PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 12, December 2007, Pages 3895–3903
S 0002-9939(07)08939-3
Article electronically published on September 10, 2007

HOMOGENEOUS POLYNOMIALS ON STRICTLY CONVEX DOMAINS

PIOTR KOT

(Communicated by Mei-Chi Shaw)

Abstract. We consider a circular, bounded, strictly convex domain $\Omega \subset \mathbb{C}^d$ with boundary of class C^2. For any compact subset K of $\partial\Omega$ we construct a sequence of homogeneous polynomials on Ω which are big at each point of K. As an application for any $E \subset \partial\Omega$ circular subset of type G_δ we construct a holomorphic function f which is square integrable on $\Omega \setminus \mathbb{D}E$ and such that $E = E^2_\Omega(f) := \left\{ z \in \partial\Omega : \int_{\mathbb{D}z} |f|^2 \, d\mathfrak{L}^2_{\mathbb{D}z} = \infty \right\}$ where $\mathbb{D}$ denotes unit disc in $\mathbb{C}$.

1. Introduction

Let Ω denote a bounded, convex and circular domain with a defining function η of class C^2. We also denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and define the exceptional set $E^2_\Omega(f) := \left\{ z \in \partial\Omega : \int_{\mathbb{D}z} |f|^2 \, d\mathfrak{L}^2_{\mathbb{D}z} = \infty \right\}$ for a holomorphic function $f \in \mathbb{O}(\Omega)$. For more information about exceptional sets see [1, 2, 3, 4, 5, 6, 7].

In the paper [12] a natural number K and a sequence $\{p_n\}_{n=0}^\infty$ of homogeneous polynomials on $\mathbb{C}^d$ were constructed so that $|p_n(z)| \leq 2$ and $\sum_{j=Km}^{K(m+1)-1} |p_n(z)| \geq 0.5$ for all z belonging to the boundary of the unit ball $\partial\mathbb{B}^d$. In the paper [7] we introduced some additional arguments in such a way that for any circular set $E \subset \partial\mathbb{B}^d$ of type G_δ and F_σ we could construct a holomorphic function f on the unit ball $\mathbb{B}^d$ so that $E^2_{\mathbb{B}^d}(f) = E$.

In this paper we construct similar homogeneous polynomials as in [7, 10, 11, 12]. Whilst these papers dealt with homogeneous polynomials on the unit ball, in this paper we construct homogeneous polynomials on Ω which is a bounded, circular and strictly convex domain with boundary of class C^2.

1.1. Geometric notions.

In the complex d-dimensional space $\mathbb{C}^d$ we consider the natural scalar product $\langle \circ, \circ \rangle$. We also consider rotation invariant pseudometrics

$$\rho(z, w) = \min_{|\lambda|=1} \|z - \lambda w\| .$$

As usual, by $B(\xi; r)$ we denote the open ball with center ξ and radius r, i.e.

$$B(\xi; r) := \left\{ z \in \mathbb{C}^d : \rho(\xi, z) < r \right\} .$$

Received by the editors September 8, 2005 and, in revised form, September 20, 2006.
2000 *Mathematics Subject Classification.* Primary 32A05, 32A40.
Key words and phrases. homogeneous polynomials, exceptional sets, highly nonintegrable holomorphic function.

We claim that there exist constants $0 < q_0 < q_1$ such that

$$q_0 r^{2d-1} \leq \mathfrak{L}^{2d}(B(\xi;r)) \leq q_1 q_0 r^{2d-1} \tag{1}$$

for $\xi \in \partial\Omega$ and $0 \leq r \leq 2R := 2\sup_{z,w\in\partial\Omega}\rho(z,w)$.

