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A CRITERION FOR THE LOGARITHMIC DIFFERENTIAL OPERATORS TO BE GENERATED BY VECTOR FIELDS

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ABSTRACT. We study divisors in a complex manifold in view of the property that the algebra of logarithmic differential operators along the divisor is generated by logarithmic vector fields. We give

- a sufficient criterion for the property,
- a simple proof of F.J. Calderón-Moreno's theorem that free divisors have the property,
- a proof that divisors in dimension 3 with only isolated quasi-homogeneous singularities have the property,
- an example of a nonfree divisor with nonisolated singularity having the property,
- an example of a divisor not having the property, and
- an algorithm to compute the V-filtration along a divisor up to a given order.

1. LOGARITHMIC COMPARISON THEOREM FOR FREE DIVISORS

Let X be a complex manifold of dimension $n \geq 2$, $\mathcal{O}$ the ring holomorphic functions on X, and $\Omega^\bullet$ the complex of holomorphic differential forms. Grothendieck's Comparison Theorem states that the De Rham system $\mathcal{O}$ is regular [Meb89, Thm. 2.3.4]. This is equivalent to the fact that, for any divisor $D \subset X$, the natural morphism

$$\Omega^\bullet(*D) = \mathrm{DR}(\mathcal{O}(*D)) \longrightarrow \mathrm{R}\, i_* i^{-1}\, \mathrm{DR}(\mathcal{O}) = \mathrm{R}\, i_* \mathbb{C}_U,$$

where i is the inclusion $U = X \backslash D \subset X$, a quasi-isomorphism. Let $\Omega^\bullet(\log D) \subset \Omega^\bullet(*D)$ be the subcomplex of logarithmic differential forms along D [Sai80, Def. 1.2]. The above statement raises the question whether the inclusion $\Omega^\bullet(\log D) \subset \Omega^\bullet(*D)$ is also a quasi-isomorphism. That is: Can one compute the cohomology of the complement of D by logarithmic differential forms along D? This turns out to be a property of D called the logarithmic comparison theorem or simply LCT. It is an open problem to characterize the divisors for which LCT holds.

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Let $\Theta = \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ be the $\mathcal{O}$-module of holomorphic vector fields on X and $\mathrm{Der}(\log D) \subset \Theta$ be the $\mathcal{O}$-submodule of logarithmic differential operators along D [Sai80, Def. 1.4]. A divisor D is called free if $\mathrm{Der}(\log D)$ is a locally free $\mathcal{O}$-module. Let $\mathcal{D}$ be the $\mathcal{O}$-algebra of differential operators on X with holomorphic coefficients, and let F be the increasing filtration on $\mathcal{D}$ by the order of differential operators. Let $\mathcal{V}^D$ be the V-filtration along D on $\mathcal{D}$ as defined in Section 2 such that $\mathcal{V}^D_0 = \mathcal{D}(\log D)$ is the $\mathcal{O}$-algebra of logarithmic differential operators along D. F.J. Calderón-Moreno [CM99, Thm. 1] proves that, for a free divisor D, $\mathcal{V}^D_0$ is generated by vector fields, that is, $\mathcal{V}^D_0 = \mathcal{O}[\mathrm{Der}(\log D)]$. Let S_D be the decreasing filtration on $\mathcal{D}$ which is locally defined by $S^k_D = f^k \cdot \mathcal{D}$ where $f \in \mathcal{O}$ such that $D = (f)$. By Corollary 3, the induced filtration S_D on $\mathcal{V}^D_0$ defined by $S^k_D\mathcal{V}^D_0 = \mathcal{V}^D_0 \cap (f^k \cdot \mathcal{D})$ reflects the embeddings $\mathcal{V}^D_k \subset \mathcal{D}$. If $\mathcal{V}^D_0 = \mathcal{O}[\mathrm{Der}(\log D)]$, then $(\mathcal{V}^D_0, S_D)$ is a filtered $(\mathcal{V}^D_0, \mathcal{V}^D)$-module.

