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MINIMAL SYSTEMS OF BINOMIAL GENERATORS AND THE INDISPENSABLE COMPLEX OF A TORIC IDEAL

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ABSTRACT. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ be a vector configuration and $I_A \subset K[x_1, \ldots, x_m]$ its corresponding toric ideal. The paper consists of two parts. In the first part we completely determine the number of different minimal systems of binomial generators of I_A. In the second part we associate to A a simplicial complex $\Delta_{\mathrm{ind}(A)}$. We show that the vertices of $\Delta_{\mathrm{ind}(A)}$ correspond to the indispensable monomials of the toric ideal I_A, while one dimensional facets of $\Delta_{\mathrm{ind}(A)}$ with minimal binomial A-degree correspond to the indispensable binomials of I_A.

1. Introduction

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be a vector configuration in $\mathbb{Z}^n$ such that the affine semigroup $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ is pointed. Recall that $\mathbb{N}A$ is *pointed* if zero is the only invertible element. Let K be a field of any characteristic; we grade the polynomial ring $K[x_1, \ldots, x_m]$ by setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

The *toric ideal* I_A associated to A is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$ (see [13]). For such binomials, we define $\deg_A(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) := \deg_A(\mathbf{x}^{\mathbf{u}})$.

In general it is possible for a toric ideal I_A to have more than one minimal system of generators. We define $\nu(I_A)$ to be the number of different minimal systems of binomial generators of the toric ideal I_A, where the sign of a binomial does not count. Minimal systems of binomial generators of toric ideals have been studied in several papers; see [3] and its introduction. A recent problem arising from algebraic statistics (see [14]), is when a toric ideal possesses a unique minimal system of binomial generators, i.e. $\nu(I_A) = 1$. To study this problem, Ohsugi and Hibi introduced in [10] the notion of indispensable binomials while Aoki, Takemura and Yoshida introduced in [1] the notion of indispensable monomials. Moreover in [10]

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a necessary and sufficient condition is given for toric ideals associated with certain finite graphs to possess unique minimal systems of binomial generators. We recall that a binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is *indispensable* if every system of binomial generators of I_A contains B or $-B$, while a monomial $\mathbf{x}^{\mathbf{u}}$ is *indispensable* if every system of binomial generators of I_A contains a binomial B such that the $\mathbf{x}^{\mathbf{u}}$ is a monomial of B.

In this article we use and extend ideas and techniques developed by Diaconis and Sturmfels (see [7]) and Takemura and Aoki (see [14]) to study minimal systems of binomial generators of the toric ideal I_A and also to investigate the notion of the indispensable complex of I_A, denoted by $\Delta_{\mathrm{ind}(A)}$. In Section 2, we construct graphs $G(\mathbf{b})$, for every $\mathbf{b} \in \mathbb{N}A$, and use them to provide a formula for $\nu(I_A)$. We give criteria for a toric ideal to be generated by indispensable binomials. In Section 3 we define $\Delta_{\mathrm{ind}(A)}$ and we show that this complex determines the indispensable monomials and binomials. As an application we characterize principal toric ideals in terms of $\Delta_{\mathrm{ind}(A)}$.

2. The number of minimal generating sets of a toric ideal

Let $A \subset \mathbb{Z}^n$ be a vector configuration so that $\mathbb{N}A$ is pointed and let $I_A \subset K[x_1, \ldots, x_m]$ be its corresponding toric ideal. A vector $\mathbf{b} \in \mathbb{N}A$ is called a *Betti A-degree* if I_A has a minimal generating set containing an element of A-degree $\mathbf{b}$. The Betti A-degrees are independent of the choice of a minimal generating set of I_A; see [4, 9, 13]. The *A-graded Betti number* $\beta_{0,\mathbf{b}}$ of I_A is the number of times $\mathbf{b}$ appears as the A-degree of a binomial in a given minimal generating set of I_A and is also an invariant of I_A.

The semigroup $\mathbb{N}A$ is pointed, so we can partially order it with the relation

$$\mathbf{c} \geq \mathbf{d} \iff \text{there is an } \mathbf{e} \in \mathbb{N}A \text{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e}.$$

For $I_A \neq \{0\}$ the minimal elements of the set $\{\deg_A(\mathbf{x}^{\mathbf{u}}) \mid \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A\} \subset \mathbb{N}A$ with respect to $\geq$ are called *minimal binomial A-degrees*. Minimal binomial A-degrees are always Betti A-degrees but the converse is not true, as Example 2.3 demonstrates. For any $\mathbf{b} \in \mathbb{N}A$ we define the ideal

$$I_{A,\mathbf{b}} := (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \gneqq \mathbf{b}) \subset I_A.$$

Definition 2.1. For a vector $\mathbf{b} \in \mathbb{N}A$ we define $G(\mathbf{b})$ to be the graph with vertices the elements of the fiber

$$\deg_A^{-1}(\mathbf{b}) = \{\mathbf{x}^{\mathbf{u}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \mathbf{b}\}$$

and edges all the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{A,\mathbf{b}}$.

