HOMOGENEOUS POLYNOMIALS
ON STRICTLY CONVEX DOMAINS

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Abstract. We consider a circular, bounded, strictly convex domain $\Omega \subset \mathbb{C}^d$ with boundary of class $C^2$. For any compact subset $K$ of $\partial \Omega$ we construct a sequence of homogeneous polynomials on $\Omega$ which are big at each point of $K$.

As an application for any $E \subset \partial \Omega$ circular subset of type $G_\delta$ we construct a holomorphic function $f$ which is square integrable on $\Omega \setminus E$ and such that $E = E^2(\Omega,f) = \{ z \in \partial \Omega : \int_{B^2(z)} |f|^2 d\mathcal{L}^2_\nu = \infty \}$ where $D$ denotes unit disc in $\mathbb{C}$.

1. Introduction

Let $\Omega$ denote a bounded, convex and circular domain with a defining function $\eta$ of class $C^2$. We also denote by $D$ the unit disc in $\mathbb{C}$ and define the exceptional set $E^2(\Omega,f) := \{ z \in \partial \Omega : \int_{D(z)} |f|^2 d\mathcal{L}^2_\nu = \infty \}$ for a holomorphic function $f \in \mathcal{O}(\Omega)$. For more information about exceptional sets see [1, 2, 3, 4, 5, 6, 7].

In the paper [12] a natural number $K$ and a sequence $\{p_n\}_{n=0}^{\infty}$ of homogeneous polynomials on $\mathbb{C}^d$ were constructed so that $|p_n(\xi)| \leq 2$ and $\sum_{j=Km}^{K(m+1)-1} |p_n(\xi)| \geq 0.5$ for all $z$ belonging to the boundary of the unit ball $\partial B^d$. In the paper [7] we introduced some additional arguments in such a way that for any circular set $E \subset \partial B^d$ of type $G_\delta$ and $F_\sigma$ we could construct a holomorphic function $f$ on the unit ball $B^d$ so that $E^2(f) = E$.

In this paper we construct similar homogeneous polynomials as in [7, 10, 11, 12]. Whilst these papers dealt with homogeneous polynomials on the unit ball, in this paper we construct homogeneous polynomials on $\Omega$ which is a bounded, circular and strictly convex domain with boundary of class $C^2$.

1.1. Geometric notions. In the complex $d$-dimensional space $\mathbb{C}^d$ we consider the natural scalar product $\langle \cdot, \cdot \rangle$. We also consider rotation invariant pseudometrics

$$\rho(z,w) = \min_{|\lambda| = 1} \|z - \lambda w\|.$$ 

As usual, by $B(\xi;r)$ we denote the open ball with center $\xi$ and radius $r$, i.e.

$$B(\xi;r) := \{ z \in \mathbb{C}^d : \rho(\xi,z) < r \}.$$
We claim that there exist constants $0 < q_0 < q_1$ such that
\begin{equation}
q_0 r^{2d-1} \leq \mathcal{L}^{2d}(B(\xi; r)) \leq q_1 r^{2d-1}
\end{equation}
for $\xi \in \partial \Omega$ and $0 \leq r \leq 2R := 2 \sup_{z, w \in \partial \Omega} \rho(z, w)$.

Since rotation does not change $\mathcal{L}^{2d}(B(\xi; r))$ we can assume that $\xi = (a, 0)$ for some $a \in \mathbb{R}_+$. In particular we can calculate
\[
\rho(\lambda, w, (a, 0)) = \min_{|\eta|=1} \| (\eta \lambda, \eta w) - (a, 0) \| = \sqrt{(|\lambda - a|^2 + \|w\|^2}.
\]
Assume for a moment that $r \leq a$. Since $\xi = (a, 0)$ we can observe that $B(\xi; r) = B_+(\xi; r) \cup B_-(\xi; r)$ where:
\[
B_+(\xi; r) := \left\{ (a + s)e^{i\phi}, w) \in \mathbb{C}^d : 0 \leq r < s < r, \phi \in [0, 2\pi], \|w\| < \sqrt{r^2 - s^2} \right\};
\]
\[
B_-(\xi; r) := \left\{ (a - s)e^{i\phi}, w) \in \mathbb{C}^d : 0 < r < s < r, \phi \in [0, 2\pi], \|w\| < \sqrt{r^2 - s^2} \right\}.
\]

