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# ON WEIGHTED $L^2$ COHOMOLOGY

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ABSTRACT. Consider an orientable manifold with countably many complete components of bounded dimension. Suppose that its rational homology is infinitely generated in some degree. Then there is no choice of weight function for which the natural map from weighted $L^2$ cohomology to de Rham cohomology is surjective in that degree.

A weighted $L^2$ space on a Riemannian manifold $M$ is obtained by replacing the volume measure $dm$ by a measure of the form $\phi^2 dm$, where $\phi$ is a positive function referred to as a weight function. If $M$ is noncompact and $\phi$ is sufficiently unbounded above or unbounded away from zero, the domains of differential operators on these spaces may differ from those on the standard $L^2$ space. The use of such spaces in connection with partial differential equations has a long history. We are concerned here with a question in $L^2$ cohomology. There is a natural homomorphism from unreduced $L^2$ cohomology computed on the weighted spaces of differential forms, $H_\phi^*$, to de Rham cohomology $H_{dR}^*$. For which manifolds does there exist a weight function such that this map is an isomorphism? The first results of this sort were apparently proved by Borel [1] and Zucker [17]. Further positive results have been obtained by a number of authors. See in particular [5], [4], [16], [11].

There is a Hodge Laplacian $\Delta_\phi$ (which is $D_\phi^2$ as defined below). Elements of its kernel in degree $k$, $\mathcal{H}_\phi^k$, are called $\phi$-harmonic. There are injections $\mathcal{H}_\phi^k \to H_\phi^k$ which are isomorphisms if and only if $\Delta_\phi^k$ has closed range. Bueler formulated a general conjecture [5]: let $M$ be complete, connected, oriented and with Ricci curvature bounded below. Let $\phi$ be a fundamental solution of the scalar heat equation. Then $\Delta_\phi$ has closed range and $\mathcal{H}_\phi^*$ is isomorphic to $H_{dR}^*$. Carron [6] has recently found examples which disprove this conjecture and that do much more. Let $S$ be a compact orientable surface of genus $\geq 2$, let $\tilde{S}$ be an infinite cyclic covering, and let $T^{n-2}$ be a torus. Let $M = \tilde{S} \times T^{n-2}$. There is no $\phi$ such that the natural map $\mathcal{H}_\phi^k \to H_{dR}^k$ is surjective in any degree $k$ with $0 < k < n$. Therefore, either $\Delta_\phi^k$ does not have closed range or $H_\phi^k \to H_{dR}^k$ is not surjective. The present paper goes one step further. Let $M$ be any orientable manifold with countably many complete components of bounded dimension and let $H_k(M;\mathbb{R})$ be infinite dimensional in any degree $k$. Then there is no $\phi$ such that $H_\phi^k \to H_{dR}^k$ is surjective. No bounded geometry hypothesis is required. The proof is not related to Carron's. It is shown that $M$ may be replaced by a union of tubular neighborhoods of submanifolds

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representing a basis for $H_k(M;\mathbb{Q})$. In this situation the result amounts to the fact that $\ell^2$ does not contain all sequences of numbers. It is still possible that Bueler's conjecture holds, or that some other choice of weight produces an isomorphism, for manifolds with finitely generated homology. This paper makes essential use of ideas of Dodziuk [9, Section 3].

We describe the analytic framework. See [5] for more information on this material.

$M$ is an oriented Riemannian manifold with countably many complete components $M_p$ of bounded dimensions $n_p$. Let $n = \max\{n_p\}$.

$\Omega_c^k$ is the complex valued smooth $k$-forms on $M$ with compact support. If $\langle u, v\rangle$ is the standard pointwise inner product of $k$-forms, $(u,v) = \int_M \langle u, v\rangle\, dm$. The associated $L^2$ norm will be written $\|u\|$.

$\phi$ is a positive smooth weight function.

$\Omega_{\phi,c}^k$ is $\Omega_c^k$ with inner product $(u,v)_\phi = \int_M \langle u,v\rangle \phi^2 dm = (\phi u, \phi v)$ and norm $\|u\|_\phi$.

$d^k$ is the exterior derivative on $\Omega_c^k$, $\delta^{k+1}$ is its formal adjoint with respect to $(\cdot,\cdot)$, and $\delta_\phi^{k+1}$ is its formal adjoint with respect to $(\cdot,\cdot)_\phi$.

$D_\phi = d + \delta_\phi$ acting on $\Omega_{\phi,c}^*$.

The closure of an operator $T$ will be written as $\bar{T}$. The domain of $\overline{D}_\phi$, $\mathcal{D}\left(\overline{D}_\phi\right)$ is the completion of $\Omega_{\phi,c}^k$ for the graph norm $\|u\|_{D_\phi,\phi} = \|D_\phi u\|_\phi + \|u\|_\phi$. More generally if $r>0$ is an integer, $\mathcal{D}\left(\overline{D_\phi^r}\right)$ is the completion for $\|u\|_{D_\phi^r,\phi} = \left\|D_\phi^r u\right\|_\phi + \|u\|_\phi$.

