THE KAC JORDAN SUPERALGEBRA:
AUTOMORPHISMS AND MAXIMAL SUBALGEBRAS

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Abstract. In this note the group of automorphisms of the Kac Jordan superalgebra is described and used to classify the maximal subalgebras.

1. Introduction

Finite dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero were classified by V. Kac in 1977 [9] (see also Kantor [10], where a missing case is added). Among these superalgebras we find the ten dimensional Kac Jordan superalgebra, $K_{10}$, which is exceptional (see [12] and [17]) and plays a significant role (see [13]).

Here we are interested in describing the group of automorphisms of $K_{10}$ and, as a consequence, in classifying the maximal subalgebras of the Kac superalgebra over an algebraically closed field of characteristic zero.

In two previous works (see [6], [7]), the authors have given a description of maximal subalgebras of finite dimensional central simple superalgebras which are either associative or associative with superinvolution and also a description of maximal subalgebras of finite dimensional special simple Jordan superalgebras over an algebraically closed field of characteristic zero. These papers are thus in the spirit of previous work by E. Dynkin ([3], [4]), M. Racine ([14], [15]) or A. Elduque ([5]) on maximal subalgebras of different classes of algebras.

Let $F$ be a field of characteristic not two. This assumption will be kept throughout the paper. Recall that a superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ ($A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ $\forall \alpha, \beta \in \mathbb{Z}_2$). If $a \in A_\alpha$ we say that $a$ is a homogeneous element and we use the notation $\bar{a} = \alpha$. Elements belonging to $A_0$ are called even elements and the ones in $A_1$ odd elements. A superalgebra $A$ is said to be nontrivial if $A_1 \neq 0$, and it is called simple if it contains no proper nontrivial graded ideals and $A^2 \neq 0$. 
A Jordan algebra is an algebra satisfying the following identities:

\begin{align*}
(1) & \quad xy = yx, \\
(2) & \quad (x^2 y)x = x^2(yx).
\end{align*}

The last identity can be written as \((x^2, y, x) = 0\), where \((x, y, z)\) denotes the associator \((xy)z - x(yz)\). Following the standard procedure, a superalgebra is a **Jordan superalgebra** if its Grassmann envelope is a Jordan algebra. In particular, a Jordan superalgebra satisfies the identities

\begin{align*}
(3) & \quad xy = (-1)^{x_0y_0}yx, \\
(4) & \quad \sum_{cyclic \ x,y,z} (-1)^{(x_0+z_0+1)t}(xy, z, t) = 0
\end{align*}

for homogeneous elements \(x, y, z, t\). (If the characteristic is not 2, 3, these two identities characterize Jordan superalgebras.)

The even part of a Jordan superalgebra is a Jordan algebra, while the odd part is a Jordan bimodule for the even part.

Associative superalgebras are just \(\mathbb{Z}_2\)-graded associative algebras, but note that Jordan superalgebras are not\(\mathbb{Z}_2\)-graded Jordan algebras.

The following examples of Jordan algebras will be useful in the sequel:

i) Let \(A\) be an associative algebra over a ground field of characteristic \(\neq 2\).

The new operation \(a \cdot b = \frac{1}{2}(ab + ba)\) defines a structure of a Jordan algebra on \(A\). It is denoted by \(A^+\).

ii) Let \(V\) be a vector space over \(F\) with a nondegenerate symmetric bilinear form \((\ ,\ ) : V \times V \rightarrow F\). The direct sum \(F1 + V\) with the product \((\lambda 1 + v)(\mu 1 + w) = (\lambda \mu + (v, w))1 + (\lambda w + \mu v)\) is a Jordan algebra, and it is called the Jordan algebra of a nondegenerate bilinear form.

Simple finite dimensional Jordan algebras over an algebraically closed field were classified by P. Jordan, J. von Neumann, E. Wigner ([8]) and by A. Albert ([I]). The examples given above are two of the four types of algebras given in that classification.

