

ON A LITTLEWOOD-PALEY TYPE INEQUALITY

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ABSTRACT. The following is proved: If u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$ and if $0 < p \leq 1$, then the inequality

$$\int_{\partial B} u^*(y)^p d\sigma \leq C_{p,N} \left(|u(0)|^p + \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x) \right)$$

holds, where u^* is the nontangential maximal function of u . This improves a recent result of Stoll. This inequality holds for polyharmonic and hyperbolically harmonic functions as well.

Let \mathbb{R}^N ($N \geq 2$) denote the N -dimensional Euclidean space. In [18], Stević proved that if u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$ and if $\frac{N-2}{N-1} \leq p < 1$, then the inequality

$$(1) \quad \sup_{0 < r < 1} M_p^p(r, u) \leq C_1 |u(0)|^p + C_2 \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x)$$

holds. Here dV denotes the Lebesgue measure in \mathbb{R}^N normalized so that $V(B) = 1$, and as usual

$$M_p^p(r, u) = \int_{\partial B} |u(ry)|^p d\sigma(y),$$

where $d\sigma$ is the normalized surface measure on the sphere ∂B . The strange condition $(N - 2)/(N - 1) \leq p \leq 1$ appears in [18] because the proof in that paper is based on the fact, due Stein and Weiss [17, 16], that $|\nabla u|^p$ is subharmonic for $p \geq (N - 2)/(N - 1)$.

In the case $N = 2$, inequality (1) was proved by Flett [2]. It holds for $1 < p < 2$ as well, while if $p > 2$, then the reverse inequality holds; these inequalities are due to Littlewood and Paley [7]. Elementary proofs of the Littlewood-Paley inequalities are given in [13] and [8, 15] ($p > 2$).

In a recent paper [19], Stoll proved a very general theorem which says, in particular, that (1) holds for every $p \in (0, 1]$. Here we improve Stoll's theorem by proving the following result. Here u^* denotes the nontangential maximal function of u , i.e.

$$u^*(y) = \sup_{|x-y| < c(1-|x|)} |u(x)|, \quad y \in \partial B,$$

where $c > 1$ is a constant.

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Theorem 1. *If u is a function harmonic in B and if $0 < p \leq 1$, then the inequality*

$$(2) \quad \int_{\partial B} u^*(y)^p d\sigma(y) \leq C_{p,N} \left(|u(0)|^p + \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x) \right)$$

holds, where C is a constant depending only on p , c , and N .

A well known theorem of Fefferman and Stein [1] enables us to replace u^* in (2) by the radial maximal function u^+ ,

$$u^+(y) = \sup_{0 < r < 1} |u(ry)|, \quad y \in \partial B.$$

Namely,

Theorem A. *If $U \geq 0$ is a function subharmonic in B and if $p > 0$, then there is a constant $C = C_{p,N,c}$ such that*

$$(3) \quad \int_{\partial B} U^*(y)^p d\sigma(y) \leq C \int_{\partial B} U^+(y)^p d\sigma(y).$$

The proof of this theorem (see [3, Theorem 3.6]), as well our proof of Theorem 1, is based on a fundamental result of Hardy and Littlewood [4] and Fefferman and Stein [1] on the subharmonic behavior of $|u|^p$. We state this result in the following way.

Lemma A. *If $U \geq 0$ is a function subharmonic in $B(a, 2\varepsilon)$ ($a \in \mathbb{R}^N$, $\varepsilon > 0$), then the inequality*

$$(4) \quad \sup_{x \in B(a, \varepsilon)} U(x)^p \leq C\varepsilon^{-N} \int_{B(a, 2\varepsilon)} U^p dV$$

holds, where C depends only on p and N .

Here $B(a, r)$ denotes the ball of radius r centered at a .

For simple proofs of Lemma A we refer to [10, 14], and for generalizations to various classes of functions, we refer to [5, 6, 9, 11, 12].

Let $P(x, y)$ denote the Poisson kernel,

$$P(x, y) = \frac{1 - |x|^2}{|x - y|^N}.$$

Since $\int_{\partial B} P(x, y) d\sigma(y) = 1$, we see that Theorem 1 is a direct consequence of Theorem A and the following:

Proposition 1. *If u is a function harmonic in B and if $0 < p \leq 1$, then the inequality*

$$(5) \quad u^+(y)^p \leq C_{p,N} \left(|u(0)|^p + \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p P(x, y) dV(x) \right), \quad |y| = 1,$$

holds, where C is a constant depending only on p and N .

Let

$$u^+(\rho y) = \sup_{0 < r < \rho} |u(ry)| = \sup_{0 < t < 1} |u(t\rho y)|, \quad 0 < \rho \leq 1, \quad y \in \partial B,$$

and let $u^+(0) = |u(0)|$.

Lemma 1. *Let $r_j = 1 - 2^{-j}$ for $j \geq 0$. If $0 < p \leq 1$ and if u is of class $C^1(B)$, then the inequality*

$$u^+(y)^p \leq C|u(0)|^p + C \sum_{j=0}^{\infty} 2^{-jp} \sup_{r_j < r < r_{j+1}} |\nabla u(ry)|^p, \quad y \in \partial B$$

holds, where C depends only on p and N .