Multiplication by $\phi$ gives a unitary $\phi : \Omega_{\phi,c}^* \to \Omega_c^*$, and $\phi$ induces a differential operator on $\Omega_c^*$ by $\tilde{D} = \phi D_\phi \phi^{-1}$. The tilde will be used generically for operators on ordinary forms produced in this way from operators on weighted forms. Suppose that $M$ has one component. Since $\tilde{D} = d + \delta +$ (zeroth order operator), $\tilde{D}^r$ is essentially selfadjoint by [8]. If $M$ has more than one component, $\tilde{D}^r = \bigoplus_p \tilde{D}_{M_p}^r$ acting on $\bigoplus_p \Omega_c^*(M_p) = \Omega_c^*$. By [14, Ex. 5.43], $\tilde{D}^r$ is essentially selfadjoint since all the $\tilde{D}_{M_p}^r$ are. Evidently $\tilde{D}^r = \phi D_\phi^r \phi^{-1}$. It follows that $\phi$ induces a unitary equivalence $\overline{D_\phi^r} \to \overline{\tilde{D}^r}$. Therefore $D_\phi^r$ is also essentially selfadjoint. In particular $\left(\bar{D}_\phi\right)^r = \overline{D_\phi^r}$.

The (unreduced) $\phi$-cohomology of $M$ is defined as follows. The closures are taken with respect to $\|\cdot\|_\phi$. Let

$$Z_\phi^k = \left\{u \in \mathcal{D}\left(\overline{d^k}\right) | \overline{d^k} u = 0\right\},\ B_\phi^k = \left\{\overline{d^{k-1}} v | v \in \mathcal{D}\left(\overline{d^{k-1}}\right)\right\}.$$

Then $H_\phi^k(M) = Z_\phi^k / B_\phi^k$. Each cohomology class has representatives which are $C^\infty$. Let $\mathcal{D}^\infty = \bigcap_r \mathcal{D}\left(\bar{D}_\phi^r\right)$, which consists of smooth forms. The inclusion $\left(\mathcal{D}^\infty, \bar{d}\right) \to \left(\mathcal{D}\left(\bar{d}\right), \bar{d}\right)$ induces an isomorphism of cohomology [3, Th. 2.12]. Thus there is a homomorphism $S^k : H_\phi^k(M) \to H_{dR}^k(M)$, where the latter group is the de Rham cohomology of $M$ based on smooth forms. For now, closures will be understood and the bar will be suppressed. Below $H_k(M)$ denotes homology with real coefficients.

**Theorem 1.** *Suppose that $H_k(M)$ is infinite dimensional. Then there is no $\phi$ such that $S^k$ is surjective.*

In particular, cohomology is never represented by $\phi$-harmonic forms. The proof will be by reductio ad absurdum. Thus assume that $S^k$ is surjective.

We claim that we may assume that $2k < n_p$ for all $p$. This is accomplished by taking the product of $M$ with a suitably weighted Euclidean space. The details will be given at the end of the paper.

The homology of $M$ is certainly countably generated. Let $\{\gamma_i\}, i \in \mathbb{N}$, be a basis for $H_k(M;\mathbb{Q}) \subset H_k(M)$ which restricts to a basis of each component. By [13, Th. III.4], there are closed oriented connected $k$-dimensional manifolds $N_i$, maps $g_i : N_i \to M$, and positive integers $m_i$ such that $m_i\gamma_i$ is the image of the fundamental class of $N_i$. (The statement of the cited theorem assumes that the space is a finite polyhedron. We may triangulate $M$ and use the fact that its homology is the direct limit of the homologies of its finite subcomplexes.) We redefine $\gamma_i$ to be $m_i\gamma_i$, so that the image is $\gamma_i$.

**Lemma 2.** *The map $\coprod_i g_i$ is homotopic to an injective smooth map $\coprod_i f_i$ such that the $f_i(N_i)$ have disjoint closed tubular neighborhoods $V_i$.*

*Proof.* Since $2k < n_p$, for all $p$, by Whitney's Embedding Theorem [15], $g_1$ is homotopic to an embedding. Let $V_1$ be any closed tubular neighborhood. Assume that $f_i$ and $V_i$ have been constructed for $i < \ell$. By transversality we can make $g_\ell(N_\ell)$ disjoint from $f_i(N_i)$ and then push it off $V_i$, for $i < \ell$. By the cited theorem, $g_\ell$ is homotopic to an embedding $f_\ell$ in $M - \bigcup_{i=1}^{\ell-1} V_i$. Let $V_\ell$ be any closed tubular neighborhood of $f_\ell(N_\ell)$ in this manifold. □