F.J. Calderón-Moreno and L. Narváez-Macarro [CMNM05, Cor. 4.2] prove that LCT holds for a free divisor D if and only if the complex

$$\mathcal{D} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}(\log D)} \mathcal{O}(D) = \mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$$

is concentrated in degree 0 and the natural multiplication morphism

$$\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathcal{O}(D) \xrightarrow{\epsilon_D} \mathcal{O}(*D)$$

is an isomorphism. Injectivity of ϵ_D is locally equivalent to $\mathrm{Ann}_{\mathcal{D}}(\frac{1}{f})$ being generated by operators of order 1 where $f \in \mathcal{O}$ such that $D = (f)$. For any divisor D, T. Torrelli proves that the latter condition already implies surjectivity of ϵ_D [Tor04, Prop. 1.3] and conjectures that it is even equivalent to LCT [Tor04, Conj. 1.11]. A problem in verifying this conjecture for a free divisor D consists in $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ not being F-strict in general [CM99, Rem. 4.2.4]. So grading by F does not reduce the problem to a commutative one. But both properties of D in question can be characterized in terms of S_D-strictness: On the one hand, exactness of $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ in degree k is equivalent to S_D-strictness of the differential of $\mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$ from degree $k-1$ to degree k. On the other hand, injectivity of ϵ_D is equivalent to S_D-strictness of the last differential of $\mathcal{D} \otimes_{\mathcal{D}(\log D)} \mathrm{Sp}^\bullet_{\mathcal{D}(\log D)}(\mathcal{O}(D))$.

A solution of the LCT problem seems to require a deeper understanding of the V-filtration in general. There are many questions:

- What are the properties of the $\mathcal{V}^D_k$?
- When is $\mathcal{V}^D_0$ generated by vector fields?
- When is $\mathcal{V}^D_0$ locally finitely generated?
- What are the properties of the embeddings $\mathcal{V}^D_k \subset \mathcal{D}$?

We shall approach the first two questions in this article.

2. V-filtration along subvarieties and divisors

Let $Y \subset X$ be a subvariety in X and let $\mathcal{I} \subset \mathcal{O}$ be its ideal. The V-filtration $\mathcal{V}^Y$ along Y is the increasing filtration on $\mathcal{D}$ defined by

$$\mathcal{V}^Y_k = \{P \in \mathcal{D} \mid \forall l \in \mathbb{Z} : P(\mathcal{I}^l) \subset \mathcal{I}^{l-k}\}$$

for all $k \in \mathbb{Z}$. We shall omit the index Y if it is clear from the context. Clearly $\mathcal{V}_k \cdot \mathcal{V}_l \subset \mathcal{V}_{k+l}$ for all $k, l \in \mathbb{Z}$. Hence $\mathcal{V}_0$ is an $\mathcal{O}$-algebra and $\mathcal{V}_k$ is a $\mathcal{V}_0$-module for all $k \in \mathbb{Z}$.

Example 1. Let $x_1, \dots, x_m, y_1, \dots, y_n$ be coordinates on $X = \mathbb{C}^{m+n}$.

(1) For the submanifold $Y = \{y = 0\}$,

$$\mathcal{V}_k^Y = \bigl\{P = \sum_{j_1 - i_1 + \cdots + j_n - i_n \le k} P_{i,j}(x, \partial_x) y_1^{i_1} \partial_{y_1}^{j_1} \cdots y_n^{i_n} \partial_{y_n}^{j_n} \in \mathcal{D}\bigr\}.$$

(2) For the normal crossing divisor $D = (y_1 \cdots y_n)$,

$$\mathcal{V}_k^D = \bigl\{P = \sum_{j_1 - i_1, \dots, j_n - i_n \le k} P_{i,j}(x, \partial_x) y_1^{i_1} \partial_{y_1}^{j_1} \cdots y_n^{i_n} \partial_{y_n}^{j_n} \in \mathcal{D}\bigr\}.$$

Denote the complement of the singularities of Y by

$$U_Y = X \setminus \operatorname{Sing}(Y) \xrightarrow{i_Y} X.$$

We shall omit the index Y if it is clear from the context. The V-filtration along a divisor has a special property.

Proposition 2. *Let $D \subset X$ be a divisor. Then $\mathcal{V}^D = (i_D)_* i_D^{-1} \mathcal{V}^D$.*

Proof. We may assume that $D = (f)$ for some $f \in \mathcal{O}$ by the local nature of the statement. Since $\mathcal{V}_k \subset \mathcal{D}$ and $\mathcal{D}$ is a locally free $\mathcal{O}$-module,

$$i_* i^{-1} \mathcal{V}_k \subset i_* i^{-1} \mathcal{D} = \mathcal{D}.$$

Since $\mathcal{O} \cdot f^{l-k}$ is a free $\mathcal{O}$-module, $P \in i_* i^{-1} \mathcal{V}_k$ implies that

$$P(g \cdot f^l) \in i_* i^{-1}(\mathcal{O} \cdot f^{l-k}) = \mathcal{O} \cdot f^{l-k}$$

for all $g \in \mathcal{O}$ and $l \in \mathbb{Z}$ and hence $P \in \mathcal{V}_k$. □