The fiber $\deg_A^{-1}(\mathbf{b})$ has finitely many elements, since the affine semigroup $\mathbb{N}A$ is pointed. If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are vertices of $G(\mathbf{b})$ such that $\gcd(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}) \neq 1$, then $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is an edge of $G(\mathbf{b})$. The next proposition follows easily from the definition.

Proposition 2.2. *Let $\mathbf{b} \in \mathbb{N}A$. Every connected component of $G(\mathbf{b})$ is a complete subgraph. The graph $G(\mathbf{b})$ is not connected if and only if $\mathbf{b}$ is a Betti A-degree.*

Example 2.3. Let

$$A = \{(2,2,2,0,0), (2,-2,-2,0,0), (2,2,-2,0,0), (2,-2,2,0,0), \\ (3,0,0,3,3), (3,0,0,-3,-3), (3,0,0,3,-3)(3,0,0,-3,3)\}.$$

Using CoCoA [6], we see that $I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$. The Betti A-degrees are $\mathbf{b}_1 = (4,0,0,0,0)$, $\mathbf{b}_2 = (6,0,0,0,0)$ and $\mathbf{b}_3 = (12,0,0,0,0)$. We note that $\mathbf{b}_3 = 2\mathbf{b}_2$, so $\mathbf{b}_3$ is not a minimal binomial A-degree. The ideals $I_{A,\mathbf{b}_1}$ and $I_{A,\mathbf{b}_2}$ are zero, while $I_{A,\mathbf{b}_3} = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8)$. The graphs $G(\mathbf{b})$ are connected for all $\mathbf{b} \in \mathbb{N}A$ except for the Betti A-degrees. In fact $G(\mathbf{b}_1)$ and $G(\mathbf{b}_2)$ consist of two connected components, $\{x_1x_2\}$ and $\{x_3x_4\}$ for $G(\mathbf{b}_1)$, $\{x_5x_6\}$ and $\{x_7x_8\}$ for $G(\mathbf{b}_2)$, while the connected components of $G(\mathbf{b}_3)$ are $\{x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3\}$ and $\{x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2\}$.

Let $n_\mathbf{b}$ denote the number of connected components of $G(\mathbf{b})$, this means that

$$G(\mathbf{b}) = \bigcup_{i=1}^{n_\mathbf{b}} G(\mathbf{b})_i,$$

and let $t_i(\mathbf{b})$ be the number of vertices of the i-component. The next proposition will be helpful in the sequel.

Proposition 2.4. *An A-degree $\mathbf{b}$ is a minimal binomial A-degree if and only if every connected component of $G(\mathbf{b})$ is a singleton.*

Proof. If $\mathbf{b}$ is a minimal binomial A-degree, then $I_{A,\mathbf{b}} = \{0\}$ and every connected component of $G(\mathbf{b})$ is a singleton. Suppose now that $\mathbf{b}$ is not minimal, i.e. $\mathbf{c} \lneqq \mathbf{b}$ for some minimal binomial A-degree $\mathbf{c}$. Thus there is a binomial $B = \mathbf{x^u} - \mathbf{x^v} \in I_A$, with $\deg_A(B) = \mathbf{c}$, and a monomial $\mathbf{x^a} \neq 1$ such that $\mathbf{b} = \mathbf{c} + \deg_A(\mathbf{x^a})$. Therefore $\mathbf{x^{a+u}}$, $\mathbf{x^{a+v}}$ are vertices of $G(\mathbf{b})$ and belong to the same component of $G(\mathbf{b})$ since $\mathbf{x^{a+u}} - \mathbf{x^{a+v}} = \mathbf{x^a}B \in I_{A,\mathbf{b}}$. $\square$

Let $\mathcal{G} \subset I_A$ be a set of binomials. We recall the definition of the graph $\Gamma(\mathbf{b})_\mathcal{G}$ ([7]) and a criterion for $\mathcal{G}$ to be a generating set of I_A (Theorem 2.5). Let $\Gamma(\mathbf{b})_\mathcal{G}$ be the graph with vertices the elements of $\deg_A^{-1}(\mathbf{b})$ and edges the sets $\{\mathbf{x^u}, \mathbf{x^v}\}$ whenever the binomial

$$\frac{(\mathbf{x^u} - \mathbf{x^v})}{\gcd(\mathbf{x^u}, \mathbf{x^v})} \quad \text{or} \quad \frac{(\mathbf{x^v} - \mathbf{x^u})}{\gcd(\mathbf{x^u}, \mathbf{x^v})}$$

belongs to $\mathcal{G}$. In [7] the following theorem was proved.