Also the following examples of Jordan superalgebras will be needed later on:

1) \(J = K_3\), the **Kaplansky superalgebra**:

\[
\begin{align*}
J_0 &= Fe, & J_1 &= Fx + Fy, \\
e^2 &= e, & e \cdot x &= \frac{1}{2}x, & e \cdot y &= \frac{1}{2}y, & x \cdot y &= e.
\end{align*}
\]

2) The family of superalgebras \(J = D_t\), with \(t \in F \setminus \{0\}\):

\[
\begin{align*}
J_0 &= Fe + Ff, & J_1 &= Fu + Fv, \\
e^2 &= e, & f^2 &= f, & e \cdot f &= 0, & e \cdot u &= \frac{1}{2}u, & e \cdot v &= \frac{1}{2}v, & f \cdot u &= \frac{1}{2}u, \\
f \cdot v &= \frac{1}{2}v, & u \cdot v &= e + tf.
\end{align*}
\]

3) \(J = K_{10}\), the **Kac superalgebra**, which will be described in more detail in the following section.

4) The **superalgebra of a superform**: Let \(V = V_0 \oplus V_1\) be a graded vector space over \(F\), and let \((\ ,\ )\) be a nondegenerate supersymmetric bilinear superform on \(V\), that is, a nondegenerate bilinear map that is symmetric on \(V_0\), skewsymmetric on \(V_1\), and \(V_0, V_1\) are orthogonal relative to \((\ ,\ )\). Now consider \(J_0 = Fe + V_0, J_1 = V_1\) with \(e \cdot x = x, v \cdot w = (v, w)e\) for every \(x \in J, v, w \in V\). This superalgebra, \(J\), is called the superalgebra of a superform.
5) $A^+$, with $A$ an associative superalgebra over a field of characteristic not 2, where the product operation in $A$ has been changed to $a_i \cdot b_j = \frac{1}{2}(a_i b_j + (-1)^{i+j} b_j a_i)$.

As we have mentioned at the beginning of this Introduction, V. Kac and I. Kantor gave the classification of nontrivial simple Jordan superalgebras. The four examples given above are also examples of simple Jordan superalgebras, assuming in 3) that the characteristic is not 3, and in 4) that $A$ is a simple associative superalgebra (see [18]).

In this paper, our purpose is to describe the group of automorphisms of $K_{10}$ and also to classify its maximal subalgebras.

2. Kac superalgebra

The Kac superalgebra, $K_{10} = J_0 \oplus J_1$, is a Jordan superalgebra with 6-dimensional even part and 4-dimensional odd part:

$$J_0 = (Fe + Fa + Fb + \sum_{i=1}^{2} Fc_i) \oplus Ff,$$
$$J_1 = \sum_{i=1}^{2} (Fp_i + Fq_i)$$

and product given in Table 1.

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$\bar{p}_1$ = $\frac{1}{2}p_1$, $p_1$ = $q_1$, $q_1$ = $2e - 3f$, $2c_1$, $b$ | $p_2$ = $\frac{1}{2}p_2$, $p_2$ = $q_2$, $q_2$ = $-2e - 3f$, $-b$, $-2c_2$ | $q_1$ = $\frac{1}{2}q_1$, $q_1$ = $-p_1$, $p_1$ = $\frac{1}{2}q_1$, $-2c_1$, $b$, $0$, $a - 2e - 3f$, $0$ | $q_2$ = $\frac{1}{2}q_2$, $q_2$ = $-p_2$, $p_1$ = $\frac{1}{2}q_2$, $-b$, $2c_2$, $-a - 2e - 3f$, $0$ |

This basis of $K_{10}$ is obtained by using the one in [16], p. 378, with $a = v_1 + v_2$, $b = v_1 - v_2$, $c_1 = v_3$, $c_2 = v_4$, $p_1 = x_1 - y_2$, $p_2 = x_2 + y_1$, $q_1 = x_1 + y_2$, $q_2 = x_2 - y_1$.

From [11] we know that $J_0 = J(V, Q) \oplus Ff$, where $J(V, Q)$ is the Jordan algebra of a nondegenerate bilinear form $Q$, $e$ is the identity element in $J(V, Q)$, and $V$ is the vector space with basis $\{a, b, c_1, c_2\}$. We will use this presentation in Section 4.

In [2] G. Benkart and A. Elduque gave a realization of $K_{10}$ which enables us to check directly that it is indeed a Jordan superalgebra. We introduce this construction, which will be very useful for us in Section 3.