Corollary 3. *Let $D = (f) \subset X$ with $f \in \mathcal{O}$ a divisor. Then*

$$\mathcal{V}_k = \begin{cases} f^{-k} \mathcal{V}_0, & k \le 0, \\ f^{-k}(\mathcal{V}_0 \cap f^k \mathcal{D}), & k \ge 1. \end{cases}$$

Proof. The equalities in question hold on U_D by Example 1 (2) and hence on X by Proposition 2. □

Denote the symbol map for F by

$$\mathcal{D} \xrightarrow{\sigma} \operatorname{gr}^F \mathcal{D}\ .$$

The decomposition $F_1\mathcal{D} = \mathcal{O} \oplus \Theta$ defines the $\mathcal{O}$-module $\operatorname{Der}(\log Y) \subset \Theta$ of logarithmic vector fields along Y by

$$F_1 \mathcal{V}_0 = \mathcal{O} \oplus \operatorname{Der}(\log Y).$$

This definition simplifies to

$$\operatorname{Der}(\log Y) = \{\theta \in \Theta \mid \theta(\mathcal{I}) \subset \mathcal{I}\}$$

by the Leibniz rule and implies involutivity of $\operatorname{Der}(\log Y)$, that is,

$$[\operatorname{Der}(\log Y), \operatorname{Der}(\log Y)] \subset \operatorname{Der}(\log Y).$$

Example 4. Let $x_1, \dots, x_m, y_1, \dots, y_n$ be coordinates on $X = \mathbb{C}^{m+n}$.

(1) For the submanifold $Y = \{y = 0\}$,

$$\mathrm{Der}(\log D) = \mathcal{O}\langle \partial_{x_1}, \dots, \partial_{x_m}\rangle + \mathcal{O}\langle y_i \partial_{y_j} \mid 1 \le i, j \le n\rangle.$$

(2) For the normal crossing divisor $D = (y_1 \cdots y_n)$,

$$\mathrm{Der}(\log D) = \mathcal{O}\langle \partial_{x_1}, \dots, \partial_{x_m}, y_1\partial_{y_1}, \dots, y_n\partial_{y_n}\rangle.$$

Let $\mathcal{O}[\mathrm{Der}(\log Y)] \subset \mathcal{D}$ be the image of the tensor algebra

$$\mathrm{T}_{\mathbb{C}} \mathrm{Der}(\log Y) \xrightarrow{\gamma_Y} \mathcal{D}.$$

Then at least $\mathcal{O}[\mathrm{Der}(\log Y)] \subset \mathcal{V}_0^Y$.

Corollary 5. *Let $D \subset X$ be a divisor. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ if and only if $\mathcal{O}[\mathrm{Der}(\log D)] = (i_D)_* i_D^{-1} \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. By Examples 1 and 4, $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on U_D. Hence the claim follows from Proposition 2. ☐

A divisor $D \subset X$ is called free if $\mathrm{Der}(\log D)$ is a locally free $\mathcal{O}$-module. By K. Saito [Sai80, Cor. 1.7], $\mathrm{Der}(\log D)$ is reflexive and hence all divisors in dimension $n = 2$ are free. By Example 4 (2), normal crossing divisors are free. In particular, any divisor D is free on U_D.

F.J. Calderón-Moreno [CM99, Thm. 1] proves that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ for a free divisor. We give a simple proof of this result.

Corollary 6. *Let $D \subset X$ be a free divisor. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. By Lemma 7 and grading by F, $\mathcal{O}[\mathrm{Der}(\log D)]$ is a locally free $\mathcal{O}$-module and hence Corollary 5 applies. ☐

Lemma 7. *Let R be a domain and let $P_1, \dots, P_n \in R \cdot T_1 \oplus \cdots \oplus R \cdot T_n = R^n$ be R-linearly independent. Then $R[P_1, \dots, P_n] \subset R[T_1, \dots, T_n]$ is a polynomial ring.*

Proof. Write $P_i = \sum_j p_{i,j} T_j$ with $p_{i,j} \in R$. Then by assumption $p = \det(p_{i,j}) \ne 0$ and hence $R_p[P_1, \dots, P_n]$ is a polynomial ring. Since R is a domain, $R \longrightarrow R_p$ is injective and hence $R[P_1, \dots, P_n]$ is a polynomial ring. ☐

In general it is not clear if, or under which conditions, $\mathcal{V}_0^Y$ is a locally finite $\mathcal{O}$-algebra. Even to compute $F_k \mathcal{V}_0^Y$ is a problem since the definition involves infinitely many conditions. The following result allows one to compute $F_k \mathcal{V}_0^D$ algorithmically.