Theorem 2.5 ([7]). *$\mathcal{G}$ is a generating set for I_A if and only if $\Gamma(\mathbf{b})_\mathcal{G}$ is connected for all $\mathbf{b} \in \mathbb{N}A$.*

We consider the complete graph $\mathcal{S}_\mathbf{b}$ with vertices the connected components $G(\mathbf{b})_i$ of $G(\mathbf{b})$, and we let $T_\mathbf{b}$ be a spanning tree of $\mathcal{S}_\mathbf{b}$; for every edge of $T_\mathbf{b}$ joining the components $G(\mathbf{b})_i$ and $G(\mathbf{b})_j$, we choose a binomial $\mathbf{x^u} - \mathbf{x^v}$ with $\mathbf{x^u} \in G(\mathbf{b})_i$ and $\mathbf{x^v} \in G(\mathbf{b})_j$. We call $\mathcal{F}_{T_\mathbf{b}}$ the collection of these binomials. Note that if $\mathbf{b}$ is not a Betti A-degree, then $\mathcal{F}_{T_\mathbf{b}} = \emptyset$.

Theorem 2.6. *The set $\mathcal{F} = \bigcup_{\mathbf{b} \in \mathbb{N}A} \mathcal{F}_{T_\mathbf{b}}$ is a minimal generating set of I_A.*

Proof. First we will prove that $\mathcal{F}$ is a generating set of I_A. From Theorem 2.5 it is enough to prove that $\Gamma(\mathbf{b})_\mathcal{F}$ is connected for every $\mathbf{b}$. We will prove the theorem by induction on $\mathbf{b}$. If $\mathbf{b}$ is a minimal binomial A-degree, the vertices of $\Gamma(\mathbf{b})_\mathcal{F}$, which are also the vertices and the connected components of $G(\mathbf{b})$, and the tree $T_\mathbf{b}$ give a path between any two vertices of $G(\mathbf{b})$. Next, let $\mathbf{b}$ be nonminimal binomial A-degree. Suppose that $\Gamma(\mathbf{b})_\mathcal{F}$ is connected for all $\mathbf{c} \lneqq \mathbf{b}$ and let $\mathbf{x^u}$, $\mathbf{x^v}$ be two vertices of $\Gamma(\mathbf{b})_\mathcal{F}$. We will show that there is a path between these two vertices.

We will consider two cases, depending on whether the vertices are in the same connected component of $G(\mathbf{b})$ or not.

(1) If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are in the same component $G(\mathbf{b})_i$ of $G(\mathbf{b})$, then $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_i \mathbf{x}^{\mathbf{d}_i}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i})$ where $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{v}_i}$ have A-degree $\mathbf{b}_i \gneqq \mathbf{b}$. From the inductive hypothesis $\Gamma(\mathbf{b}_i)_{\mathcal{F}}$ is connected and there is a path from $\mathbf{x}^{\mathbf{u}_i}$ to $\mathbf{x}^{\mathbf{v}_i}$. This gives a path from $\mathbf{x}^{\mathbf{d}_i}\mathbf{x}^{\mathbf{u}_i}$ to $\mathbf{x}^{\mathbf{d}_i}\mathbf{x}^{\mathbf{v}_i}$ and joining these paths we find a path from $\mathbf{x}^{\mathbf{u}}$ to $\mathbf{x}^{\mathbf{v}}$ in $\Gamma(\mathbf{b})_{\mathcal{F}}$.

(2) If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ belong to different components of $G(\mathbf{b})$, we use the tree $T_{\mathbf{b}}$ to find a path between the two components. In each component we use the previous case and/or the induction hypothesis to move between vertices if needed. The join of these paths provides a path from $\mathbf{x}^{\mathbf{u}}$ to $\mathbf{x}^{\mathbf{v}}$ in $\Gamma(\mathbf{b})_{\mathcal{F}}$.