Let the space of restrictions of elements of $\mathcal{D}\left(D^r_\phi\right)$ to $V_i$ be $\mathcal{D}\left(D^r_\phi\right)_{V_i}$. It has the norm $\|u\|_{D^r_\phi,\phi,i}$ which is the same as $\|u\|_{D^r_\phi,\phi}$ except that the integral is evaluated on $V_i$. The restrictions $q_i : \mathcal{D}\left(D^r_\phi\right) \to \mathcal{D}\left(D^r_\phi\right)_{V_i}$ clearly have norm 1. Since the $V_i$ are disjoint, $\|u\|^2_{D^r_\phi,\phi} \geq \sum_i \|q_i u\|^2_{D^r_\phi,\phi,i}$. Therefore there is a bounded operator $r : \mathcal{D}\left(D^r_\phi\right) \to \widehat{\bigoplus}_i \mathcal{D}\left(D^r_\phi\right)_{V_i}$ (the Hilbert sum), $u \to \sum_i q_i u$ [14, Ex. 5.43].

We will use some properties of Sobolev spaces of $k$-forms. The basic objects are the local spaces $W^k_{r,loc}(M)$. For a compact codimension zero submanifold with boundary $W$ of $M$, $W^k_r(W)$ is the space of restrictions of elements of $W^k_{r,loc}(M)$ to $W$. Since $W^k_0(W)$ is just the $L^2$ space of forms, the norm will be written as $\|u\|_W$. See [7] or [10] for background information.

By local elliptic regularity $\mathcal{D}\left(\tilde{D}^r\right) \subset W^*_{r,loc}(M)$, so $\mathcal{D}\left(D^r_\phi\right) \subset \phi^{-1}W^*_{r,loc}(M)$. Essentially by definition, $\phi^{-1}W^*_{r,loc}(M) = W^*_{r,loc}(M)$, so $\mathcal{D}\left(D^r_\phi\right) \subset W^*_{r,loc}(M)$. Restricting to $V_i$, if $u \in \mathcal{D}\left(D^r_\phi\right)$, then $q_i u \in W^*_r(V_i)$. Since $\phi$ is bounded away from zero on $V_i$, the unweighted and weighted graph norms satisfy

$$\|u\|_{D^r_\phi} \leq L_i \|u\|_{D^r_\phi,\phi,i} \tag{1}$$

on $\mathcal{D}\left(D^r_\phi\right)_{V_i}$ for some constants $L_i$. Now let $U_i$ be a closed tubular neighborhood of $N_i$ contained in the interior of $V_i$. The following elliptic estimate may be found in [12, Th. 5.11.1]. Let $T$ be an elliptic operator of order $r$ on $M$. Then there is a constant $K_i$ such that for all $u \in W^*_r(V_i)$,

$$\|u\|_{W^*_r(U_i)} \leq K_i \left(\|Tu\|_{V_i} + \|u\|_{V_i}\right).$$

Taking $T = D^r_\phi$, we interpret this as saying that restriction induces a bounded operator from $\mathcal{D}\left(D^r_\phi\right)_{V_i}$ with the norm $\|u\|_{D^r_\phi,i}$ to $W^*_r(U_i)$. Combining this with (1), restriction from $\mathcal{D}\left(D^r_\phi\right)_{V_i}$ with the norm $\|u\|_{D^r_\phi,\phi,i}$ to $W^*_r(U_i)$ has the bound $L_iK_i$. Of course, these bounds depend on $i$.

The final step is to evaluate forms on the fundamental classes of the $N_i$. Choose $r > \frac{n}{2}$ so that by Sobolev's Theorem $W^k_r(U_i)$ is continuously embedded in the $C^0$ forms. Define a linear functional $c_i$ on $W^k_r(U_i)$ by $c_i(u) = \int_{N_i} u$.

$$|c_i(u)| = \left|\int_{N_i} u\right| \leq \underset{x\in N_i}{Sup}\,|u(x)|\,Vol(N_i) \leq C_i\,\|u\|_{W^k_r(U_i)}$$

for some constants $C_i$, so that $c_i$ is bounded. Then the composition from $u \in \mathcal{D}\left(D^r_\phi\right)_{V_i}$ to $c_i$ has norm less than or equal to $L_iK_iC_i$. Let $\tau_i = (L_iK_iC_i)^{-1}$and let $\mathbb{C}_{\tau_i}$ be $\mathbb{C}$ with the norm $\|z\|_{\tau_i} = \tau_i\,|z|$. Then the map $\widehat{\bigoplus}_i\,\mathcal{D}\left(D^r_\phi\right)_{V_i} \to \widehat{\bigoplus}_i\mathbb{C}_{\tau_i}$, $(u_i) \to (c_i(u_i))$ has norm $\leq 1$. Composing with $r$ gives a bounded map $\mathcal{D}\left(D^r_\phi\right) \to \widehat{\bigoplus}_i\mathbb{C}_{\tau_i}$.