Proposition 8. *Let $x_1, \dots, x_n$ be coordinates on $X = \mathbb{C}^n$. Let $D = (f) \subset X$ with $f \in \mathcal{O}$ be a divisor. Then, for $P \in F_d\mathcal{D}$, $P \in \mathcal{V}_k^D$ if and only if*

$$\forall \alpha \in \mathbb{N}^n, l \in \mathbb{N} : |\alpha| + l \le d \Rightarrow P(x^\alpha f^l) \in \mathcal{O} \cdot f^{l-k}. \tag{1}$$

Proof. Let $0 \ne P \in F_d\mathcal{D}$ and assume that condition (1) holds. For $l \in \mathbb{N}$, the vector space $\mathbb{C}[x_1, \dots, x_n]_{\le d-l}$ is invariant under $x \mapsto Ax + a$ for $a \in \mathbb{C}^n$ and $A \in \mathrm{GL}_n(\mathbb{C})$. Hence, at a smooth point y of D, condition (1) holds for coordinates $x_1, \dots, x_n$ at y such that $\partial_{x_n}(f)(y) \ne 0$. Then $y_1, \dots, y_{n-1}, t = x_1, \dots, x_{n-1}, f$ are coordinates at y such that

$$\forall \beta \in \mathbb{N}^{n-1}, l \in \mathbb{N} : |\beta| + l \le d \Rightarrow P_y(y^\beta t^l) \in \mathcal{O}_y \cdot t^{l-k}.$$

Write $P_y = \sum_{|\beta|+l\le d} p_{\beta,l}\partial_y^\beta\partial_t^l$ with $p_{\beta,l}\in\mathcal{O}_y$ and choose $\gamma\in\mathbb{N}^{n-1}$ and $m\in\mathbb{N}$ such that $|\gamma|+m$ is minimal with $p_{\gamma,m}\neq 0$. Then

$$\gamma!m!p_{\gamma,m} = P(y^\gamma t^m)\in\mathcal{O}_y\cdot t^{m-k},$$

and hence $p_{\gamma,m}\partial_y^\gamma\partial_t^m\in\mathcal{V}_{k,y}$ by Example 1 (2). By increasing induction on $|\gamma|+m$, this implies that $P_y\in\mathcal{V}_{k,y}$ for all $y\in U_D$ and hence $P\in\mathcal{V}_k$ by Proposition 2. $\square$

Example 9. Let x,y,z be coordinates on $\mathbb{C}^3$ and

$$f = xyz(x+y+z)(x+2y+3z).$$

Then $D=(f)\subset\mathbb{C}^3$ is a central generic hyperplane arrangement. Let

$$Q = (x+y+z)(x+2y+3z)(3zy^2\partial_y^2 + (x+4y-3z)yz\partial_y\partial_z - 4yz^2\partial_z^2).$$

Then $Q\in F_2\mathcal{V}_0^D$ by a Singular [GPS05] computation using Proposition 8. We shall see in Example 13 that $Q\notin F_2\mathcal{O}[\mathrm{Der}(\log D)]$.

There is another special property of the V-filtration along a divisor.

Proposition 10. *Let $D\subset X$ be a divisor. Then* $\mathrm{depth}_x(\mathcal{V}_k^D)\ge 2$ *for all $x\in X$ and $k\in\mathbb{Z}$.*

Proof. Let $x\in X$ and $D_x=(f)$ with $f\in\mathcal{O}_x$. Since $\mathcal{O}_x$ is torsion free and $\mathrm{depth}(\mathcal{O}_x)\ge 2$, there is an $\mathcal{O}_x$-sequence $a_1,a_2\in\mathfrak{m}_x$ such that a_1 is different from all irreducible factors of f. Let $P\in\mathcal{V}_{k,x}$ with $a_2\cdot P\in a_1\cdot\mathcal{V}_{k,x}\subset a_1\cdot\mathcal{D}_x$. Then $P\in a_1\cdot\mathcal{D}_x$ since $\mathcal{D}_x$ is a free $\mathcal{O}_x$-module. But $P(g\cdot f^l)\in\mathcal{O}_x\cdot f^{l-k}$ implies that $(a_1^{-1}\cdot P)(g\cdot f^l)\in\mathcal{O}_x\cdot f^{l-k}$ by the choice of a_1 for all $g\in\mathcal{O}$ and $l\in\mathbb{Z}$, and hence $P\in a_1\cdot\mathcal{V}_{0,x}$. Then $a_1,a_2\in\mathfrak{m}_x$ is a $\mathcal{V}_{k,x}$-sequence, and hence $\mathrm{depth}_x(\mathcal{V}_k)\ge 2$. $\square$