First we pick up the Kaplansky superalgebra, $K_3$, over the field $F$ and we define the following supersymmetric bilinear form (that is, the even and odd parts are
orthogonal and the form is symmetric in the even part and alternating in the odd part):

\[(e|e) = \frac{1}{2}, \quad (x|y) = 1, \quad ((K_3)_0| (K_3)_1) = 0.\]

Consider now the vector space over \(F\): \(F \cdot 1 \oplus (K_3 \otimes_K K_3)\) and define on it the product:

\[(a \otimes b)(c \otimes d) = (-1)^{bc}(ac \otimes bd - \frac{3}{4}(a|c)(b|d)1)\]

with \(a, b, c, d \in K_3\) homogeneous elements, and where 1 is a formal identity element. Then, \(K_{10}\) is isomorphic to \(F \cdot 1 \oplus (K_3 \otimes K_3)\) by means of the linear map given by:

\[
\begin{align*}
    e & \mapsto \frac{3}{2} \cdot 1 - 2e \otimes e, \\
    f & \mapsto -\frac{1}{2} \cdot 1 + 2e \otimes e, \\
    a & \mapsto -4x \otimes x - y \otimes y, \\
    b & \mapsto -4x \otimes x + y \otimes y, \\
    c_1 & \mapsto 2x \otimes y, \\
    c_2 & \mapsto -2y \otimes x, \\
    p_1 & \mapsto 4x \otimes e - 2e \otimes y, \\
    p_2 & \mapsto -4e \otimes x - 2y \otimes e, \\
    q_1 & \mapsto 4x \otimes e + 2e \otimes y, \\
    q_2 & \mapsto -4e \otimes x + 2y \otimes e.
\end{align*}
\]

If we denote by \(W\) the vector space \((K_3)_1\), we notice that \(V = W \otimes W\) is endowed with the bilinear nondegenerate symmetric form given by \(b(s \otimes t, u \otimes v) = (s|u)(t|v)\). We note also that the corresponding decomposition of \((K_{10})_0\) into the direct sum of two simple ideals according to the one given above is \((K_{10})_0 = I \oplus J\), where \(I = F \cdot (-1/2 + 2e \otimes e)\) and \(J = F \cdot (3/2 - 2e \otimes e) \oplus V\).

3. AUTOMORPHISMS OF \(K_{10}\)

Here by automorphism we mean automorphism of graded algebras.

It is easy to compute the group of automorphisms of the examples 1), 2) and 4) of Jordan superalgebras given in Section 1. So, with the notation of Section 2, \(\text{Aut}(K_3) = \text{Sp}(W)\), the symplectic group (or also \(\text{Aut}(K_3) = \text{SL}(W)\), the special linear group). In addition, \(\text{Aut}(D_1) = \text{Sp}(U)\), where \(U\) is the vector space generated by \(\{u,v\}\). If \(J\) is the superalgebra of a superform, then \(\text{Aut}(J) = O(n,F) \oplus \text{Sp}(m,F)\), where \(O(n,F)\) denotes the orthogonal group of \(V_0\) with \(\dim V_0 = n\), related to \(\ell|V_0 \times V_0\), and \(\text{Sp}(m,F)\) denotes the symplectic group of \(V_1\), with \(\dim V_1 = m\), related to \(\ell|V_1 \times V_1\).

In what follows in this section it will be assumed that \(F\) is a field of characteristic \(\neq 2,3\) such that \(F^2 = F\).

Let \(C_2 = \{1,e\}\) be the cyclic group of order 2 \((e^2 = 1)\) and consider the wreath product \(G = \text{Sp}(W) \rtimes C_2\) (that is, \(G\) is the semidirect product \((\text{Sp}(W) \times \text{Sp}(W)) \rtimes C_2\) and \((f,g)e = e(g,f)\) for any \(f,g \in \text{Sp}(W)\)).
The following maps are clearly group homomorphisms:

\[ \Psi : \text{Aut}(K_{10}) \rightarrow O(V, b) \]
\[ \varphi \mapsto \varphi|V \]

\[ \tilde{\Psi} : G \rightarrow O(V, b) \]
\[ (f, g) \mapsto f \otimes g : \rightarrow W \otimes W \]
\[ s \otimes t \mapsto f(s) \otimes g(t) \]
\[ \epsilon \mapsto \hat{\epsilon} : W \otimes W \rightarrow W \otimes W \]
\[ s \otimes t \mapsto -t \otimes s \]

\[ \Phi : G \rightarrow \text{Aut}(K_{10}) \]
\[ (f, g) \mapsto \Phi(f, g) : \left\{ \begin{array}{ll}
1 & \mapsto 1 \\
a \otimes b & \mapsto \tilde{f}(a) \otimes \tilde{g}(b)
\end{array} \right. \]
\[ \epsilon \mapsto \delta : \left\{ \begin{array}{ll}
1 & \mapsto 1 \\
a \otimes b & \mapsto (-1)^{a} b \otimes a
\end{array} \right. \]

where for any \( f \in Sp(W) \), \( \tilde{f} \) denotes the automorphism of \( K_{3} \) such that \( \tilde{f}(e) = e \), \( f(w) = f(w) \forall w \in W \).

