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ON A LITTLEWOOD-PALEY TYPE INEQUALITY

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ABSTRACT. The following is proved: If u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$ and if $0 < p \leq 1$, then the inequality

$$\int_{\partial B} u^*(y)^p \, d\sigma \leq C_{p,N} \left(|u(0)|^p + \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x) \right)$$

holds, where u^* is the nontangential maximal function of u. This improves a recent result of Stoll. This inequality holds for polyharmonic and hyperbolically harmonic functions as well.

Let $\mathbb{R}^N$ $(N \geq 2)$ denote the N-dimensional Euclidean space. In [18], Stević proved that if u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$ and if $\frac{N-2}{N-1} \leq p < 1$, then the inequality

$$\sup_{0<r<1} M_p^p(r,u) \leq C_1 |u(0)|^p + C_2 \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x) \tag{1}$$

holds. Here dV denotes the Lebesgue measure in $\mathbb{R}^N$ normalized so that $V(B) = 1$, and as usual

$$M_p^p(r,u) = \int_{\partial B} |u(ry)|^p \, d\sigma(y),$$

where $d\sigma$ is the normalized surface measure on the sphere ∂B. The strange condition $(N-2)/(N-1) \leq p \leq 1$ appears in [18] because the proof in that paper is based on the fact, due Stein and Weiss [17, 16], that $|\nabla u|^p$ is subharmonic for $p \geq (N-2)/(N-1)$.

In the case $N = 2$, inequality (1) was proved by Flett [2]. It holds for $1 < p < 2$ as well, while if $p > 2$, then the reverse inequality holds; these inequalities are due to Littlewood and Paley [7]. Elementary proofs of the Littlewood-Paley inequalities are given in [13] and [8, 15] $(p > 2)$.

In a recent paper [19], Stoll proved a very general theorem which says, in particular, that (1) holds for every $p \in (0, 1]$. Here we improve Stoll's theorem by proving the following result. Here u^* denotes the nontangential maximal function of u, i.e.

$$u^*(y) = \sup_{|x-y|<c(1-|x|)} |u(x)|, \qquad y \in \partial B,$$

where $c > 1$ is a constant.

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Theorem 1. *If u is a function harmonic in B and if $0 < p \le 1$, then the inequality*

$$\int_{\partial B} u^*(y)^p \, d\sigma(y) \le C_{p,N} \left(|u(0)|^p + \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x) \right) \tag{2}$$

holds, where C is a constant depending only on p, c, and N.

A well known theorem of Fefferman and Stein [1] enables us to replace u^* in (2) by the radial maximal function u^+,

$$u^+(y) = \sup_{0<r<1} |u(ry)|, \qquad y \in \partial B.$$

Namely,

Theorem A. *If $U \ge 0$ is a function subharmonic in B and if $p > 0$, then there is a constant $C = C_{p,N,c}$ such that*

$$\int_{\partial B} U^*(y)^p \, d\sigma(y) \le C \int_{\partial B} U^+(y)^p \, d\sigma(y). \tag{3}$$

The proof of this theorem (see [3, Theorem 3.6]), as well our proof of Theorem 1, is based on a fundamental result of Hardy and Littlewood [4] and Fefferman and Stein [1] on the subharmonic behavior of $|u|^p$. We state this result in the following way.

Lemma A. *If $U \ge 0$ is a function subharmonic in $B(a, 2\varepsilon)$ $(a \in \mathbb{R}^N,\ \varepsilon > 0)$, then the inequality*

$$\sup_{x \in B(a,\varepsilon)} U(x)^p \le C\varepsilon^{-N} \int_{B(a,2\varepsilon)} U^p \, dV \tag{4}$$

holds, where C depends only on p and N.

Here $B(a, r)$ denotes the ball of radius r centered at a.

For simple proofs of Lemma A we refer to [10, 14], and for generalizations to various classes of functions, we refer to [5, 6, 9, 11, 12].

Let $P(x, y)$ denote the Poisson kernel,

$$P(x,y) = \frac{1-|x|^2}{|x-y|^N}.$$

Since $\int_{\partial B} P(x, y) \, d\sigma(y) = 1$, we see that Theorem 1 is a direct consequence of Theorem A and the following:

Proposition 1. *If u is a function harmonic in B and if $0 < p \le 1$, then the inequality*

$$u^+(y)^p \le C_{p,N} \left(|u(0)|^p + \int_B (1-|x|)^{p-1} |\nabla u(x)|^p P(x,y) \, dV(x) \right), \qquad |y| = 1, \tag{5}$$

holds, where C is a constant depending only on p and N.

Let

$$u^+(\rho y) = \sup_{0<r<\rho} |u(ry)| = \sup_{0<t<1} |u(t\rho y)|, \qquad 0 < \rho \le 1,\ y \in \partial B,$$

and let $u^+(0) = |u(0)|$.