3. Symmetric algebra of logarithmic vector fields

The condition in Corollary 5 is difficult to verify in general. Therefore we focus on a case in which it still holds after grading by F. Consider the commutative diagram of graded algebras

$$\begin{array}{ccc} & \mathrm{T}_{\mathbb{C}}\,\mathrm{Der}(\log Y) & \\ {\scriptstyle\Sigma}\swarrow & & \searrow{\scriptstyle\mathrm{gr}\,\gamma_Y} \\ \mathrm{Sym}_{\mathcal{O}}\,\mathrm{Der}(\log Y) & \xrightarrow{\alpha_Y} & \mathrm{gr}^F\,\mathcal{O}[\mathrm{Der}(\log Y)] \\ {\scriptstyle\pi_Y}\downarrow & & \uparrow \\ \mathrm{Rees}_{\mathcal{O}}\,\mathrm{Der}(\log Y) & \xrightarrow{\cong} & \mathcal{O}[\sigma(\mathrm{Der}(\log Y)]. \end{array}$$

Lemma 11. *If α_Y is an isomorphism, then*

$$\mathrm{Sym}_{\mathcal{O}}\,\mathrm{Der}(\log Y) = (i_Y)_* i_Y^{-1}\,\mathrm{Sym}_{\mathcal{O}}\,\mathrm{Der}(\log Y)$$

implies that $\mathcal{O}[\mathrm{Der}(\log Y)] = (i_Y)_* i_Y^{-1}\mathcal{O}[\mathrm{Der}(\log Y)]$.

Proof. The following diagram is commutative.

$$\begin{array}{ccc}
\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) & \longrightarrow & i_* i^{-1} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y) \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle i_* i^{-1}\alpha_Y} \\
\operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)] & \hookrightarrow & i_* i^{-1} \operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)] \\
& \searrow \quad \nearrow & \\
& \operatorname{gr}^F i_* i^{-1} \mathcal{O}[\operatorname{Der}(\log Y)] &
\end{array}$$

Then the claim follows by induction on $\deg(P)$ for $P \in i_* i^{-1} \mathcal{O}[\operatorname{Der}(\log Y)]$. □

Lemma 12. *α_Y is an isomorphism if and only if π_Y is injective.*

Proof. Assume that π_Y is injective. An element of $\operatorname{gr}^F \mathcal{O}[\operatorname{Der}(\log Y)]$ is of the form $\sigma(\gamma_Y(P))$ where $P \in \mathrm{T}_{\mathbb{C}} \operatorname{Der}(\log Y)$. Write $P = P_0 \oplus \cdots \oplus P_d$ where $d = \deg(P)$. If $\sigma(\gamma_Y(P)) \notin \operatorname{im} \alpha_Y$, then $(\operatorname{gr} \gamma_Y)(P_d) = (\operatorname{gr} \gamma_Y)(P) = 0$ and hence $P_d \in \ker \Sigma$ by injectivity of π_Y. By the definition of $\operatorname{Sym}_{\mathcal{O}}$, this implies that P_d is in the two-sided ideal generated by the relations $\xi \otimes \eta - \eta \otimes \xi$ and $\xi \otimes (a\eta) - (a\xi) \otimes \eta$ where $\xi, \eta \in \operatorname{Der}(\log Y)$ and $a \in \mathcal{O}$. But

$$\gamma_Y(\xi \otimes \eta - \eta \otimes \xi) = \xi\eta - \eta\xi = [\xi, \eta] \in \operatorname{Der}(\log Y)$$

by involutivity of $\operatorname{Der}(\log Y)$, and

$$\gamma_Y(\xi \otimes (a\eta) - (a\xi) \otimes \eta) = \xi a \eta - a \xi \eta = [\xi, a]\eta = \xi(a)\eta \in \operatorname{Der}(\log Y).$$

This means that

$$\begin{aligned}
\xi \otimes \eta - \eta \otimes \xi &\equiv [\xi, \eta] \mod \ker \gamma_Y, \\
\deg(\xi \otimes \eta - \eta \otimes \xi) &> \deg([\xi, \eta]), \\
\xi \otimes (a\eta) - (a\xi) \otimes \eta &\equiv \xi(a)\eta \mod \ker \gamma_Y, \\
\deg(\xi \otimes (a\eta) - (a\xi) \otimes \eta) &> \deg(\xi(a)\eta).
\end{aligned}$$

Hence $\gamma_Y(P) = \gamma_Y(P')$ and $\deg(P) < \deg(P')$ for some $P' \in \mathrm{T}_{\mathbb{C}} \operatorname{Der}(\log Y)$. Then the claim follows by induction on $d = \deg(P)$. □

Example 13. Let D and Q be as in Example 9. Then a Singular [GPS05] computation shows that π_D is injective and that

$$\sigma(Q) \notin \alpha_D\big(\operatorname{Sym}^2_{\mathcal{O}} \operatorname{Der}(\log D)\big).$$

By Lemma 12, this implies that $Q \notin F_2\mathcal{O}[\operatorname{Der}(\log D)]$ and hence, by Example 9, that $\mathcal{O}[\operatorname{Der}(\log D)] \subsetneq \mathcal{V}^D_0$.