We may calculate
\[
\mathcal{L}^{2d}(B(\xi; r)) = \int_{B_+(\xi; r)} 1 d\mathcal{L}^{2d} + \int_{B_-(\xi; r)} 1 d\mathcal{L}^{2d} = 2 \pi a^2 r^{d-2} \int_0^r s^{d-2} ds = 4 \pi a^2 r^{d-2} \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k r^{2(d-1-k)} s^{2k} ds,
\]
Since $0 < r < 2R$ and $0 \in \Omega$ the above equality implies that there exist constants $0 < q_0 < q_1$ such that \((1)\) holds. Let us observe that if $r \leq \min_{\xi \in \partial \Omega} \|\xi\|$ we have $\mathcal{L}^{2d}(B(\xi; r)) = r^{2d-1} \|\xi\| \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k (2k+1)^{(d-1-k)}$ for $\xi \in \partial \Omega$. However for us it suffices to use \((1)\).

Additionally let us assume that $\mathcal{L}^{2d}(B(0; 2R)) \leq q_1 q_0$.

A subset $A \subset \mathbb{C}^d$ is called $\alpha$-separated if $\rho(z_1, z_2) > \alpha$ for all distinct elements $z_1$ and $z_2$ of $A$. It is clear that for $\alpha > 0$ each $\alpha$-separated subset of $\partial \Omega$ is finite.

If $g : \mathbb{C}^d \to \mathbb{C}$ is a function of class $C^2$, then we denote $g_\xi = \left( \frac{\partial g}{\partial z_1}(\xi), \ldots, \frac{\partial g}{\partial z_d}(\xi) \right)$ and
\[
H_g(P, w) := \frac{1}{2} \sum_{j, k=1}^d \frac{\partial^2 g}{\partial z_j \partial z_k}(P) w_j w_k + \frac{1}{2} \sum_{j, k=1}^d \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(P) \overline{w}_j \overline{w}_k + \sum_{j, k=1}^d \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(P) u_j \overline{w}_k.
\]
Let us recall that $\eta$ is a defining function of class $C^2$ for $\Omega$. Let $X$ be a compact, circular set. Assume that $X$ contains only strictly convex points of $\partial \Omega$, i.e. if $\xi \in X$, then $H_g(\xi, w) > 0$ where $w \neq 0$ and $\Re\{\langle w, \nabla \eta \rangle\} = 0$. 

\[\pi_m := \mathcal{L}^m \{ w \in \mathbb{C}^m : \|w\| < 1 \} \]
2. Homogeneous polynomials

All homogeneous polynomials of degree \( n \) constructed in this paper have the following form:
\[
p_n(z) = \sum_{\xi \in A} \langle z, \nu_\xi \rangle^n
\]
where \( A \) is a finite subset of \( \partial \Omega \) and \( \nu_\xi = \frac{1}{\langle \xi, \nu_\xi \rangle} \).

We begin with a very important estimation of \( |\langle z, \nu_\xi \rangle| \).

**Lemma 2.1.** There exist constants \( c_1, c_2 > 0 \) such that
\[
c_1 \rho^2(z, \xi) \leq 1 - |\langle z, \nu_\xi \rangle| \leq c_2 \rho^2(z, \xi)
\]
for \( \xi \in X \), \( z \in \partial \Omega \).

**Proof.** Since \( \Omega \) is a circular and convex domain
\[
|\langle z, \nu_\xi \rangle| - |\langle \xi, \nu_\xi \rangle| \leq |\langle z, \nu_\xi \rangle| - \Re \langle \xi, \nu_\xi \rangle \leq \max_{|\lambda|=1} \Re \lambda z - \xi, \nu_\xi \rangle \leq 0.
\]
First we prove that for \( \xi \in \partial \Omega \) we have the following property:
\[
\langle \xi, \nu_\xi \rangle \in \mathbb{R}^+.
\]
Let \( \lambda_0 \) be such that \( \langle \xi, \lambda_0 \nu_\xi \rangle \in \mathbb{R}^+ \) and \( |\lambda_0| = 1 \). Observe that
\[
\Re \langle z - \xi, \lambda_0 \nu_\xi \rangle \leq |\langle z, \nu_\xi \rangle| - |\langle \xi, \nu_\xi \rangle| \leq 0
\]
for \( z \in \partial \Omega \). Since \( \partial \Omega \) is of class \( C^2 \) we have \( \nu_\xi = \lambda_0 \nu_\xi \). In particular \( \lambda_0 = 1 \).

