at P.

The following is a minor variant of Corollary 4.19 in [B].

Lemma 4 (Equivariant Riemann-Roch formula)**.** *If D is a G-invariant nonspecial divisor, then the (virtual) character of $L(D)$ is given by*

$$\chi(L(D)) = (1-g_Y)\chi(k[G]) + \deg_{eq}(D) - \chi(\tilde\Gamma_G). \tag{15}$$

We derive the following from the equivariant Riemann-Roch formula and Theorem 2. The notation is as in Section 1.

Proposition 5. *If $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure, then it decomposes into irreducible $\mathbb{Q}[G]$-modules as*

$$\tilde{\Gamma}_G \simeq \bigoplus_j \frac{1}{m_j^2 d_j}\Big(\sum_\ell (\dim(V_j) - \dim(V_j^{H_\ell}))\frac{R_\ell}{2}\Big)V_j.$$

Proof. The ramification module does not depend on the divisor, so we compare Theorem 2 with the equivariant Riemann-Roch formula in the case where D is a pull-back. If $D = \pi^*(D_0)$ is the pull-back of a divisor $D_0 \in \mathrm{Div}(Y)$, then the equivariant degree $\deg_{eq}(D)$ has a very simple form. On each orbit, r is a multiple of e_P, so every character of the cyclic group G_P appears in the sum being induced in the definition of $\deg_{eq}(D)$. Thus, the equivariant degree on this orbit is induced from a multiple of the regular representation of G_P. The equivariant degree of the whole divisor is then

$$\deg_{eq}(D) = \deg(D_0)\chi(k[G]). \tag{16}$$

(This is also a special case of Corollary 4.9 in [B].)

The first two terms of the equivariant Riemann-Roch formula then become

$$(\deg D_0 + 1 - g_Y)\chi(k[G]).$$

This is clearly the character of a $\mathbb{Q}[G]$-module, so $L(D)$ will have a $\mathbb{Q}[G]$-module structure if and only if $\tilde{\Gamma}_G$ does. The rest of the proposition follows from Theorem 2. □

Proposition 5 has also been proven by Köck [Ko], using a different method.

Corollary 6. *Suppose that $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure. Let W be an irreducible $F[G]$-module. Then the multiplicity of the character of W in $\tilde{\Gamma}_G$ is*

$$\sum_\ell (\dim(W) - \dim(W^{H_\ell}))\frac{R_\ell}{2}. \tag{17}$$

Proof. The same as the proof of Theorem 2 from Theorem 1. □

Remark 2. Again, the rationality criterion is necessary. If $\tilde{\Gamma}_G$ does not have a $\mathbb{Q}[G]$-module structure, we get an average of multiplicities, similar to (14):

$$\frac{1}{d_j}\sum_{r=1}^{d_j}\langle \chi_{jr}, \tilde{\Gamma}_G\rangle = \sum_{\ell=1}^{M}(\dim(W_{jr}) - \dim(W_{jr}^{H_\ell}))\frac{R_\ell}{2} \tag{18}$$

with notation as in (1).

4. Examples

Example 1. Let k be an algebraically closed field whose characteristic is not 2 or 3. Consider the nonsingular projective curve X which is the closure of

$$\{(x,y,t) \in k^3 \mid y^2 = x(x-2)(x-4),\ t^2 = x+4\}.$$

This has an action of $G = C_2 \times C_2$ given by

$$\begin{aligned} \alpha &: (x,y,t) \longmapsto (x,-y,t), \\ \beta &: (x,y,t) \longmapsto (x,y,-t), \\ \alpha\beta &: (x,y,t) \longmapsto (x,-y,-t). \end{aligned}$$

The cover $X \to X/\langle\beta\rangle$ is a degree-two cover of an elliptic curve, ramified at the two points with $x = -4$, so X has genus 2. The quotient $Y = X/G$ is the projective x-line.

The divisor

$$D = (0, 0, 2) + (0, 0, -2) + (-4, 8\sqrt{3}, 0) + (-4, -8\sqrt{3}, 0)$$

is G-invariant, and $2D$ is the pullback of the divisor $D_0 = \{x = 0, x = -4\}$ on Y. The Riemann-Roch theorem implies that $\dim L(2D) = 7$.