Note that \( \Psi \circ \Phi = \tilde{\Psi} \).

**Lemma 3.1.** \( \tilde{\Psi} \) is onto with \( \text{Ker} \tilde{\Psi} = \{ \pm (id, id) \} \).

**Proof.** Let \( \{u, v\} \) be a symplectic basis of \( W \) (so \( (u \mid v) = 1 \)). Then if \( \alpha = u \otimes s + v \otimes t \) is an isotropic vector in \( V \) (that is, \( b(\alpha, \alpha) = 0 \)), then \( 0 = b(\alpha, \alpha) = (s \mid t) \), so \( s, t \) are linearly dependent. This shows that

\[ \{\text{isotropic vectors of } V\} = \{s \otimes t \mid s, t \in W\}. \]

Now let us show that \( \tilde{\Psi} \) is onto. For any \( \varphi \in O(V, b), \varphi(u \otimes u) \) is isotropic \( (u \otimes u) \) is, so there are \( s, t \in W \) with \( \varphi(u \otimes u) = s \otimes t \neq 0 \).

Take \( f, g \in Sp(W) \) with \( f(s) = u, g(t) = u \). Hence \( \varphi \in \text{im} \tilde{\Psi} \) if and only if

\[ (f \otimes g) \circ \varphi = \tilde{\Psi}(f, g) \circ \varphi \in \text{im} \tilde{\Psi}, \]

and, therefore, it can be assumed that \( \varphi(u \otimes u) = u \otimes u \).

Now \( \varphi(v \otimes u) = s \otimes t \) for some \( s, t \in W \) with

\[ 1 = b(u \otimes u, v \otimes v) = b(\varphi(u \otimes u), \varphi(v \otimes v)) = b(u \otimes u, s \otimes t) = (u \mid s)(u \mid t), \]

and there exists \( 0 \neq \alpha \in F \) such that \( (u \mid s) = \alpha, (u \mid t) = \alpha^{-1} \) or \( (u \mid \alpha^{-1}s) = 1, (u \mid \alpha t) = 1 \). Thus there exist \( f, g \in Sp(W) \) such that \( f(u) = u, f(v) = \alpha^{-1}s, g(u) = u, g(v) = \alpha t \), and changing \( \varphi \) to \( (f^{-1} \otimes g^{-1}) \circ \varphi \) we may assume that \( \varphi(u \otimes u) = u \otimes u, \varphi(v \otimes v) = v \otimes v \).

Then \( \varphi(u \otimes v) \) is isotropic and orthogonal to both \( u \otimes u \) and \( v \otimes v \), so it is a scalar multiple of either \( u \otimes v \) or \( v \otimes u \). By using \( \tilde{\Psi}(e) = \hat{\epsilon} \) we may assume that \( \varphi(u \otimes v) = \gamma u \otimes v \) for some \( 0 \neq \gamma \in F \) and then, necessarily, \( \varphi(v \otimes u) = \gamma^{-1} v \otimes u \). Let \( \mu \in F = F^{2} \) with \( \mu^{2} = \gamma \). Then \( \varphi = f \otimes g \) where \( f(u) = \mu u, f(v) = \mu^{-1} v, g(u) = \mu^{-1} u, g(v) = \mu v \), so that \( \varphi \in \text{im} \tilde{\Psi} \).

The assertion about the kernel is clear. \( \square \)
Lemma 3.2. $\Psi$ is onto with $\text{Ker} \Psi = \{id, \tau\}$, where $\tau$ is the grading automorphism $(\tau(z) = (-1)^z z)$ for any homogeneous $z \in K_{10}$.