In the next step we prove that there exist constants \( c_3, c_4 > 0 \) such that for \( z \in \partial \Omega \) and \( \xi \in X \) we have:
\[
c_3 \|z - \xi\|^2 \leq |\Re \langle z - \xi, \nu_\xi \rangle| \leq c_4 \|z - \xi\|^2.
\]

Due to [8, Lemma 3.1.6] there exist a defining function \( \tilde{\eta} \) of class \( C^2 \) for \( \Omega \) and constants \( c_5, c_6 > 0 \) such that \( c_5 \|w\|^2 \leq H_\tilde{\eta}(\xi, w) \leq c_6 \|w\|^2 \) for \( \xi \in X \) and \( w \in C^d \).

Let
\[
\phi(\xi, h) := \langle h, \nu_\xi \rangle + \langle h, \nu_\xi \rangle - \tilde{\eta}(\xi + h).
\]

Since \( \tilde{\eta} \) is of class \( C^2 \) we have \( \tilde{\eta}(\xi + h) = \tilde{\eta}(\xi) + \langle h, \nu_\xi \rangle + \langle h, \nu_\xi \rangle + H_\tilde{\eta}(\xi, h) + f(\xi, h) \|h\|^2 \) where \( f \) is a continuous function such that \( \lim_{|h| \to 0} f(\xi, h) = 0 \). Observe that
\[
2\Re \langle z - \xi, \nu_\xi \rangle = \phi(\xi, z - \xi) \text{ for } z \in \partial \Omega \text{ and } \xi \in X.
\]
In particular we may estimate
\[
\frac{2\Re \langle z - \xi, \nu_\xi \rangle}{\|z - \xi\|^2} = \frac{-H_\tilde{\eta}(\xi, z - \xi) - f(\xi, z - \xi) \|z - \xi\|^2}{\|z - \xi\|^2} \leq -c_5 - f(\xi, z - \xi)
\]
and
\[
\frac{2\Re \langle z - \xi, \nu_\xi \rangle}{\|z - \xi\|^2} \geq -c_6 - f(\xi, z - \xi).
\]
The above inequalities imply that there exist constants \( c_7, c_8 > 0 \) such that
\[
c_7 \|z - \xi\|^2 \leq \left| \Re \langle z - \xi, \nu_\xi \rangle \right| \leq c_8 \|z - \xi\|^2.
\]
Since \( \eta, \tilde{\eta} \) are defining functions for \( \Omega \) there exists a continuous, positive function \( g \) such that

\[
\left( \frac{\partial \eta}{\partial x_1}, \frac{\partial \eta}{\partial y_1}, \ldots, \frac{\partial \eta}{\partial x_d}, \frac{\partial \eta}{\partial y_d} \right) = g(\xi) \left( \frac{\partial \tilde{\eta}}{\partial x_1}, \frac{\partial \tilde{\eta}}{\partial y_1}, \ldots, \frac{\partial \tilde{\eta}}{\partial x_d}, \frac{\partial \tilde{\eta}}{\partial y_d} \right).
\]

In particular \( \Re\langle z - \xi, \eta_k \rangle = g(\xi)\Re\langle z - \xi, \tilde{\eta}_k \rangle \) and there exist constants \( c_3, c_4 > 0 \) such that (5) holds.

Now we prove the main conclusion. Let constants \( c_3, c_4 > 0 \) be such that \( 5 \) holds. Let \( \lambda_1, \lambda_2 \) be such that \( \min_{|\alpha|=1} |\Re\langle \lambda z - \xi, \eta_k \rangle| = |\Re\langle \lambda_1 z - \xi, \eta_k \rangle| \) and \( \rho(z, \xi) = \min_{|\alpha|=1} \|\lambda z - \xi\| = \|\lambda_2 z - \xi\| \). By (5) we may estimate

\[
c_3 \|\lambda_2 z - \xi\|^2 \leq c_3 \|\lambda_1 z - \xi\|^2 \leq |\Re\langle \lambda_1 z - \xi, \eta_k \rangle| \\
\leq |\Re\langle \lambda_2 z - \xi, \eta_k \rangle| \leq c_4 \|\lambda_2 z - \xi\|^2.
\]