By the following general statement, injectivity of π_Y is equivalent to $\mathcal{O}$-torsion freeness of $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log Y)$.

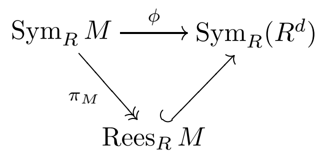
Lemma 14. *Let R be a domain and M a finitely presented torsion free R-module. Then the following are equivalent:*

(1) $\operatorname{Sym}_R M$ *is R-torsion free;*
(2) $\operatorname{Sym}_R M$ *is a domain;*
(3) $\operatorname{Sym}_R M \xrightarrow{\pi_M} \operatorname{Rees}_R M$ *is injective.*

Proof. Assume that $\operatorname{Sym}_R M$ is R-torsion free. Let $K = Q(R)$ be the fraction field of R. Then $M \otimes_R K \cong K^d$ where $d = \operatorname{rk}(M)$. By choosing a basis of K^d and clearing denominators, one can embed $M \subset R^d$. Then

$$\operatorname{Sym}_R(M) \otimes_R K \cong \operatorname{Sym}_{R\otimes_R K}(M \otimes_R K) \cong \operatorname{Sym}_K(K^d)$$

is a domain, and hence $\operatorname{Sym}_R M$ is a domain since R is a domain. Applying Sym_R to the inclusion $M \subset R^d$ yields



and $\ker(\phi) \otimes_R K = 0$ since

$$\operatorname{Sym}_R(M) \otimes_R K \cong \operatorname{Sym}_K(K^d) \cong \operatorname{Sym}_R(R^d) \otimes_R K.$$

A presentation

$$R^m \xrightarrow{(a_{i,j})} R^n \longrightarrow M \longrightarrow 0$$

of M defines an isomorphism

$$\operatorname{Sym}_R M \cong R[T_1, \dots, T_n]/J$$

where $J = \langle \sum_j a_{i,j} T_j \rangle$ is a prime ideal since $\operatorname{Sym}_R M$ is a domain. Since $\operatorname{Sym}_R(R^d)$ is a domain, $\ker \phi$ lifts to a prime ideal $Q \subset R[T_1, \dots, T_n]$. Then $J \subset Q$, $Q \cap R = 0$, and $J \otimes_R K = Q \otimes_R K$ implies $J = Q$ and hence $\ker \pi_M = \ker \phi = 0$. □

Example 15. Let $D_4 \subset \mathbb{C}^4$ be the central generic hyperplane arrangement defined in Section 5. Then one can compute that the coordinates are zero divisors on $\operatorname{Sym}^2_{\mathcal{O}} A_4$ at 0. By Lemmata 12, 14, and 24, this implies that α_{D_4} is not an isomorphism.

A divisor $D \subset X$ is called Euler homogeneous if locally $\chi(f) = f$ for some $\chi \in \operatorname{Der}(\log D)$ and $f \in \mathcal{O}$ such that $D = (f)$. In this case, χ is called an Euler vector field and

$$\operatorname{Der}(\log D) \cong \mathcal{O} \cdot \chi \oplus \operatorname{Ann}_\Theta(f).$$

If $\operatorname{Der}(\log D) \cong \mathcal{O} \cdot \chi \oplus A$, then $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) \cong \operatorname{Sym}_{\mathcal{O}}(A)[\chi]$. For an Euler homogeneous divisor D, this implies that

$$\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) \cong \operatorname{Sym}_{\mathcal{O}}(\operatorname{Ann}_\Theta(f))[\chi].$$

Proposition 16. *Let $D \subset X$ be a divisor such that $\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D)$ is $\mathcal{O}$-torsion free. Then $\mathcal{V}_0^D = \mathcal{O}[\operatorname{Der}(\log D)]$ follows from*

$$\operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D) = (i_D)_* i_D^{-1} \operatorname{Sym}_{\mathcal{O}} \operatorname{Der}(\log D).$$

If D is Euler homogeneous and $A = \operatorname{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \operatorname{Der}(\log D)$, then the latter is equivalent to $\operatorname{Sym}_{\mathcal{O}} A = (i_D)_* i_D^{-1} \operatorname{Sym}_{\mathcal{O}} A$.