Lemma 1. *Let $r_j = 1 - 2^{-j}$ for $j \geq 0$. If $0 < p \leq 1$ and if u is of class $C^1(B)$, then the inequality*

$$u^+(y)^p \leq C|u(0)|^p + C\sum_{j=0}^{\infty} 2^{-jp} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p, \qquad y \in \partial B$$

holds, where C depends only on p and N.

Proof. We start from the inequality

$$u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq \sup_{0<t<1} |u(tr_{j+1}y) - u(tr_jy)|^p.$$

By Lagrange's theorem,

$$\begin{aligned} |u(tr_{j+1}y) - u(tr_jy)| &\leq (r_{j+1} - r_j) \sup_{0<r<r_{j+1}} |\nabla u(ry)| \\ &\leq 2^{-j} \sup_{0<r<r_{j+1}} |\nabla u(ry)|, \end{aligned}$$

and hence

$$u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq 2^{-jp} \sup_{0<r<r_{j+1}} |\nabla u(ry)|^p,$$

which implies

$$\begin{aligned} u^+(y)^p - |u(0)|^p &= \sum_{j=0}^{\infty} (u^+(r_{j+1}y)^p - u^+(r_jy)^p) \\ &\leq \sum_{j=0}^{\infty} 2^{-jp} \sup_{0<r<r_{j+1}} |\nabla u(ry)|^p. \end{aligned}$$

On the other hand, by summation by parts we see that if $\{A_j\}_0^\infty$ is a nondecreasing sequence of real numbers, then

$$\sum_{j=0}^{\infty} 2^{-jp} A_{j+1} \leq CA_0 + C\sum_{j=0}^{\infty} 2^{-jp}(A_{j+1} - A_j),$$

where C depends only on p. By taking $A_j = \sup_{0\leq r\leq r_j} |\nabla u(ry)|^p$ and using the inequalities

$$A_{j+1} - A_j \leq \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p,$$

we get the desired result. □

Proof of Proposition 1. By Lemma A with $U = |\nabla u|$, $a = a_j := (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$2^{-jp} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p \leq C2^{-jp}2^{jN} \int_{B(a_j,2^{-j-1})} |\nabla u(x)|^p \, dV(x). \tag{6}$$

On the other hand, simple calculation shows that $|x - a_j| \leq 2^{-j-1}$ implies

$$2^{-j-2} \leq 1 - |x|, \qquad |x - y| \leq 2^{-j+1}.$$

Hence

$$2^{-j}2^{jN} \leq 2^{N+2}\, P(x,y), \quad \text{for } x \in B(a_j, 2^{-j-1}).$$

From this and (6) we get

$$2^{-jp} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p \le C2^{-j(p-1)} \int_{r_{j-1}\le|x|\le r_{j+2}} P(x,y)\,|\nabla u(x)|^p\,dV(x)$$
$$\le 2^{1-p}C \int_{r_{j-1}\le|x|\le r_{j+2}} (1-|x|)^{p-1}P(x,y)\,|\nabla u(x)|^p\,dV(x)$$

$(r_{-1}=0)$ where we have used the inclusion

$$\{x\colon |x-a_j|\le 2^{-j-1}\}\subset\{x\colon r_{j-1}\le|x|\le r_{j+2}\}.$$

Now the desired conclusion is easily obtained by using Lemma 1. □

Remarks.

Remark 1. As the above proofs show, the validity of Proposition 1 depends only on the inequality

$$\sup_{x\in B(a,\varepsilon)} |\nabla u(x)|^p \le C\varepsilon^{-N}\int_{B(a,2\varepsilon)} |\nabla u|^p\,dV. \tag{7}$$

It was proved in [10] that (7) is implied by

$$|\nabla u(x)| \le K\varepsilon^{-1}\sup_{z\in B(a,\varepsilon)}|u(z)-u(x)|. \tag{8}$$

More precisely: *Let $u\in C^1(B)$ and let $0<p<\infty$. If there is a constant K such that (8) holds whenever $B(x,\varepsilon)\subset B$, then there is a constant $C=C_{p,N,K}$ such that (7) holds whenever $B(x,2\varepsilon)\subset B$.*

Remark 2. Condition (8), and hence (7), is satisfied in a wide class of functions containing, in particular, polyharmonic, hyperbolically harmonic, and convex functions [11]. Recall that u is called polyharmonic if $\Delta^k u\equiv 0$ for some integer $k\ge 1$, and u is hyperbolically harmonic if

$$\Delta_h u(x) := (1-|x|^2)^2[\Delta u(x)+2(N-2)(1-|x|^2)^{-1}x\cdot\nabla u(x)]\equiv 0.$$

Note also that the class of hyperbolically harmonic functions is invariant under Möbius transformations of the ball.

Remark 3. The proof of Theorem A given in [3] (see also [14, theorem 7.1.8]) shows that (3) is implied by (4). On the other hand, as was proved in [10], condition (8) implies (4).

It follows from the above remarks that the following generalization of Theorem 1 holds.

Theorem 2. *If u is a function polyharmonic, hyperbolically harmonic, or convex in B and if $0<p\le 1$, then inequality (1) holds.*

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