By (4) we have \( |\Re\langle \lambda_1 z - \xi, \eta_k \rangle| = \langle \xi, \eta_k \rangle - |\langle z, \eta_k \rangle| \). In particular we may estimate

\[
c_3 \frac{|\langle \xi, \eta_k \rangle - |\langle z, \eta_k \rangle|}{|\langle \xi, \eta_k \rangle|} \leq 1 - \frac{|\langle \xi, \eta_k \rangle|}{|\langle \xi, \eta_k \rangle|} \leq c_4 \rho^2(z, \xi).
\]

Since \( X \) is a compact set and \( \langle \xi, \eta_k \rangle > 0 \) it is enough to define \( c_1 = \inf_{\xi \in X} \frac{c_3}{\langle \xi, \eta_k \rangle} \) and \( c_2 = \sup_{\xi \in X} \frac{c_4}{\langle \xi, \eta_k \rangle} \).

In order to control the values of the constructed polynomials we need some information about \( \alpha \)-separated sets.

**Lemma 2.2.** Suppose that \( A = \{\xi_1, \ldots, \xi_s\} \) is a \( 2\alpha t \)-separated subset of \( \partial \Omega \). For \( z \in \partial \Omega \) let

\[
A_k(z) := \{ \xi \in A : \alpha k t \leq \rho(z, \xi) \leq \alpha(k+1)t \}.
\]

Then the set \( A_k(z) \) has at most \( q_1(k+2)^{2d-1} \) elements. The set \( A_0 \) has at most 1 element and \( s \leq q_1(\alpha t)^{1-2d} \).

**Proof.** Observe that \( B(\xi_1; \alpha t) \cap B(\xi_1; \alpha t) = \emptyset \) for \( \xi_1 \neq \xi_2 \in A \). Moreover

\[
\bigcup_{\xi \in A_k(z)} B(\xi; \alpha t) \subset B(z; \alpha(k+2)t).
\]

Let \( d_k \) be a number of elements in \( A_k(z) \). In particular

\[
d_k q_0(\alpha t)^{2d-1} \leq \sum_{\xi \in A_k(z)} \mathcal{L}^{2d} (B(\xi; \alpha t)) \leq \mathcal{L}^{2d} (B(z; \alpha(k+2)t)) \leq q_1 q_0(\alpha(k+2)t)^{2d-1}.
\]

We conclude that \( d_k \leq q_1(k+2)^{2d-1} \). Moreover if \( \xi_j, \xi_k \in A_0(z) \), then \( \rho(\xi_j, \xi_k) \leq \rho(z, \xi_j) + \rho(z, \xi_k) < 2\alpha t \) so \( \xi_j = \xi_k \) and \( d_0 \leq 1 \).

Since \( \Omega \subset B(0, R) \) (see section 1.1) we may assume that \( \alpha t \leq R \). In particular

\[
\bigcup_{\xi \in A_k(z)} B(\xi; \alpha t) \subset B(0; 2R)
\]

and we may estimate (see section 1.1)

\[
s q_0(\alpha t)^{2d-1} \leq \sum_{\xi \in A} \mathcal{L}^{2d} (B(\xi; \alpha t)) \leq \mathcal{L}^{2d} (B(0; 2R)) \leq q_1 q_0.
\]

**Lemma 2.3.** If \( A \subset \partial \Omega \) is \( \alpha \)-separated, then for each \( \beta > \alpha \) there exists an integer \( K = K(\alpha, \beta) \) such that \( A \) can be partitioned into \( K \) disjoint \( \beta t \)-separated sets.
Lemma 2.5. Let $x \in (0, 1)$. To prove the left inequality let $f(x) := x + \ln(1 - x)$. Since $f'(x) = \frac{1}{1-x} < 0$ we have $f(x) < f(0) = 0$ and $\frac{1}{x}\ln(1 - x) < -1$.

To prove right inequality let $g(x) = \frac{x}{1-x} - \ln(1 - x)$. Since $g'(x) = \frac{1}{(1-x)^2} < 0$, then $g(x) < g(0) = 0$ and $-1 < \frac{1}{x}\ln(1 - x)$. □

We can conclude that $s \leq q_1 \left( \frac{x}{a} + 1 \right)^{2d-1}$. Now it suffices to choose a natural number $K$ so that $q_1 \left( \frac{x}{a} + 1 \right)^{2d-1} \leq K$.