Since rotation does not change $\mathfrak{L}^{2d}(B(\xi;r))$ we can assume that $\xi = (a,0)$ for some $a \in \mathbb{R}_+$. In particular we can calculate

$$\rho((\lambda,w),(a,0)) = \min_{|\eta|=1}\|(\eta\lambda,\eta w)-(a,0)\| = \sqrt{(|\lambda|-a)^2+\|w\|^2}.$$

Assume for a moment that $r \leq a$. Since $\xi = (a,0)$ we can observe that $B(\xi;r) = B_-(\xi;r)\cup B_+(\xi;r)$ where:

$$\begin{aligned} B_+(\xi;r) &:= \left\{((a+s)e^{i\phi},w)\in\mathbb{C}^d : 0\leq s<r, \phi\in[0,2\pi], \|w\|<\sqrt{r^2-s^2}\right\}; \\ B_-(\xi;r) &:= \left\{((a-s)e^{i\phi},w)\in\mathbb{C}^d : 0< s<r, \phi\in[0,2\pi], \|w\|<\sqrt{r^2-s^2}\right\}. \end{aligned}$$

We may calculate:[1]

$$\begin{aligned} \mathfrak{L}^{2d}(B(\xi;r)) &= \int_{B_+(\xi;r)} 1 d\mathfrak{L}^{2d} + \int_{B_-(\xi;r)} 1 d\mathfrak{L}^{2d} \\ &= \int_0^{2\pi}\int_0^r\int_{\|w\|<\sqrt{r^2-s^2}} (a+s)+(a-s) d\mathfrak{L}^{2d-2}(w) ds d\phi \\ &= 4\pi a \pi_{2d-2}\int_0^r \left(r^2-s^2\right)^{d-1} ds \\ &= 4\pi a \pi_{2d-2}\int_0^r \sum_{k=0}^{d-1}\begin{pmatrix} d-1 \\ k\end{pmatrix}(-1)^k r^{2(d-1-k)}s^{2k} ds \\ &= 4\pi a \pi_{2d-2}\sum_{k=0}^{d-1}\begin{pmatrix} d-1 \\ k\end{pmatrix}\frac{(-1)^k}{2k+1} r^{2d-1}. \end{aligned}$$

Since $0 < r < 2R$ and $0 \in \Omega$ the above equality implies that there exist constants $0 < q_0 < q_1$ such that (1) holds. Let us observe that if $r \leq \min_{\xi\in\partial\Omega}\|\xi\|$ we have $\mathfrak{L}^{2d}(B(\xi;r)) = r^{2d-1}\|\xi\|\sum_{k=0}^{d-1}\frac{(-1)^k 4\pi\pi_{2d-2}(d-1)!}{(2k+1)k!(d-1-k)!}$ for $\xi \in \partial\Omega$. However for us it suffices to use (1).

Additionally let us assume that $\mathfrak{L}^{2d}(B(0;2R)) \leq q_1 q_0$.

A subset $A \subset \mathbb{C}^d$ is called α-separated if $\rho(z_1,z_2) > \alpha$ for all distinct elements z_1 and z_2 of A. It is clear that for $\alpha > 0$ each α-separated subset of $\partial\Omega$ is finite.

If $g : \mathbb{C}^d \to \mathbb{C}$ is a function of class C^2, then we denote $g_\xi = \left(\frac{\partial g}{\partial z_1}(\xi),...,\frac{\partial g}{\partial z_d}(\xi)\right)$ and

$$H_g(P,w) := \frac{1}{2}\sum_{j,k=1}^d \frac{\partial^2 g}{\partial z_j \partial z_k}(P)\, w_j w_k + \frac{1}{2}\sum_{j,k=1}^d \frac{\partial^2 g}{\partial \overline{z}_j \partial \overline{z}_k}(P)\, \overline{w}_j \overline{w}_k + \sum_{j,k=1}^d \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(P)\, w_j \overline{w}_k.$$

Let us recall that η is a defining function of class C^2 for Ω. Let X be a compact, circular set. Assume that X contains only strictly convex points of $\partial\Omega$, i.e. if $\xi \in X$, then $H_\eta(\xi,w) > 0$ where $w \neq 0$ and $\Re(\langle w, \overline{\eta_\xi}\rangle) = 0$.

[1] $\pi_m := \mathfrak{L}^m(\{w \in \mathbb{C}^m : \|w\| < 1\})$.

2. Homogeneous polynomials

All homogeneous polynomials of degree n constructed in this paper have the following form:

$$p_n(z) = \sum_{\xi \in A} \langle z, \nu_\xi \rangle^n \tag{2}$$

where A is a finite subset of $\partial\Omega$ and $\nu_\xi = \frac{1}{\langle \xi, \overline{\eta_\xi} \rangle} \overline{\eta_\xi}$.