Proof. We start from the inequality

$$u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq \sup_{0 < t < 1} |u(tr_{j+1}y) - u(tr_jy)|^p.$$

By Lagrange's theorem,

$$\begin{aligned} |u(tr_{j+1}y) - u(tr_jy)| &\leq (r_{j+1} - r_j) \sup_{0 < r < r_{j+1}} |\nabla u(ry)| \\ &\leq 2^{-j} \sup_{0 < r < r_{j+1}} |\nabla u(ry)|, \end{aligned}$$

and hence

$$u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq 2^{-jp} \sup_{0 < r < r_{j+1}} |\nabla u(ry)|^p,$$

which implies

$$\begin{aligned} u^+(y)^p - |u(0)|^p &= \sum_{j=0}^{\infty} (u^+(r_{j+1}y)^p - u^+(r_jy)^p) \\ &\leq \sum_{j=0}^{\infty} 2^{-jp} \sup_{0 < r < r_{j+1}} |\nabla u(ry)|^p. \end{aligned}$$

On the other hand, by summation by parts we see that if $\{A_j\}_0^\infty$ is a nondecreasing sequence of real numbers, then

$$\sum_{j=0}^{\infty} 2^{-jp} A_{j+1} \leq CA_0 + C \sum_{j=0}^{\infty} 2^{-jp} (A_{j+1} - A_j),$$

where C depends only on p . By taking $A_j = \sup_{0 \leq r \leq r_j} |\nabla u(ry)|^p$ and using the inequalities

$$A_{j+1} - A_j \leq \sup_{r_j < r < r_{j+1}} |\nabla u(ry)|^p,$$

we get the desired result. □

Proof of Proposition 1. By Lemma A with $U = |\nabla u|$, $a = a_j := (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$(6) \quad 2^{-jp} \sup_{r_j < r < r_{j+1}} |\nabla u(ry)|^p \leq C 2^{-jp} 2^{jN} \int_{B(a_j, 2^{-j-1})} |\nabla u(x)|^p dV(x).$$

On the other hand, simple calculation shows that $|x - a_j| \leq 2^{-j-1}$ implies

$$2^{-j-2} \leq 1 - |x|, \quad |x - y| \leq 2^{-j+1}.$$

Hence

$$2^{-j} 2^{jN} \leq 2^{N+2} P(x, y), \quad \text{for } x \in B(a_j, 2^{-j-1}).$$

From this and (6) we get

$$\begin{aligned} 2^{-jp} \sup_{r_j < r < r_{j+1}} |\nabla u(ry)|^p &\leq C 2^{-j(p-1)} \int_{r_{j-1} \leq |x| \leq r_{j+2}} P(x, y) |\nabla u(x)|^p dV(x) \\ &\leq 2^{1-p} C \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{p-1} P(x, y) |\nabla u(x)|^p dV(x) \end{aligned}$$

($r_{-1} = 0$) where we have used the inclusion

$$\{x: |x - a_j| \leq 2^{-j-1}\} \subset \{x: r_{j-1} \leq |x| \leq r_{j+2}\}.$$

Now the desired conclusion is easily obtained by using Lemma 1. \square

Remarks.

Remark 1. As the above proofs show, the validity of Proposition 1 depends only on the inequality

$$(7) \quad \sup_{x \in B(a, \varepsilon)} |\nabla u(x)|^p \leq C \varepsilon^{-N} \int_{B(a, 2\varepsilon)} |\nabla u|^p dV.$$

It was proved in [10] that (7) is implied by

$$(8) \quad |\nabla u(x)| \leq K \varepsilon^{-1} \sup_{z \in B(a, \varepsilon)} |u(z) - u(x)|.$$

More precisely: *Let $u \in C^1(B)$ and let $0 < p < \infty$. If there is a constant K such that (8) holds whenever $B(x, \varepsilon) \subset B$, then there is a constant $C = C_{p, N, K}$ such that (7) holds whenever $B(x, 2\varepsilon) \subset B$.*

Remark 2. Condition (8), and hence (7), is satisfied in a wide class of functions containing, in particular, polyharmonic, hyperbolically harmonic, and convex functions [11]. Recall that u is called polyharmonic if $\Delta^k u \equiv 0$ for some integer $k \geq 1$, and u is hyperbolically harmonic if

$$\Delta_h u(x) := (1 - |x|^2)^2 [\Delta u(x) + 2(N - 2)(1 - |x|^2)^{-1} x \cdot \nabla u(x)] \equiv 0.$$

Note also that the class of hyperbolically harmonic functions is invariant under Möbius transformations of the ball.

Remark 3. The proof of Theorem A given in [3] (see also [14, theorem 7.1.8]) shows that (3) is implied by (4). On the other hand, as was proved in [10], condition (8) implies (4).

It follows from the above remarks that the following generalization of Theorem 1 holds.

Theorem 2. *If u is a function polyharmonic, hyperbolically harmonic, or convex in B and if $0 < p \leq 1$, then inequality (1) holds.*

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