Next, we will show that no proper subset $\mathcal{F}'$ of $\mathcal{F}$ generates I_A. Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{F} \backslash \mathcal{F}'$, and let $\deg_A(B) = \mathbf{b}$. Since B is an element of $\mathcal{F}_{T_{\mathbf{b}}}$, it corresponds to an edge $\{G(\mathbf{b})_i, G(\mathbf{b})_j\}$ of $T_{\mathbf{b}}$, and the monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ belong to different components of $G(\mathbf{b})$. Suppose that there was a path $\{\mathbf{x}^{\mathbf{u}_1} = \mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{u}_2}, \ldots, \mathbf{x}^{\mathbf{u}_t} = \mathbf{x}^{\mathbf{v}}\}$ in $\Gamma(\mathbf{b})_{\mathcal{F}'}$ joining the vertices $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$. Certainly there are monomials $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{u}_{i+1}}$ that are in different connected components of $G(\mathbf{b})$. Since $\gcd(\mathbf{x}^{\mathbf{u}_i}, \mathbf{x}^{\mathbf{u}_{i+1}}) \neq 1$ implies that the monomials $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{u}_{i+1}}$ are in the same connected component of $G(\mathbf{b})$, we conclude that $\gcd(\mathbf{x}^{\mathbf{u}_i}, \mathbf{x}^{\mathbf{u}_{i+1}}) = 1$ for some i. In this case the binomial $\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{u}_{i+1}}$ is in $\mathcal{F}'$, it has A-degree $\mathbf{b}$, and it corresponds to an edge of $T_{\mathbf{b}}$. By considering these binomials and corresponding edges, we obtain a path in $T_{\mathbf{b}}$ joining the components $G(\mathbf{b})_i, G(\mathbf{b})_j$ and not containing $\{G(\mathbf{b})_i, G(\mathbf{b})_j\}$ of $T_{\mathbf{b}}$. This is a contradiction since $T_{\mathbf{b}}$ is a tree. □

The converse is also true; let $\mathcal{G} = \bigcup_{\mathbf{b} \in \mathbb{N}A} \mathcal{G}_{\mathbf{b}}$ be a minimal generating set for I_A where $\mathcal{G}_{\mathbf{b}}$ consists of the binomials in $\mathcal{G}$ of A-degree $\mathbf{b}$. We will show that $\mathcal{G}_{\mathbf{b}}$ determines a spanning tree $T_{\mathbf{b}}$ of $\mathcal{S}_{\mathbf{b}}$.

Theorem 2.7. *Let $\mathcal{G} = \bigcup_{\mathbf{b} \in \mathbb{N}A} \mathcal{G}_{\mathbf{b}}$ be a minimal generating set for I_A. The binomials of $\mathcal{G}_{\mathbf{b}}$ determine a spanning tree $T_{\mathbf{b}}$ of $\mathcal{S}_{\mathbf{b}}$.*

Proof. Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\mathbf{b}}$. The monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are in different connected components of $G(\mathbf{b})$; otherwise B is not a part of a minimal generating set of I_A. Therefore B indicates an edge in $\mathcal{S}_{\mathbf{b}}$. Let $T_{\mathbf{b}}$ be the union over $B \in \mathcal{G}_{\mathbf{b}}$ of these edges. Here $T_{\mathbf{b}}$ is tree of $\mathcal{S}_{\mathbf{b}}$, since if $T_{\mathbf{b}}$ contains a cycle, we can delete a binomial from $\mathcal{G}$ and still generate the ideal I_A, contradicting the minimality of $\mathcal{G}$. Theorem 2.5 guarantees that the tree $T_{\mathbf{b}}$ is spanning. □

An immediate corollary of Theorems 2.6 and 2.7 concerns the indispensable monomials.

Corollary 2.8. *The monomial $\mathbf{x}^{\mathbf{u}}$ is an indispensable monomial of A-degree $\mathbf{b}$ if and only if $\{\mathbf{x}^{\mathbf{u}}\}$ is a component of $G(\mathbf{b})$.*

We use Theorems 2.6 and 2.7 to compute $\nu(I_A)$, the number of minimal generating sets of I_A. For each $\mathbf{b} \in \mathbb{N}A$ the number of possible spanning trees $T_{\mathbf{b}}$ depends on $n_{\mathbf{b}}$, the number of connected components of $G(\mathbf{b})$. For a given spanning tree $T_{\mathbf{b}}$ the number of possible binomial sets $\mathcal{F}_{T_{\mathbf{b}}}$ (up to a sign) depends on $t_i(\mathbf{b})$, the number of vertices of $G(\mathbf{b})_i$. These numbers determine $\nu(I_A)$. We note that the sum $t_1(\mathbf{b}) + \cdots + t_{n_{\mathbf{b}}}(\mathbf{b})$ is equal to $|\deg_A^{-1}(\mathbf{b})|$, the cardinality of the fiber set $\deg_A^{-1}(\mathbf{b})$. We also point out that $|\mathcal{F}_{T_{\mathbf{b}}}| = n_{\mathbf{b}} - 1$ and that $|\mathcal{F}_{T_{\mathbf{b}}}| = \beta_{0,\mathbf{b}}$, the A-graded Betti number of I_A.