We begin with a very important estimation of $|\langle z, \nu_\xi \rangle|$.

Lemma 2.1. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 \rho^2(z, \xi) \leq 1 - |\langle z, \nu_\xi \rangle| \leq c_2 \rho^2(z, \xi) \tag{3}$$

for $\xi \in X$, $z \in \partial\Omega$.

Proof. Since Ω is a circular and convex domain

$$|\langle z, \overline{\eta_\xi} \rangle| - |\langle \xi, \overline{\eta_\xi} \rangle| \leq |\langle z, \overline{\eta_\xi} \rangle| - \Re \langle \xi, \overline{\eta_\xi} \rangle \leq \max_{|\lambda|=1} \Re \langle \lambda z - \xi, \overline{\eta_\xi} \rangle \leq 0.$$

First we prove that for $\xi \in \partial\Omega$ we have the following property:

$$\langle \xi, \overline{\eta_\xi} \rangle \in \mathbb{R}_+. \tag{4}$$

Let λ_0 be such that $\langle \xi, \lambda_0 \overline{\eta_\xi} \rangle \in \mathbb{R}_+$ and $|\lambda_0| = 1$. Observe that

$$\Re \langle z - \xi, \lambda_0 \overline{\eta_\xi} \rangle \leq |\langle z, \overline{\eta_\xi} \rangle| - |\langle \xi, \overline{\eta_\xi} \rangle| \leq 0$$

for $z \in \partial\Omega$. Since $\partial\Omega$ is of class C^2 we have $\overline{\eta_\xi} = \lambda_0 \overline{\eta_\xi}$. In particular $\lambda_0 = 1$.

In the next step we prove that there exist constants $c_3, c_4 > 0$ such that for $z \in \partial\Omega$ and $\xi \in X$ we have:

$$c_3 \|z - \xi\|^2 \leq |\Re \langle z - \xi, \overline{\eta_\xi} \rangle| \leq c_4 \|z - \xi\|^2. \tag{5}$$

Due to [9, Lemma 3.1.6] there exist a defining function $\widetilde{\eta}$ of class C^2 for Ω and constants $c_5, c_6 > 0$ such that $c_5 \|w\|^2 \leq H_{\widetilde{\eta}}(\xi, w) \leq c_6 \|w\|^2$ for $\xi \in X$ and $w \in \mathbb{C}^d$. Let

$$\phi(\xi, h) := \left\langle h, \overline{\widetilde{\eta_\xi}} \right\rangle + \overline{\left\langle h, \overline{\widetilde{\eta_\xi}} \right\rangle} - \widetilde{\eta}(\xi + h).$$

Since $\widetilde{\eta}$ is of class C^2 we have $\widetilde{\eta}(\xi + h) = \widetilde{\eta}(\xi) + \left\langle h, \overline{\widetilde{\eta_\xi}} \right\rangle + \overline{\left\langle h, \overline{\widetilde{\eta_\xi}} \right\rangle} + H_{\widetilde{\eta}}(\xi, h) + f(\xi, h) \|h\|^2$ where f is a continuous function such that $\lim_{h \to 0} f(\xi, h) = 0$. Observe that $2\Re \left\langle z - \xi, \overline{\widetilde{\eta_\xi}} \right\rangle = \phi(\xi, z - \xi)$ for $z \in \partial\Omega$ and $\xi \in X$. In particular we may estimate

$$\frac{2\Re \left\langle z - \xi, \overline{\widetilde{\eta_\xi}} \right\rangle}{\|z - \xi\|^2} = \frac{-H_{\widetilde{\eta}}(\xi, z - \xi) - f(\xi, z - \xi) \|z - \xi\|^2}{\|z - \xi\|^2} \leq -c_5 - f(\xi, z - \xi)$$

and

$$\frac{2\Re \left\langle z - \xi, \overline{\widetilde{\eta_\xi}} \right\rangle}{\|z - \xi\|^2} \geq -c_6 - f(\xi, z - \xi).$$

The above inequalities imply that there exist constants $c_7, c_8 > 0$ such that

$$c_7 \|z - \xi\|^2 \leq \left| \Re \left\langle z - \xi, \overline{\widetilde{\eta_\xi}} \right\rangle \right| \leq c_8 \|z - \xi\|^2.$$