First, let us use Theorem 2 to decompose $L(2D)$ into irreducibles. The cyclic subgroups of G are the trivial group H_1 and each of the two-element subgroups generated by α, β, and $\alpha\beta$. Let us call the last three H_α, H_β, and $H_{\alpha\beta}$. Each is in its own conjugacy class.

The cover $X \to X/G$ has 5 branch points: three with inertia group H_α (at $x = 0, 2, 4$), one with inertia group H_β (at $x = -4$), and one with inertia group $H_{\alpha\beta}$ (at $x = \infty$). This means that

$$R_\alpha = 3, \quad R_\beta = 1, \quad R_{\alpha\beta} = 1.$$

The group G has character table

	1	α	β	$\alpha\beta$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Each irreducible representation is one dimensional, and every $\mathbb{C}[G]$-module is a $\mathbb{Q}[G]$-module, so d_j and the Schur index m_j are both 1. The dimension $\dim(V_j^{H_\ell})$ is 1 if the character of V_j is 1 on the generator and 0 otherwise. Consequently, we get:

$$\begin{aligned}
n_1 &= (2 + 1 - 0) - 0 = 3,\\
n_2 &= (2 + 1 - 0) - \frac{1}{2}(R_\beta + R_{\alpha\beta}) = 3 - 1 = 2,\\
n_3 &= (2 + 1 - 0) - \frac{1}{2}(R_\alpha + R_{\alpha\beta}) = 3 - 2 = 1,\\
n_4 &= (2 + 1 - 0) - \frac{1}{2}(R_\alpha + R_\beta) = 3 - 2 = 1.
\end{aligned}$$

Thus the character of $L(2D)$ is $3\chi_1 + 2\chi_2 + \chi_3 + \chi_4$.

Now let us consider $L(D)$. The Riemann-Roch theorem tells that this will be a three-dimensional space. Since D is not a pullback from Y, we cannot use Theorem 2. However, the ramification module does have a $\mathbb{Q}[G]$-module structure, so we can use Proposition 5 with the equivariant Riemann-Roch formula. The calculations above tell us that the ramification module has character $\chi_2 + 2\chi_3 + 2\chi_4$.

Now we need to calculate the equivariant degree of D. The divisor consists of two reduced orbits, the orbit of $(0, 0, 2)$ and the orbit of $(-4, 8\sqrt{3}, 0)$. At the first point the inertia group is H_α, and at the second point the inertia group is H_β. In both cases the ramification character is the nontrivial character of C_2. Adding the induced characters of G gives us $\deg_{eq}(D) = \chi_2 + \chi_3 + 2\chi_4$.

Adding the pieces of the equivariant Riemann-Roch formula, we get the character of $L(D)$ to be $\chi_1 + \chi_2 + \chi_4$. In fact, one can check that the three functions $\{1, \frac{1}{t}, \frac{y}{xt}\}$

form a basis for $L(D)$, and G acts on the three basis elements by the three respective characters. $\square$

Example 2. Spectral curves. The curve in Example 1 is one of the family of spectral curves of the integrable system used to solve $\mathcal{N} = 2$ Seiberg-Witten theory in the case of the gauge group $SU(2)$ with matter in the adjoint representation [D1]. Many similar examples can be constructed using the spectral curves of other integrable systems. In general, the *cameral curves* (Galois closures of spectral curves) of an integrable system with a Lie algebra structure will have an automorphism group which is the Weyl group of the Lie algebra [D1, D2]. It is known that for these groups, which include the symmetric groups S_n, any representation has a rational structure [[BZ], section 3.4]. This gives a large class of curves for which the rationality criterion is automatic, and therefore Theorem 2 gives the correct multiplicity for all irreducible representations.