Theorem 2.9. *For a toric ideal I_A we have that*

$$\nu(I_A) = \prod_{\mathbf{b}\in\mathbb{N}A} t_1(\mathbf{b})\cdots t_{n_\mathbf{b}}(\mathbf{b})(t_1(\mathbf{b})+\cdots+t_{n_\mathbf{b}}(\mathbf{b}))^{n_\mathbf{b}-2}$$

where $n_\mathbf{b}$ is the number of connected components of $G(\mathbf{b})$ and $t_i(\mathbf{b})$ is the number of vertices of the connected component $G(\mathbf{b})_i$ of the graph $G(\mathbf{b})$.

Proof. Let d_i be the degree of $G(\mathbf{b})_i$ in a spanning tree $T_\mathbf{b}$, i.e. the number of edges of $T_\mathbf{b}$ incident with $G(\mathbf{b})_i$. We have that $\sum_{i=1}^{n_\mathbf{b}} d_i = 2n_\mathbf{b}-2$. There are

$$\frac{(n_\mathbf{b}-2)!}{(d_1-1)!(d_2-1)!\cdots(d_{n_\mathbf{b}}-1)!}$$

such spanning trees; see for example the proof of Cayley's formula in [8]. For fixed $T_\mathbf{b}$ with degrees d_i, there are $(t_i(\mathbf{b}))^{d_i}$ choices for the monomials for the edges involving the vertex $G(\mathbf{b})_i$. This implies that the number of possible binomial sets $\mathcal{F}_{T_\mathbf{b}}$ is $(t_1(\mathbf{b}))^{d_1}\cdots(t_{n_\mathbf{b}}(\mathbf{b}))^{d_{n_\mathbf{b}}}$. Therefore the total number of all possible $\mathcal{F}_{T_\mathbf{b}}$ is

$$\sum_{d_1+\cdots+d_{n_\mathbf{b}}=2n_\mathbf{b}-2} \frac{(n_\mathbf{b}-2)!}{(d_1-1)!(d_2-1)!\cdots(d_{n_\mathbf{b}}-1)!}(t_1(\mathbf{b}))^{d_1}\cdots(t_{n_\mathbf{b}}(\mathbf{b}))^{d_{n_\mathbf{b}}}$$
$$= t_1(\mathbf{b})\cdots t_{n_\mathbf{b}}(\mathbf{b})(t_1(\mathbf{b})+\cdots+t_{n_\mathbf{b}}(\mathbf{b}))^{n_\mathbf{b}-2}. \qquad \square$$

We point out that if $t_i(\mathbf{b}) = 1$ for all i, then the number of possible spanning trees of $G(\mathbf{b})$ is ${n_\mathbf{b}}^{n_\mathbf{b}-2}$ (Cayley's formula; see [5]). We also note that if $n_\mathbf{b} = 1$, for some $\mathbf{b}\in\mathbb{N}A$, then the factor $t_1(\mathbf{b})(t_1(\mathbf{b}))^{-1}$ in the above product has value 1. Thus the contributions to $\nu(I_A)$ come only from Betti A-degrees $\mathbf{b}\in\mathbb{N}A$. On the other hand we have a unique choice for a generator of degree $\mathbf{b}$ when $n_\mathbf{b} = 2$ and $t_1(\mathbf{b}) = t_2(\mathbf{b}) = 1$. Thus in these cases $G(\mathbf{b})$ consists of two isolated vertices and by Proposition 2.4, $\mathbf{b}$ is minimal. These remarks prove the following:

Corollary 2.10. *Let $B = \mathbf{x^u}-\mathbf{x^v}\in I_A$ with A-degree $\mathbf{b}$. Here B is indispensable if and only if the graph $G(\mathbf{b})$ consists of two connected components, $\{\mathbf{x^u}\}$ and $\{\mathbf{x^v}\}$. Moreover $\mathbf{b}$ is a minimal binomial A-degree.*

Corollary 2.11. *Suppose that the Betti A-degrees $\mathbf{b}_1,\ldots,\mathbf{b}_q$ of I_A are minimal binomial A-degrees. Then*

$$\nu(I_A) = (\beta_{0,\mathbf{b}_1}+1)^{\beta_{0,\mathbf{b}_1}-1}\cdots(\beta_{0,\mathbf{b}_q}+1)^{\beta_{0,\mathbf{b}_q}-1}.$$

Proof. By Proposition 2.4, the connected components of $G(\mathbf{b}_i)$ are singletons. It follows that $t_j(\mathbf{b}_i) = 1$ and that $n_{\mathbf{b}_i} = \sum t_j(\mathbf{b}_i)$. Moreover $\beta_{0,\mathbf{b}_i} = |\mathcal{F}_{\mathbf{b}_i}| = n_{\mathbf{b}_i} - 1$. $\square$

The next theorem provides a necessary and sufficient condition for a toric ideal to be generated by its indispensable binomials. It is a generalization of Corollary 2.1 in [14].