Since $\eta, \widetilde{\eta}$ are defining functions for Ω there exists a continuous, positive function g such that

$$\left(\frac{\partial \eta}{\partial x_1}, \frac{\partial \eta}{\partial y_1}, ..., \frac{\partial \eta}{\partial x_d}, \frac{\partial \eta}{\partial y_d}\right) = g(\xi)\left(\frac{\partial \widetilde{\eta}}{\partial x_1}, \frac{\partial \widetilde{\eta}}{\partial y_1}, ..., \frac{\partial \widetilde{\eta}}{\partial x_d}, \frac{\partial \widetilde{\eta}}{\partial y_d}\right).$$

In particular $\Re\left\langle z-\xi, \overline{\eta_\xi}\right\rangle = g(\xi)\Re\left\langle z-\xi, \overline{\widetilde{\eta}_\xi}\right\rangle$ and there exist constants $c_3, c_4 > 0$ such that (5) holds.

Now we prove the main conclusion. Let constants $c_3, c_4 > 0$ be such that (5) holds. Let λ_1, λ_2 be such that $\min_{|\lambda|=1} |\Re\left\langle \lambda z-\xi, \overline{\eta_\xi}\right\rangle| = |\Re\left\langle \lambda_1 z-\xi, \overline{\eta_\xi}\right\rangle|$ and $\rho(z,\xi) = \min_{|\lambda|=1}\|\lambda z-\xi\| = \|\lambda_2 z-\xi\|$. By (5) we may estimate

$$\begin{aligned} c_3\|\lambda_2 z-\xi\|^2 \le c_3\|\lambda_1 z-\xi\|^2 &\le |\Re\left\langle \lambda_1 z-\xi, \overline{\eta_\xi}\right\rangle| \\ &\le |\Re\left\langle \lambda_2 z-\xi, \overline{\eta_\xi}\right\rangle| \le c_4\|\lambda_2 z-\xi\|^2. \end{aligned}$$

By (4) we have $|\Re\left\langle \lambda_1 z-\xi, \overline{\eta_\xi}\right\rangle| = \left\langle \xi, \overline{\eta_\xi}\right\rangle - |\left\langle z, \overline{\eta_\xi}\right\rangle|$. In particular we may estimate

$$\frac{c_3}{\left\langle \xi, \overline{\eta_\xi}\right\rangle}\rho^2(z,\xi) \le \frac{\left\langle \xi, \overline{\eta_\xi}\right\rangle - |\left\langle z, \overline{\eta_\xi}\right\rangle|}{\left\langle \xi, \overline{\eta_\xi}\right\rangle} = 1 - |\left\langle z, \nu_\xi\right\rangle| \le \frac{c_4}{\left\langle \xi, \overline{\eta_\xi}\right\rangle}\rho^2(z,\xi).$$

Since X is a compact set and $\left\langle \xi, \overline{\eta_\xi}\right\rangle > 0$ it is enough to define $c_1 = \inf_{\xi\in X}\frac{c_3}{\left\langle \xi, \overline{\eta_\xi}\right\rangle}$ and $c_2 = \sup_{\xi\in X}\frac{c_4}{\left\langle \xi, \overline{\eta_\xi}\right\rangle}$. □

In order to control the values of the constructed polynomials we need some information about α-separated sets.

Lemma 2.2. *Suppose that $A = \{\xi_1, ..., \xi_s\}$ is a $2\alpha t$-separated subset of $\partial\Omega$. For $z \in \partial\Omega$ let*

$$A_k(z) := \{\xi \in A : \alpha k t \le \rho(z,\xi) \le \alpha(k+1)t\}.$$

Then the set $A_k(z)$ has at most $q_1(k+2)^{2d-1}$ elements. The set A_0 has at most 1 element and $s \le q_1(\alpha t)^{1-2d}$.