Proof. Since $\Psi \circ \Phi = \hat{\Psi}$ and $\hat{\Psi}$ is onto, so is $\Psi$. Moreover, if $\varphi \in \text{Aut}(K_{10})$ belongs to the kernel of $\Psi$, then $\varphi([K_{10}]_0) = Id$, so $\varphi(z_0 \cdot z_1) = z_0 \cdot \varphi(z_1)$ for any $z_0 \in ([K_{10}]_0)$ and $z_1 \in (K_{10})_{1}$.

For any $s \in W$, $s \otimes e$ is, up to scalars, the unique odd element annihilated by $s \otimes W (\subseteq (K_{10})_{0})$; hence $\varphi(s \otimes e) = \beta s \otimes e$ for some $0 \neq \beta \in F$.

Take $t \in W$ with $(s \mid t) = 1$. Then $\forall z \in W$, $(t \otimes z) \cdot (s \otimes e) = \frac{1}{2} e \otimes z$, so $\varphi(e \otimes z) = 2(t \otimes z) \cdot \varphi(s \otimes e) = \beta (e \otimes z)$ for any $z \in W$. It follows that $\varphi([K_{10}]_1) = \varphi(e \otimes W \oplus W \otimes e) = \beta 1d$ and, since $\varphi$ is an automorphism, $\beta^2 = 1$, so $\varphi$ is either the identity or the grading automorphism $\tau$. \hfill \Box

Theorem 3.3. $\Phi$ is an isomorphism.

Proof. Since $\Phi(-id, -id) = \tau$, $\Phi(\text{Ker} \hat{\Psi}) = \text{Ker} \Psi$. Now the assertion follows from the fact that $\Psi \circ \Phi = \hat{\Psi}$, together with Lemmata 1 and 2. \hfill \Box

4. Maximal subalgebras of $K_{10}$

In what follows the word subalgebra will be used in the graded sense, so any subalgebra is graded. Also in this paragraph we consider that $K_{10}$ is a superalgebra over $F$, an algebraically closed field of characteristic $\neq 2, 3$.

Theorem 4.1. Any maximal subalgebra of $K_{10}$ is, up to an automorphism of $K_{10}$, one of the following:

(i) $(K_{10})_0$.

(ii) The subalgebra with basis $\{e, f, a, p_1, p_2\}$.

(iii) The subalgebra with basis $\{e, f, a + b, c_1, p_1, q_1, p_2 + q_2\}$.

(iv) The subalgebra with basis $\{e, f, a, b, c_1, p_1\}$.

Proof. Note that if $B$ is a maximal subalgebra of $K_{10}$, $1 = e + f \in B$, as $F1 + B$ is a subalgebra of $K_{10}$ and $B$ is an ideal in $F1 + B$. If $f \notin B$, then $B_0 \oplus Ff = \pi_I(B_0) \oplus \pi_J(B_0)$, which is a subalgebra of $(K_{10})_0$ (recall that $(K_{10})_0$ is the direct sum of the ideals $I$ and $J = Ff$; here $\pi_I$ and $\pi_J$ denote the corresponding projections), and so the subalgebra generated by $B$ and $f$ equals $B \oplus Ff$. Since $B$ is maximal, $B \oplus Ff = K_{10}$, so, in particular, $B_1 = (K_{10})_1$, but $(K_{10})_1$ generates $K_{10}$, so $B = K_{10}$, a contradiction. Therefore $f \in B$, and hence $e = 1 - f \in B$ too.

We now use the description of $K_{10}$ given at the beginning of Section 2 and due to D. King. Then $B_0 = Fe + Ff + V_0$, with $V_0$ a vector subspace of $V$ and so, in order to study the maximal subalgebras of $K_{10}$, we analyze the possible dimensions of $V_0$.

1. If $V_0 = V$, then $B = (K_{10})_0$ is a maximal subalgebra of $K_{10}$.

2. If $\dim V_0 = 1$, $Q$ is either $2$ or $0$.

If $\dim(Q|_{V_0}) = 1$, using Witt’s Theorem and Theorem 3.3 we can suppose that $V_0 = Fa$. Now $B_1 = P_B + Q_B$ with $P_B = \{p \in B : pa = p\}$, $Q_B = \{q \in B : qa = -q\}$. If $\dim B_2 = 2$ we can check that $B = \{e, f, a, p_1, p_2\}$ is a maximal subalgebra of $K_{10}$. Likewise if $\dim Q_B = 2$, $B = \{e, f, a, q_1, q_2\}$ is a maximal subalgebra of $K_{10}$, but there is an automorphism of $K_{10}$ applying $\{e, f, a, p_1, p_2\}$ into $\{e, f, a, q_1, q_2\}$ (the automorphism given by the isometry that applies $a$ to $-a$ and fixes $b, c_1, c_2$). If $\dim P_B = \dim Q_B = 1$, then we can check that $\dim V_0 > 1$, a contradiction.
$P_B = 0$, then $B \subseteq \langle e, f, a, q_1, q_2 \rangle$, which is a maximal subalgebra, and we are in the case above.