Proof. This follows from Corollary 5, Lemmata 11, 12, and 14, and the preceding remarks. □

4. Depth and torsion of symmetric algebras

Using a theorem of G. Scheja [Sch61] on extension of coherent analytic sheaves, we shall give sufficient conditions for $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ in terms of the depth and torsion of the symmetric algebras in Proposition 16.

Theorem 17. *Let $D \subset X$ be a divisor such that $\mathrm{Sym}_{\mathcal{O}} \mathrm{Der}(\log D)$ is $\mathcal{O}$-torsion free. Let $Z \subset \mathrm{Sing}(D)$ be a closed subset such that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on $X \backslash Z$. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on X if*

$$\mathrm{depth}_z\big(\mathrm{Sym}_{\mathcal{O}}^k \mathrm{Der}(\log D)\big) \geq \dim_z(Z) + 2$$

for all $z \in Z$ and $k \in \mathbb{N}$. In particular, this holds if D is Euler homogeneous, $A = \mathrm{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \mathrm{Der}(\log D)$, and

$$\mathrm{depth}_z\big(\mathrm{Sym}_{\mathcal{O}}^k A\big) \geq \dim_z(Z) + 2$$

for all $z \in Z$ and $k \in \mathbb{N}$.

Proof. This follows from [Sch61, Satz I-III] and Proposition 16. □

We shall apply a criterion by C. Huneke [Hun81] for the torsion freeness of symmetric algebras.

Proposition 18. *Let R be a Noetherian domain and let*

$$0 \longrightarrow R \xrightarrow{(a_1,\ldots,a_m)^t} R^m \longrightarrow M \longrightarrow 0$$

be a resolution of M. If $\mathrm{grade}(I) \geq k+1$ for $I = \langle a_1, \ldots, a_m \rangle$, then

$$\mathrm{depth}(I, \mathrm{Sym}_R(M)) \geq k.$$

Proof. We proceed by induction on k. By [Hun81, Prop. 2.1], $\mathrm{grade}(I) \geq 2$ implies that $\mathrm{Sym}_R(M)$ is R-torsion free. If $k \geq 2$, then $\mathrm{grade}(I/a) \geq k$ for some $a \in I$ and hence

$$0 \longrightarrow R/a \xrightarrow{([a_1],\ldots,[a_m])^t} (R/a)^m \longrightarrow M/a \longrightarrow 0$$

is a resolution of M. Since $\mathrm{Sym}_{R/a}(M/a) \cong \mathrm{Sym}_R(M)/a$, the induction hypothesis applies. □

Theorem 19. *Let $D \subset X$ be an Euler homogeneous divisor and $A = \mathrm{Ann}_\Theta(f)$ or $A \oplus \mathcal{O} \cdot \chi \cong \mathrm{Der}(\log D)$. Let $Z \subset \mathrm{Sing}(D)$ be a closed subset such that $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on $X \backslash Z$. For $z \in Z$, let*

$$0 \longrightarrow \mathcal{O}_z \xrightarrow{(a_{z,1},\ldots,a_{z,m})^t} \mathcal{O}_z^m \longrightarrow A_z \longrightarrow 0$$

be a resolution of A_z such that

$$\mathrm{grade}(\langle a_{z,1}, \ldots, a_{z,m} \rangle) \geq \dim_z(Z) + 3.$$

Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$ on X.

Proof. This follows from Theorem 17, Proposition 18, and [Hun81, Prop. 2.1]. □

Corollary 20. *Let X be a complex manifold of dimension 3 and let $D \subset X$ be a divisor with only isolated quasi-homogeneous singularities. Then $\mathcal{V}_0^D = \mathcal{O}[\mathrm{Der}(\log D)]$.*

Proof. We may assume that $X \subset \mathbb{C}^3$ is an open neighbourhood of 0, $D = (f)$ with $f \in \mathcal{O}$, and $\mathrm{Sing}(D) = \{0\}$. Let x_1, x_2, x_3 be coordinates on X. Then $\partial(f) = \partial_1(f), \partial_2(f), \partial_3(f) \in \mathfrak{m}_0$ is an $\mathcal{O}_0$-sequence. Hence the Koszul-complex

$$\begin{array}{ccccccccccc} 0 \to \mathcal{O}_0 & \xrightarrow{\partial(f)^t} & \mathcal{O}_0^3 & & \longrightarrow & & \mathcal{O}_0^3 & \longrightarrow & \mathcal{O}_0 & \longrightarrow & \mathcal{O}_0/\langle\partial(f)\rangle \to 0 \\ & & & \searrow & & \nearrow & & & & & \\ & & & & \mathrm{Ann}_{\Theta_0}(f) & & & & & & \\ & & & \nearrow & & \searrow & & & & & \\ & & 0 & & & & 0 & & & & \end{array}$$

is exact and induces a resolution of $\mathrm{Ann}_{\Theta_0}(f)$. Then the claim follows from Theorem 19 with $Z = \mathrm{Sing}(D)$. □

Our criterion also applies to some cases of nonisolated singularities.