The restriction $H^k_{dR}(M) \to H^k_{dR}\left(\bigcup_i U_i\right)$ is surjective, being the (complexified) transpose of the injection $H_k\left(\bigcup_i U_i\right) \to H_k(M)$. Thus a set of forms in $\mathcal{D}^\infty$ representing all of $H^k_{dR}(M)$ would restrict to a set representing all of $H^k_{dR}\left(\bigcup_i U_i\right)$. Evaluation of elements of this group on the $N_i$ is well-defined, by Stokes's Theorem, and gives an isomorphism with $\prod_i \mathbb{C}_i$. Therefore $\widehat{\bigoplus}_i\mathbb{C}_{\tau_i}$ would contain *all* sequences of complex numbers, which is impossible. In fact, it does not contain $\left(\tau_i^{-1}\right)$. This completes the proof of Theorem 1 under the assumption $2k < n_p$ for all $p$.

We now justify this assumption. Bars will again denote closures. We will form the product of $M$ with weight $\phi$ and $\mathbb{R}^{2k+1}$ with its usual metric and a particular weight $\psi$. This choice satisfies the dimensional requirement. Consider the general situation: manifolds $M_1$, $M_2$ of dimensions $n_1$ and $n_2$ with weights $\phi$ and $\psi$. Equip $M_1 \times M_2$ with the weight $\phi \otimes \psi$. Let $d$ be the exterior derivative of $M_1 \times M_2$ acting on $\Omega^k_{\phi\otimes\psi,c}(M_1 \times M_2)$. The spaces $\mathcal{E}^k = \bigoplus_{i+j=k} \Omega^i_{\phi,c}(M_1) \otimes \Omega^j_{\psi,c}(M_2)$ embed isometrically into $\Omega^k_{\phi\otimes\psi,c}(M_1 \times M_2)$ using the isomorphism of exterior algebras $\Lambda_{n_1}\hat{\otimes}\Lambda_{n_2} \cong \Lambda_{n_1+n_2}$, $\hat{\otimes}$ the graded tensor product. Denote the exterior derivatives of $M_1$, $M_2$, and $M_1 \times M_2$ by $d_1$, $d_2$, and $d$. The restriction of $d$ to $\mathcal{E}^k$ is given by $\bigoplus_{i+j=k}\left(d^i_1 \otimes I + (-1)^i\,I \otimes d^j_2\right)$. There is a homomorphism $H^i_\phi(M_1) \otimes H^j_\psi(M_2) \to H^k_{\phi\otimes\psi}(M_1 \times M_2)$. Choose representatives $u$, $v$ of given cohomology classes which are in $\mathcal{D}^\infty$. We then check directly that $u \otimes v$ is a smooth closed form in $\mathcal{D}\left(\bar{d}\right)$. Its class in $H^k_{\phi\otimes\psi}(M_1 \times M_2)$ is independent of the choices. The following diagram is obviously commutative:

$$\begin{array}{ccc}
\bigoplus_{i+j=k}\left(H^i_\phi(M_1) \otimes H^j_\psi(M_2)\right) & \xrightarrow{\oplus\left(S^i_1\otimes S^j_2\right)} & \bigoplus_{i+j=k}\left(H^i_{dR}(M_1) \otimes H^j_{dR}(M_2)\right) \\
\downarrow & & \downarrow \\
H^k_{\phi\otimes\psi}(M_1 \times M_2) & \xrightarrow{S} & H^k_{dR}(M_1 \times M_2).
\end{array}$$

On the right we use any smooth representatives of the classes. From [2, Prop. II.9.12] we have that the right arrow is an isomorphism provided that $H^*_{dR}(M_2)$

is finitely generated. (These authors use a description of the map equivalent to $\pi_1^* u \wedge \pi_2^* v$.) (The left arrow is also an isomorphism if $H_\psi^*(M_2)$ is finitely generated [3, Th. 2.14], but this is not needed.)

Let $M_1 = M$ and $M_2 = \mathbb{R}^{2k+1}$. Let $\psi = e^{-|x|}$, smoothed near the origin. An application of [16, Th. 3.1, Rem. p. 161] shows that $s_2$ is an isomorphism. The diagram reduces to

$$\begin{array}{ccc} H_\phi^k(M) & \xrightarrow{s_1^k} & H_{dR}^k(M) \\ \downarrow & & \downarrow\cong \\ H_{\phi\otimes\psi}^k\left(M \times \mathbb{R}^{2k+1}\right) & \xrightarrow{s^k} & H_{dR}^k\left(M \times \mathbb{R}^{2k+1}\right). \end{array}$$

If the top arrow were surjective, the bottom one would be as well, contradicting the statement already proved.

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