If rank$(Q|_{V_0}) = 0$, using Witt’s Theorem and Theorem 3.3 we can suppose that $V_0 = Fc_1$. But $c_1B_1 \subseteq Fp_1 + Fq_1$ and $B_1B_1 \subseteq Fc_1 + Fc_1 + Ff$, so $B_1 \subseteq \langle p_1, q_1 \rangle$. It follows that $B = \langle e, f, c_1, p_1, q_1 \rangle$ is not maximal since $B \subseteq \langle e, f, a, b, c_1, p_1, q_1 \rangle$.

(3) If dim $V_0 = 2$, then rank$(Q|_{V_0}) = 0$, 1 or 2. If rank$(Q|_{V_0}) = 2$, we can suppose that $V_0 = Fa + Fb$, again by Witt’s Theorem and Theorem 3.3. As in (2), we have that $B_1 = P_B + Q_B$. If dim $P_B = 2$, then since $bP_B = Q_B$, it follows that dim $Q_B = 2$, and then $B = K_{10}$, a contradiction. If dim $P_B = 1$, then dim $Q_B = 1$, but since $B_1B_1 \subseteq B_0$, then either $B_1 = \langle p_1, q_2 \rangle$ or $B_1 = \langle p_2, q_1 \rangle$. This is a contradiction with $bB_1 \subseteq B_1$. If dim $P_B = 0$, then dim $Q_B = 0$ and $B$ is not a maximal subalgebra because $B \subseteq (K_{10})_0$.

If rank$(Q|_{V_0}) = 1$, then $V_0 = Fa + Fc_1$ and again $B_1 = P_B + Q_B$. If dim $P_B = 2$, then dim $Q_B \geq 1$ (because $c_1p_1 = 0$ and $c_1p_2 = q_1$) and so $B = K_{10}$, a contradiction. If dim $P_B = 1$, then dim $Q_B = 1$ or 0. If dim $Q_B = 1$, from $B_1B_1 \subseteq B_0$ we have that $Q_B = Fq_1$ and for $0 \neq p \in P_B$ it follows that $pq_1 \in V_0 = Fa + Fc_1$, and therefore $p \in Fp_1$. But in this case $B$ is not maximal since $B \subseteq \langle e, f, a, b, c_1, p_1, q_1 \rangle$. Also, if dim $Q_B = 0$, then $P_B = Fp_1$ and again $B$ is not maximal. Now if dim $P_B = 0$, then dim $Q_B = 1$, with $Q_B = \langle q_1 \rangle$. But again $B$ is not maximal because $B \subseteq \langle e, f, a, b, c_1, p_1, q_1 \rangle$.

Finally, if rank$(Q|_{V_0}) = 0$, we can suppose that $V_0 = F \cdot (a + b) + Fc_1$, by Witt’s Theorem and Theorem 3.3. If $x = \alpha p_1 + \beta p_2 + \sigma q_1 + \gamma q_2 \in B_1$, then $c_1x = \beta p_1 + \gamma p_1 \in B_1$ and $(a + b)(c_1x) = (\gamma - \beta)(p_1 - q_1) \subseteq B_1$. Therefore if $\gamma \neq \beta$, then $\langle p_1, q_1 \rangle \subseteq B_1$. So either $\langle p_1, q_1 \rangle \subseteq B_1$ or $B_1 \subseteq Fp_1 + Fq_1 + F(p_2 + q_2)$. Moreover with $\gamma \neq \beta$, $q_1x \notin B_0$. Therefore $\gamma = \beta$ and $B = \langle e, f, a + b, c_1, p_1, q_1, p_2 + q_2 \rangle$, which is a maximal subalgebra.