Example 21. Let $D_3 \subset \mathbb{C}^3$ be the central generic hyperplane arrangement defined in Section 5. Then D_3 is not a free divisor and has a nonisolated singularity at 0. By Lemma 23 and Proposition 24, $A_3 \cong \mathcal{O}^3/\mathcal{O}\cdot(x_1, x_2, x_3)$ and $A_3 \oplus \mathcal{O}\cdot\chi \cong \mathrm{Der}(\log D_3)$. Then, by Examples 1 (2) and 4 (2) on $\mathrm{Sing}(D_3)\backslash\{0\}$ and Theorem 19 for $A = A_3$ and $Z = \{0\}$, $\mathcal{V}_0^{D_3} = \mathcal{O}[\mathrm{Der}(\log D_3)]$.

Our approach may fail in dimension $n > 3$ even for quasi-homogeneous isolated singularities.

Example 22. Let x_1, x_2, x_3, x_4 be coordinates on $\mathbb{C}^4$ and

$$f = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Then $D = (f) \subset \mathbb{C}^4$ has a quasi-homogeneous isolated singularity at 0. One can compute that the coordinates are zero divisors on $\mathrm{Sym}^2_{\mathcal{O}} \mathrm{Ann}_{\mathcal{O}}(f)$ at 0. By Lemmata 12 and 14 this implies that α_D is not an isomorphism.

5. Example of generic hyperplane arrangements

We shall provide some background for the examples in the previous sections. Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{C}^n$ and

$$f_n = x_1 \cdots x_n(x_1 + \cdots + x_n).$$

Then $D_n = (f_n) \subset \mathbb{C}^n$ is a central generic hyperplane arrangement. Let $\chi = \sum_i x_i\partial_i$ be the Euler vector field,

$$\eta_{i,j} = x_i x_j(\partial_i - \partial_j) \in \mathrm{Der}(\log D_n)$$

for $i < j$, and $A_n = \mathcal{O}\langle\eta_{i,j}\rangle$. By J. Wiens [Wie01, Thm. 3.4],

$$\mathrm{Der}(\log D_n) = \mathcal{O}\cdot\chi + A_n$$

with a minimal number of generators. Let

$$\sigma_{i,j,k} = x_i\eta_{j,k} - x_j\eta_{i,k} + x_k\eta_{i,j} \in \mathrm{syz}(\eta_{i,j})$$

for $i < j < k$ and choose a monomial ordering refining $\partial_1 < \cdots < \partial_n$.

Lemma 23. *$(\eta_{i,j})$ is a standard basis of A_n and* $\mathrm{syz}(\eta_{i,j}) = \langle\sigma_{i,j,k}\rangle$.

Proof. This follows from Buchberger's criterion [GP02, Thm. 1.7.3]. □

Proposition 24. $\mathrm{Der}(\log D_n) = \mathcal{O}\cdot\chi \oplus A_n$.

Proof. It suffices to verify that no syzygy of χ and the $\eta_{i,j}$ involves χ. One can obtain the syzygies from a standard basis computation [GP02, Alg. 2.5.4]. The first s-polynomials $x_k\chi - \eta_{k,n}$ and $x_j\eta_{i,k} - x_i\eta_{j,k}$ have a zero ∂_n component. Hence only a sequence of s-polynomials starting with $x_k\chi - \eta_{k,n}$ can contribute to syzygies involving χ, and the coefficient of χ remains a monomial. Each element in such a sequence has exactly one monomial involving x_n. Since the $\partial_2, \ldots, \partial_{n-1}$ are leading components of the $\eta_{i,j}$, the sequence terminates with a nonzero element $a_k\partial_1 \equiv x^{\alpha_k}\chi \mod A_n$. By the same reason, $\mathcal{O} \cdot \partial_1 \oplus A_n$ is a direct sum and hence $x^{\alpha_j}a_k = x^{\alpha_k}a_j$. This implies that the coefficient of χ in any syzygy is zero. $\square$

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