Example 3. Here is an elementary example where the group has representations which are not rational, but the ramification module is rational. Let $X = \mathbb{P}^1$ and let G be a cyclic group of prime order q. Let a be a generator of G, and let a act on X by $z \longmapsto \zeta z$, where ζ is a primitive qth root of unity. The cyclic subgroups of G are the trivial group and G itself; the irreducible representations of G over $\mathbb{Q}$ are the one-dimensional trivial representation and a $(q-1)$-dimensional representation V. Let ψ be the character of G over $\mathbb{C}$ whose value on a is ζ; then the irreducible characters of G over $\mathbb{C}$ are the tensor powers $\psi, \psi^2, \dots, \psi^{q-1}, \psi^q = \mathbf{1}$. The character of V is $\psi + \psi^2 + \cdots + \psi^{q-1}$.

The cover $X \to X/G$ is totally ramified at 0 and ∞. The ramification module in this case is a $\mathbb{Q}[G]$-module, so we can use either Proposition 5 or Corollary 6 to find that

$$\tilde{\Gamma}_G = \psi + \psi^2 + \cdots + \psi^{q-1} = V.$$

Example 4. Here is an example that illustrates what can happen when the rationality condition is not met.

Let X be the Klein quartic

$$\{\ (x, y, z) \in \mathbb{P}^2 \mid x^3y + y^3z + z^3x = 0\ \}.$$

We assume that k contains both cube roots of unity and 7^{th} roots of unity; let ω be a primitive cube root of unity and ζ be a primitive seventh root of unity. Let G be the group generated by

$$\begin{aligned} \sigma : (x : y : z) &\longmapsto (y : z : x), \\ \tau : (x : y : z) &\longmapsto (\zeta x : \zeta^4 y : \zeta^2 z). \end{aligned}$$

The group G of automorphisms generated by these two actions is the semidirect product $C_7 \rtimes C_3$. (This is not the full automorphism group of this curve; see example 5 below.) X has genus 2, and the quotient $Y = X/G$ has genus 0 [E].

The group G has character table:[1]

	e	σ	τ	σ^{-1}	τ^{-1}
χ_1	1	1	1	1	1
χ_2	1	ω^2	1	ω	1
χ_3	1	ω	1	ω^2	1
χ_4	3	0	$\zeta^3+\zeta^5+\zeta^6$	0	$\zeta+\zeta^2+\zeta^4$
χ_5	3	0	$\zeta+\zeta^2+\zeta^4$	0	$\zeta^3+\zeta^5+\zeta^6$

There are two conjugacy classes of nontrivial cyclic subgroups, with representatives generated by σ and τ. Let $H_3 = \langle\sigma\rangle$ and $H_7 = \langle\tau\rangle$. The irreducible representations over $\mathbb{Q}$ have characters χ_1, $\chi_2+\chi_3$, and $\chi_4+\chi_5$. Each has Schur index 1.

The points of X fixed by H_7 are $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, and $P_3 = (0:0:1)$. These form one orbit under G, so $R_7 = 1$. There are seven points in the orbit of $(1:\omega:\omega^2)$ and seven points in the orbit of $(1:\omega^2:\omega)$, all fixed by cyclic groups of order 3. Since these form two orbits, we have $R_3 = 2$.

We now compute

$$\sum_{\ell=1}^{M}(\dim(W)-\dim(W^{H_\ell}))\frac{R_\ell}{2}, \tag{19}$$

as in (18), for the irreducible representations over $\mathbb{C}$. We find that

$$\begin{aligned}\frac{1}{2}\langle\chi_2+\chi_3,\tilde{\Gamma}_G\rangle &= 1,\\ \frac{1}{2}\langle\chi_4+\chi_5,\tilde{\Gamma}_G\rangle &= \frac{7}{2}.\end{aligned}$$

These give the average multiplicities. In fact one can compute directly that $\tilde{\Gamma}_G = \chi_2+\chi_3+3\chi_4+4\chi_5$.

Example 5. The Klein quartic in Example 4 is also known as the modular curve $X(7)$, whose full automorphism group is $PSL_2(GF(7))$. In general, the ramification module for the modular curve $X(N)$ will be rational if and only if $N \equiv 1 \pmod 4$. The ramification modules for $X(N)$ have been computed for all N in [JK].

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