Proposition 2.4. We have the following inequalities for $0 < x < 1$:

(6) 
$(1 - x)^{\frac{1}{2}} < e^{-1} < (1 - x)^{\frac{1}{3}}$.

Proof. Let $x \in (0, 1)$. To prove the left inequality let $f(x) := x + \ln(1 - x)$. Since $f'(x) = \frac{1}{1-x} < 0$ we have $f(x) < f(0) = 0$ and $\frac{1}{x}\ln(1 - x) < -1$.

To prove right inequality let $g(x) = \frac{x}{1-x} - \ln(1 - x)$. Since $g'(x) = \frac{1}{(1-x)^2} < 0$, then $g(x) < g(0) = 0$ and $-1 < \frac{1}{x}\ln(1 - x)$. □

Now we are ready to state some estimations for polynomials of the form (2).

Lemma 2.5. Let $0 < c_1 < c_2$ be constants from Lemma 2.1. For a given $a \in (0, 0.5)$ there exist constants $C > 2$ and $N_0 \in \mathbb{N}$ such that for all integers $N \geq N_0$, for each $C/c_1N$-separated subset $A$ of $X$ and each integer $m$ with $N \leq m \leq 2N$ the polynomial $p_m(z) := \sum_{\xi \in \Lambda} (z, \nu_\xi)^m$ satisfies

1. If $z \in \partial \Omega$, $Q(z) := \left\{ \xi \in A : \rho(z, \xi) \geq \frac{C}{2c_1N} \right\}$, then $\sum_{\xi \in \Lambda} |(z, \nu_\xi)|^m < a$.
2. If $z \in \partial \Omega$, then $Q(z) \setminus A$ has at most one element.
3. If $\xi_0 \in A$, $z \in \partial \Omega$ are such that $\rho(z, \xi_0) \leq \frac{a}{c_1N}$, then
   (a) $Q(z) = A \setminus \{ \xi_0 \}$,
   (b) $\sum_{\xi \in \Lambda} |(z, \nu_\xi)|^m > 1 - 2a^2$,
   (c) $|p_m(z)| > 1 - 2a^2 - a$.
4. $|p_m(z)| \leq \sum_{\xi \in A} |(z, \nu_\xi)|^m < 1 + a$ for all $z \in \partial \Omega$.

Proof. There exists a constant $C > 2$ large enough that for $k \in \mathbb{N}_+$ we have

(7) 
$\sum_{k=1}^{\infty} q_1(k + 2)^{2d-1} \exp \left( \frac{k^2}{4} \right) < a$.

Due to Proposition 2.4 we can estimate:

\[
\lim_{N \to \infty} \exp \left( \frac{-2a^2}{1 - a^2N^{-1}} \right) = \exp \left( -2a^2 \right) > 1 - 2a^2.
\]

In particular we can choose $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

(8) 
$\exp \left( \frac{-2a^2}{1 - a^2N^{-1}} \right) > 1 - 2a^2$. 

Let \( z \in \partial \Omega \), \( Q(z) := \left\{ \xi \in A : \rho(z, \xi) \geq \frac{C}{2\sqrt{c_1}N} \right\} \) and
\[
A_k(z) = \left\{ \xi \in A : \frac{kC}{2\sqrt{c_1}N} \leq \rho(z, \xi) < \frac{(k+1)C}{2\sqrt{c_1}N} \right\}.
\]

Due to Lemma 2.2 the set \( A_0(z) \) has at most 1 element and
\[
\#A_k(z) \leq q_1(k+2)^{2d-1}.
\]
Since \( Q(z) \setminus A = A_0(z) \) we have the property (2). Due to Lemma 2.1 for \( \xi \in A_k(z) \) we have
\[
|\langle z, \nu_\xi \rangle| \leq 1 - c_1 \rho^2(z, \xi) \leq 1 - \frac{k^2C^2}{4N}.
\]

Now we observe the property (3b) for \( N \)
\[
|\langle z, \nu_\xi \rangle| \geq 1 - c_2 \rho^2(z, \xi) \geq 1 - \frac{a^2}{N}.
\]