Theorem 2.12. *The ideal I_A is generated by its indispensable binomials if and only if the Betti A-degrees $\mathbf{b}_1,\ldots,\mathbf{b}_q$ of I_A are minimal binomial A-degrees and $\beta_{0,\mathbf{b}_i} = 1$.*

Proof. Suppose that I_A is generated by indispensable binomials. Then $\nu(I_A) = 1$ and therefore, from Theorem 2.9, $t_j(\mathbf{b}_i) = 1$ and $n_{\mathbf{b}_i} = 2$, for all j, i. Thus $\beta_{0,\mathbf{b}_i} = 1$. Now Proposition 2.4 together with the fact that $t_j(\mathbf{b}_i) = 1$ implies that all $\mathbf{b}_i$ are minimal binomial A-degrees. $\square$

We point out that the above theorem implies that in the case that a toric ideal I_A is generated by indispensable binomials, no two minimal generators can have the same A-degree. We compute $\nu(I_A)$ in the following example.

Example 2.13. Let $A = \{a_0 = k, a_1 = 1, \ldots, a_n = 1\} \subset \mathbb{N}$ be a set of $n+1$ natural numbers with $k > 1$ and $I_A \subset K[x_0, x_1, \ldots, x_n]$, the corresponding toric ideal. The ideal I_A is minimally generated by the binomials $x_0 - x_1^k, x_1 - x_2, \ldots, x_{n-1} - x_n$. The Betti A-degrees are $\mathbf{b}_1 = 1$ and $\mathbf{b}_2 = k$, while the A-graded Betti numbers are $\beta_{0,1} = n - 1$ and $\beta_{0,k} = 1$. Also $G(1)$ consists of n vertices, each one being a connected component, and $G(k)$ has two connected components, the singleton $\{x_0\}$ and the complete graph on the $\binom{k+n-1}{n-1}$ vertices $x_1^k, x_1^{k-1}x_2, \ldots, x_n^k$. Thus

$$\nu(I_A) = n^{n-2}\binom{k+n-1}{n-1}.$$

Example 2.14. Generic toric ideals were introduced in [12] by Peeva and Sturmfels. The term generic is justified due to a result of Barany and Scarf (see [2]) in integer programming theory which shows that, in a well defined sense, almost all toric ideals are generic. Given a vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the *support* of α, denoted by $\mathrm{supp}(\alpha)$, is the set $\{i \in \{1, \ldots, m\} \mid \alpha_i \neq 0\}$. For a monomial $\mathbf{x}^{\mathbf{u}}$ we define $\mathrm{supp}(\mathbf{x}^{\mathbf{u}}) := \mathrm{supp}(\mathbf{u})$. A toric ideal $I_A \subset K[x_1, \ldots, x_m]$ is called *generic* if it is minimally generated by binomials with full support, i.e.

$$I_A = (\mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r} - \mathbf{x}^{\mathbf{v}_r})$$

where $\mathrm{supp}(\mathbf{u}_i) \cup \mathrm{supp}(\mathbf{v}_i) = \{1, \ldots, m\}$ for every $i \in \{1, \ldots, r\}$. In [12] it was shown that the minimal binomial generating set of I_A is unique and thun I_A is generated by its indispensable binomials. This also follows as an easy application of Theorem 2.12.

3. The indispensable complex of a vector configuration

Consider a vector configuration $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in $\mathbb{Z}^n$ with $\mathbb{N}A$ pointed and the toric ideal $I_A \subset K[x_1, \ldots, x_m]$. In [11] it is proved that a binomial B is indispensable if and only if either B or $-B$ belongs to the reduced Gröbner base of I_A for any lexicographic term order on $K[x_1, \ldots, x_m]$. In [1] it is shown that a monomial M is indispensable if the reduced Gröbner base of I_A, with respect to any lexicographic term order on $K[x_1, \ldots, x_m]$, contains a binomial B such that M is a monomial of B. We are going to provide a more efficient way to check if a binomial is indispensable and, respectively, for a monomial. Namely we will give a criterion that provides the indispensable binomials and monomials with only the information from one specific generating set of I_A.

We let $\mathcal{M}_A$ be the monomial ideal generated by all $\mathbf{x}^{\mathbf{u}}$ for which there exists a nonzero $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$; in other words given a vector $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, the monomial $\mathbf{x}^{\mathbf{u}}$ belongs to $\mathcal{M}_A$ if and only if there exists $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{N}^m$ such that $\mathbf{v} \neq \mathbf{u}$, i.e. $v_i \neq u_i$ for some i, and $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$. We note that if $\{B_1 = \mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}, \ldots, B_s = \mathbf{x}^{\mathbf{u}_s} - \mathbf{x}^{\mathbf{v}_s}\}$ is a generating set of I_A, then $\mathcal{M}_A = (\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_s}, \mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_s})$. Let $T_A := \{M_1, \ldots, M_k\}$ be the unique minimal monomial generating set of $\mathcal{M}_A$.