(4) If dim $V_0 = 3$, then rank$(Q|_{V_0}) = 2$ or 3. If rank$(Q|_{V_0}) = 3$, then we can suppose by Witt’s Theorem and Theorem 3.3 that $V_0 = Fa + Fb + F \cdot (c_1 + c_2)$. Again $B_1 = P_B + Q_B$. If dim $P_B = 2$, then dim $Q_B = 2$ (because $bP_B \subseteq Q_B$) and $B_1 = (K_{10})_1$, a contradiction. So dim $P_B = 1 = \dim Q_B$, but then, since $(c_1 + c_2)B_1 \subseteq B_1$ and $B_1B_1 \subseteq B_0$, it follows that either $B_1 = \langle p_2, q_1 \rangle$ or $B_1 = \langle p_1, q_2 \rangle$, and this contradicts $bB_1 \subseteq B_1$.

If rank$(Q|_{V_0}) = 2$, then $V_0 = Fa + Fb + Fc_1$ and $B_1 = P_B + Q_B$, as usual. If dim $P_B = 2$, then dim $Q_B = 2$, a contradiction. So dim $P_B = 1$ and dim $Q_B = 1$. Since $P_BQ_B \subseteq V_0$ we have that $B_1 = \langle p_1, q_1 \rangle$ and $B = \langle e, f, a + b, c_1, p_1, q_1 \rangle$, which is a maximal subalgebra of $K_{10}$.

Next we describe the structure of each one of the types of maximal subalgebras that we have obtained for $K_{10}$:

(i) $(K_{10})_0$ is semisimple, and we have seen that $K_{10} \cong J(V, Q) \oplus F$.
(ii) $\langle e, f, a, p_1, p_2 \rangle$ is semisimple and isomorphic to $F \oplus D_{-6}$, where $F \cong (2e - a)$ and $\langle f, 2e + a \rangle \cong (p_1, p_2) \cong D_{-6}$.
(iii) $B = \langle e, f, a + b, c_1, p_1, q_1, p_2 + q_2 \rangle$ has the following radical:

$$R = \langle c_1, a + b, p_1, q_1 \rangle,$$

and the quotient of the subalgebra $B$ by its radical is isomorphic to $D_{-3}$:

$$B/R = \langle e, f, \frac{p_1 + q_1}{2}, p_2 + q_2 \rangle \cong D_{-3}.$$
Corollary 4.2. The maximal subalgebras of $K_{10}$ are, up to conjugation by automorphisms, the following:

(i) $\langle K_{10} \rangle_0$.

(ii) The fixed subalgebra by the automorphism $\varphi$.

(iii) The subalgebra $F \cdot 1 \oplus (M \otimes K_3)$, where $M = Fe + Fx$ is a maximal subalgebra of $K_3$.

(iv) The subalgebra $F \cdot 1 + F(e \otimes e) + (x \otimes K_3) + (K_3 \otimes y)$.

Proof. It is easy to compute the fixed elements by $\varphi$ in terms of the original basis of $K_{10}$ given in the introduction. This subalgebra is $\langle e, f, c_1 + c_2, p_1 - q_2, p_2 - q_1, f, e \rangle$ and, now, using Proposition 3.1, we know that there exists an automorphism of $K_{10}$ that applies $\langle e, f, a, p_1, p_2 \rangle$ into $\langle e, f, c_1 + c_2, p_1 - q_2, p_2 - q_1 \rangle$ (notice that $F(c_1 + c_2)$ and $Fa$ are isometric relative to the bilinear form $Q$ of $V$, and moreover $(p_1 - q_2, p_2 - q_1) = \{x \in (K_{10})_0 \mid (c_1 + c_2)x = -x \}$). So the maximal subalgebras of type (ii) in Theorem 4.1 are conjugate to the fixed subalgebra of elements fixed by $\varphi$.

Also the maximal subalgebras of type (iii) correspond, through the isomorphism in Theorem 2.1 to $\{1, e \otimes e, x \otimes x, e \otimes x, x \otimes e, e \otimes y, e \otimes x, x \otimes e, e \otimes y\}$, that is, to subalgebras $F \cdot 1 (M \otimes K_3)$ with $M$ a maximal subalgebra of $K_3$, while the maximal subalgebras of type (iv) correspond, through the isomorphism in Theorem 2.1 to $F \cdot 1 + F(e \otimes e) + (x \otimes K_3) + (K_3 \otimes y)$. $\square$

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