Now let \( \xi_0 \in A, z \in \partial \Omega \) be such that \( \rho(z, \xi_0) \leq \frac{a}{2\sqrt{c_2}N} < \frac{C}{2\sqrt{c_1}N} \). Since \( A_0(z) \) has at most 1 element we have \( A_0(z) = \{\xi_0\} \), which gives the property (3a). Moreover we have:
\[
|\langle z, \nu_{\xi_0} \rangle| \geq 1 - c_2 \rho^2(z, \xi_0) \geq 1 - \frac{a^2}{N}.
\]

Now we observe the property (3b) for \( N \geq N_0 \):
\[
|\langle z, \nu_{\xi_0} \rangle|^m \geq \left(1 - \frac{a^2}{N} \right)^{2N} \geq \exp \left( -\frac{a^2N^{-1/2}N}{1 - a^2N^{-1}} \right) > 1 - 2a^2.
\]

Moreover we may conclude the property (3c):
\[
|p_m(z)| \geq |\langle z, \nu_{\xi_0} \rangle|^m - \sum_{\xi \in Q(z)} |\langle z, \nu_\xi \rangle|^m > 1 - 2a^2 - a,
\]
which finishes the proof.

We are ready for main result of this paper.

**Theorem 2.6.** There exists \( K \in \mathbb{N} \) such that for \( 0 < \epsilon < 1 \) and for each pair of compact, circular and disjoint sets \( D, T \) such that \( T \subset X \), \( D \subset \partial \Omega \), we can choose \( m_0 = m_0(D,T,\epsilon) \in \mathbb{N} \) and a sequence \( p_m \) of homogeneous polynomials of degree \( m \) which satisfy

1. \( |p_m(z)| \leq 2 \) for all \( z \in \partial \Omega \), \( m > m_0 \),
2. \( \sum_{k=Km}^{K(m+1)-1} |p_k(z)|^2 \geq 0.25 \) for all \( z \in T \), \( m > m_0 \),
3. \( \sum_{k=Km}^{K(m+1)-1} |p_k(z)|^2 \leq 2^{-(Km)^{1-\epsilon}} \) for all \( z \in D \), \( m > m_0 \).
Proof. Let $0 < c_1 < c_2$ be from Lemma 2.1. For $a = \frac{1}{4}$ we can choose $C$ from Lemma 2.5. Let $K = K(\alpha, \beta)$ be from Lemma 2.3 for $\alpha = \frac{1}{4\sqrt{c_2}}$ and $\beta = \frac{C}{\sqrt{c_2}}$. For $N = Km$ fix a maximal $1/(4\sqrt{c_2N})$-separated subset $A \subset T$. Using Lemma 2.3 we can divide $A$ into at most $K$ disjoint $C/\sqrt{c_1N}$-separated subsets $A_0, A_1, \ldots, A_{K-1}$. We define

$$p_{Km+j}(z) := \sum_{\xi \in A_j} (z, \nu_\xi)^{Km+j}$$

for $j = 0, 1, \ldots, K-1$. From Lemma 2.5 we infer that there exists $m_0$ so high that for $m > m_0$ we have $|p_{Km+j}(z)| < 1 + a = \frac{5}{4} < 2$ for all $z \in \partial \Omega$ and $|p_{Km+j}(z)| > 1 - 2a^2 - a = \frac{3}{50} > 0.5$ for $z \in \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right)$.

Since $A = \bigcup_{j=0}^{K-1} A_j$ is a maximal $1/(4\sqrt{c_2N})$-separated subset of $T$ we conclude that

$$\bigcup_{j=0}^{K-1} \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right) = \bigcup_{\xi \in A} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right) \supset T,$$

and from this follows that

$$\sum_{j=Km}^{K(m+1)-1} |p_j(z)|^2 \geq 0.25 \text{ for all } z \in T, m > m_0.$$