Proposition 3.1. *The indispensable monomials of I_A are precisely the elements of T_A.*

Proof. First we will prove that the elements of T_A are indispensable monomials. Let $\{B_1, \dots, B_s\}$ be a minimal generating set of I_A. Set $M_j := \mathbf{x}^{\mathbf{u}}$ for $j \in \{1, \dots, k\}$. Since $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is in I_A for some $\mathbf{v}$, it follows that there is an $i \in \{1, \dots, s\}$ and a monomial N of B_i such that N divides $\mathbf{x}^{\mathbf{u}}$ and thus $\mathbf{x}^{\mathbf{u}} = N$.

Conversely consider an indispensable monomial $\mathbf{x}^{\mathbf{u}}$ of I_A and assume that it is not an element of T_A. Then $\mathbf{x}^{\mathbf{u}} = M_j \mathbf{x}^{\mathbf{c}}$ for some $j \in \{1, \dots, k\}$ and $\mathbf{c} \neq \mathbf{0}$. By our previous argument M_j is indispensable. Without loss of generality we may assume that $B_1 = M_j - \mathbf{x}^{\mathbf{z}}$. If $B_j = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, then

$$B'_j := \mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{z}} - \mathbf{x}^{\mathbf{v}} = B_j - \mathbf{x}^{\mathbf{c}} B_1 \in I_A$$

and therefore $I_A = (B_1, \dots, B_{j-1}, B'_j, B_{j+1}, \dots, B_s)$. This way we can eliminate $\mathbf{x}^{\mathbf{u}}$ from all the elements of the generating set of I_A, a contradiction to the fact that $\mathbf{x}^{\mathbf{u}}$ is indispensable. □

Definition 3.2. We define the indispensable complex $\Delta_{\mathrm{ind}(A)}$ to be the simplicial complex with vertices the elements of T_A and faces all subsets of T_A consisting of monomials with the same A-degree.

By Proposition 3.1 the indispensable monomials are the vertices of $\Delta_{\mathrm{ind}(A)}$. The connected components consist of the vertices of the same A-degree and are simplices of $\Delta_{\mathrm{ind}(A)}$; actually they are facets. Different connected components have different A-degrees. We compute $\Delta_{\mathrm{ind}(A)}$ in the following example.

Example 3.3. In Example 2.13 we have that $\mathcal{M}_A = (x_0, x_1, \dots, x_n)$ and also the facets of $\Delta_{\mathrm{ind}(A)}$ are $\{x_0\}$ and $\{x_1, \dots, x_n\}$.

It follows easily that whenever $\deg_A(\mathbf{x}^{\mathbf{u}})$ is a minimal binomial A-degree, then $\mathbf{x}^{\mathbf{u}} \in T_A$. The converse is not true in general. Indeed in Example 2.13, x_0 belongs to T_A but $\deg_A(x_0)$ is not minimal. Next we give a criterion that determines the indispensable binomials.

Theorem 3.4. *A binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is indispensable if and only if $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is a 1-dimensional facet of $\Delta_{\mathrm{ind}(A)}$ and $\deg_A(B)$ is a minimal binomial A-degree.*

Proof. Let $\mathbf{b} = \deg_A(B)$. Suppose that $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is a 1-dimensional facet of $\Delta_{\mathrm{ind}(A)}$ and $\mathbf{b}$ is a minimal binomial A-degree. By Proposition 2.4, minimality of $\mathbf{b}$ implies that the elements of $\deg_A^{-1}(\mathbf{b})$ which are the vertices of $G(\mathbf{b})$ are vertices of $\Delta_{\mathrm{ind}(A)}$ and the connected components of $G(\mathbf{b})$ are singletons. Since $\Delta_{\mathrm{ind}(A)}$ contains only two vertices of A-degree $\mathbf{b}$, $G(\mathbf{b})$ consists of two connected components, $\{\mathbf{x}^{\mathbf{u}}\}$ and $\{\mathbf{x}^{\mathbf{v}}\}$, and B is indispensable by Corollary 2.10. The other direction is done by reversing the implications. □

Theorem 3.4 shows that the toric ideal I_A of Example 2.13 has no indispensable binomials for $n > 2$. Indeed in this case the indispensable complex of I_A contains no 1-simplices that are facets.

We remark that to check the minimality of the A-degree $\mathbf{b}$ of the binomial $B \in I_A$, it is enough to compare $\mathbf{b}$ with the A-degrees of the vertices of $\Delta_{\mathrm{ind}(A)}$. Thus given any generating set of I_A, one can compute T_A and construct the simplicial complex $\Delta_{\mathrm{ind}(A)}$. The elements of T_A are the indispensable monomials and the 1-dimensional facets of $\Delta_{\mathrm{ind}(A)}$ of minimal binomial A-degree are the indispensable binomials.