Without loss of generality we can assume that $m_0$ is so large that $\rho(z, w) > \sqrt{1/c_1N^\epsilon}$ for all $z \in D$ and $w \in T$. Due to Lemma 2.2 we have

$$\#A_j \leq q_1 \left(\frac{\sqrt{c_1N}}{C}\right)^{2d-1}.$$

If $\xi \in A$ and $z \in D$, then on the basis of Lemma 2.1 we have

$$|(z, \nu_\xi)| \leq 1 - c_1\rho^2(z, \xi) \leq 1 - \frac{1}{N^\epsilon}.$$

Now for $m_0$ large enough, $m > m_0$, $N = Km$ and $z \in D$ we may estimate

$$\sum_{j=0}^{K-1} |p_{Km+j}(z)|^2 \leq \sum_{j=0}^{K-1} \sum_{\xi \in A_j} |(z, \nu_\xi)|^{2Km+j} \leq \sum_{\xi \in A} |(z, \nu_\xi)|^N N^r N^{-1-r} \leq \frac{1}{2 N^{1-r}}.$$

As an application we can present the following result:

**Theorem 2.7.** Assume that $\Omega$ is a circular, bounded and strictly convex domain with the boundary of class $C^2$. Then for any circular subset $E \subset \partial \Omega$ of type $G_\delta$ there exists a holomorphic function $f$ which is square integrable on $\Omega \setminus \bar{E}$ and such that $E = E_\delta(f) := \{z \in \partial \Omega : \int_{\partial E} |f|^2 d\Omega_{\partial E} = \infty\}$. 
Proof. Let $\sigma$ be the natural measure on $\partial \Omega$. On the basis of [8, Theorem 2.6] there exist sequences $\{D_i\}_{i \in \mathbb{N}}$, $\{T_i\}_{i \in \mathbb{N}}$ of compact, circular sets in $\partial \Omega$ such that:

1. $\bigcup_{i \in \mathbb{N}} D_i = \partial \Omega \setminus E$ and $D_j \subset D_{j+1}$ for $j \in \mathbb{N}$,
2. $T_j \cap D_j = \emptyset$ for $j \in \mathbb{N}$,
3. $E = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} T_i$,
4. $\sigma(\partial \Omega \setminus (E \cup D_j)) \leq 2^{-j}$.

Since $\Omega$ is a strictly convex domain then $X = \partial \Omega$. Let $K$ be a natural number from Theorem 2.6. We can use Theorem 2.6 once again and conclude that there exist a sequence of natural numbers $\{m_j\}_{j \in \mathbb{N}}$ and a sequence of homogeneous polynomials $\{p_m\}_{m \in \mathbb{N}}$ such that:

1. $m_j < m_{j+1}$ for $j \in \mathbb{N}$,
2. $|p_m(z)| \leq 2$ for all $z \in \partial \Omega$, $m > m_0$,
3. $\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \geq 0.25$ for all $z \in T_j$,
4. $\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \leq 2^{-j}$ for all $z \in D_j$.

Now we can define

$$f = \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} \sqrt{k+1} p_k.$$

Observe that for $z \in \partial \Omega$ we have

$$\int_{\partial \mathbb{D}} |f|^2 d\mathbb{S}_z^2 = \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2.$$

In particular for $z \in E$ we have

$$\int_{\partial \mathbb{D}} |f|^2 d\mathbb{S}_z^2 \geq \sum_{j \in T_j} 0.25 = \infty.$$

If $z \in \partial \Omega \setminus E$, then there exists $j_0$ such that $z \in D_j$ for $j \geq j_0$. In particular

$$\int_{\partial \mathbb{D}} |f|^2 d\mathbb{S}_z^2 \leq \sum_{j=1}^{j_0} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 + \sum_{j=j_0}^{\infty} 2^{-j} < \infty.$$

Now we prove that $f$ is square integrable on $\Omega \setminus D$. There exists $M > 0$ such that

$$\int_{\Omega \setminus D} |f|^2 d\mathbb{S}^{2d} \leq M \int_{\partial \Omega \setminus E} \int_{\partial \mathbb{D}} |f|^2 d\mathbb{S}_z^2 d\sigma(z).$$

In particular we may estimate

$$\int_{\Omega \setminus D} |f|^2 d\mathbb{S}^{2d} \leq M \sum_{j=1}^{\infty} \int_{\partial \Omega \setminus E} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k|^2 d\sigma$$

$$\leq M \sum_{j=1}^{\infty} 2^{-j} \sigma(D_j) + M \sum_{j=1}^{\infty} 4K\sigma(\partial \Omega \setminus (E \cup D_j))$$

$$\leq M\sigma(\partial \Omega) + 4KM < \infty. \quad \square$$

References


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