Example 3.5. Let

$$A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}.$$

Using CoCoA [6], we see that I_A is minimally generated by $x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2$. Moreover $T_A = \{M_1 = x_1x_6,\ M_2 = x_2x_4,\ M_3 = x_3x_5,\ M_4 = x_3x_6^2,\ M_5 = x_4^2x_5,\ M_6 = x_2x_6^2,\ M_7 = x_4x_5^2,\ M_8 = x_3^2x_6,\ M_9 = x_1x_4^2,\ M_{10} = x_2^2x_6,\ M_{11} = x_1x_5^2,\ M_{12} = x_1^2x_5,\ M_{13} = x_2^2x_3,\ M_{14} = x_1^2x_4,\ M_{15} = x_2x_3^2, M_{16} = x_2x_3x_6,\ M_{17} = x_1x_4x_5\}$. It follows that $\Delta_{\mathrm{ind}(A)}$ is a simplicial complex on 17 vertices and its connected components are the facets

$$\{M_1, M_2, M_3\}, \{M_4, M_5\}, \{M_6, M_7\}, \{M_8, M_9\},$$
$$\{M_{10}, M_{11}\}, \{M_{12}, M_{13}\}, \{M_{14}, M_{15}\}, \{M_{16}, M_{17}\}.$$

The A-degrees of the components are accordingly

$$(2,2,2), (2,2,5), (1,4,4), (4,1,4),$$
$$(2,5,2), (4,4,1), (5,2,2), (3,3,3).$$

All of them are minimal binomial A-degrees and thus I_A has seven indispensable binomials corresponding to the 1-dimensional facets. We see that all nonzero A-graded Betti numbers equal 1, except from $\beta_{0,(2,2,2)}$, which equals 2. From Corollary 2.11 we take that $\nu(I_A) = 3$.

The next corollary gives a necessary condition for a toric ideal to be generated by the indispensable binomials.

Corollary 3.6. *Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be a vector configuration in $\mathbb{Z}^n$. If I_A is generated by the indispensable binomials, then every connected component of $\Delta_{\mathrm{ind}(A)}$ is* 1*-simplex.*

Proof. Let $\{B_1, \ldots, B_s\}$ be a minimal generating set of I_A consisting of indispensable binomials $B_i = \mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}$. We note that the monomials of the B_i are all indispensable and form T_A. Thus if a face of $\Delta_{\mathrm{ind}(A)}$ contains $\mathbf{x}^{\mathbf{u}_i}$, it also contains $\mathbf{x}^{\mathbf{v}_i}$. By Theorem 3.4, $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is a facet of $\Delta_{\mathrm{ind}(A)}$. ☐

The next example shows that the converse of Corollary 3.6 does not hold.

Example 3.7. We return to Example 2.3. The simplicial complex $\Delta_{\mathrm{ind}(A)}$ consists of only two 1-simplices $\{x_1x_2, x_3x_4\}$, $\{x_5x_6, x_7x_8\}$, the indispensable binomials are $x_1x_2 - x_3x_4$, $x_5x_6 - x_7x_8$ and $I_A \neq (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8)$.

When $\Delta_{\mathrm{ind}(A)}$ is a 1-simplex, the next proposition shows that I_A is principal and therefore generated by an indispensable binomial.

Proposition 3.8. *The simplicial complex $\Delta_{\mathrm{ind}(A)}$ is a* 1*-simplex if and only if I_A is a principal ideal.*

Proof. One direction of this proposition is trivial. For the converse assume that $\Delta_{\mathrm{ind}(A)} = \{\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{v}_1}\}$ and let $B_1 := \mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}$. We will show that $I_A = (B_1)$. Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ be the binomial of minimal binomial A-degree such that $B \in I_A \setminus (B_1)$. Since $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{u}_1}$ and $\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{d}}\mathbf{x}^{\mathbf{v}_1}$, where $\mathbf{x}^{\mathbf{c}} \neq \mathbf{x}^{\mathbf{d}}$, and none of them equals 1, we have that

$$\mathbf{x}^{\mathbf{v}_1}(\mathbf{x}^{\mathbf{c}} - \mathbf{x}^{\mathbf{d}}) = \mathbf{x}^{\mathbf{c}} B_1 - B.$$

Therefore $0 \neq \mathbf{x}^{\mathbf{c}} - \mathbf{x}^{\mathbf{d}} \in I_A$, while $\deg_A(\mathbf{x}^{\mathbf{c}}) \lneqq \deg_A(\mathbf{x}^{\mathbf{u}})$, a contradiction. ☐

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