Proof. Observe that $B(\xi_1;\alpha t)\cap B(\xi_1;\alpha t) = \emptyset$ for $\xi_1 \ne \xi_2 \in A$. Moreover

$$\bigcup_{\xi\in A_k(z)} B(\xi;\alpha t) \subset B(z;\alpha(k+2)t).$$

Let d_k be a number of elements in $A_k(z)$. In particular

$$d_k q_0(\alpha t)^{2d-1} \le \sum_{\xi\in A_k(z)} \mathfrak{L}^{2d}\left(B(\xi;\alpha t)\right) \le \mathfrak{L}^{2d}\left(B(z;\alpha(k+2)t)\right) \le q_1 q_0(\alpha(k+2)t)^{2d-1}.$$

We conclude that $d_k \le q_1(k+2)^{2d-1}$. Moreover if $\xi_{j,}, \xi_k \in A_0(z)$, then $\rho(\xi_j,\xi_k) \le \rho(z,\xi_j) + \rho(z,\xi_k) < 2\alpha t$ so $\xi_j = \xi_k$ and $d_0 \le 1$.

Since $\Omega \subset B(0,R)$ (see section 1.1) we may assume that $\alpha t \le R$. In particular

$$\bigcup_{\xi\in A_k(z)} B(\xi;\alpha t) \subset B(0;2R)$$

and we may estimate (see section 1.1)

$$s q_0(\alpha t)^{2d-1} \le \sum_{\xi\in A} \mathfrak{L}^{2d}\left(B(\xi;\alpha t)\right) \le \mathfrak{L}^{2d}\left(B(0;2R)\right) \le q_1 q_0. \qquad \square$$

Lemma 2.3. *If $A \subset \partial\Omega$ is αt-separated, then for each $\beta > \alpha$ there exists an integer $K = K(\alpha,\beta)$ such that A can be partitioned into K disjoint βt-separated sets.*

Proof. Let us select from A a maximal βt-separated subset A_1. Next from $A \setminus A_1$ we select a maximal βt-separated subset A_2. We continue this way until we exhaust A. Let A_s be the last non-empty set in this procedure. Let $\xi \in A_s$. Observe that $B(\xi;\beta t) \cap A_k \neq \emptyset$ for $k = 1, ..., s-1$. In particular $B(\xi;\beta t)$ contains at least s different elements $\{\xi_1, ..., \xi_s\}$ from A. Since $B(\xi_j;\alpha t) \cap B(\xi_k;\alpha t) = \emptyset$ for $j \neq k$ and $B(\xi_j;\alpha t) \subset B(\xi;(\beta+\alpha)t)$, then

$$sq_0(\alpha t)^{2d-1} \leq \sum_{j=1}^{s} \mathfrak{L}^{2d}\left(B(\xi_j;\alpha t)\right) \leq \mathfrak{L}^{2d}\left(B(\xi;(\beta+\alpha)t)\right) \leq q_1 q_0((\beta+\alpha)t)^{2d-1}.$$

We can conclude that $s \leq q_1\left(\frac{\beta}{\alpha}+1\right)^{2d-1}$. Now it suffices to choose a natural number K so that $q_1\left(\frac{\beta}{\alpha}+1\right)^{2d-1} \leq K$. □

Proposition 2.4. *We have the following inequalities for* $0 < x < 1$*:*

$$(1-x)^{\frac{1}{x}} < e^{-1} < (1-x)^{\frac{1}{x}-1}. \tag{6}$$

Proof. Let $x \in (0,1)$. To prove the left inequality let $f(x) := x + \ln(1-x)$. Since $f'(x) = \frac{-x}{1-x} < 0$ we have $f(x) < f(0) = 0$ and $\frac{1}{x}\ln(1-x) < -1$.

To prove right inequality let $g(x) = \frac{-x}{1-x} - \ln(1-x)$. Since $g'(x) = \frac{-x}{(1-x)^2} < 0$, then $g(x) < g(0) = 0$ and $-1 < \frac{1-x}{x}\ln(1-x)$. □

Now we are ready to state some estimations for polynomials of the form (2).

Lemma 2.5. *Let* $0 < c_1 < c_2$ *be constants from Lemma 2.1. For a given* $a \in (0, 0.5)$ *there exist constants* $C > 2$ *and* $N_0 \in \mathbb{N}$ *such that for all integers* $N \geq N_0$*, for each* $C/\sqrt{c_1 N}$*-separated subset* A *of* X *and each integer* m *with* $N \leq m \leq 2N$ *the polynomial* $p_m(z) := \sum_{\xi\in A}\langle z, \nu_\xi\rangle^m$ *satisfies*

1. *If* $z \in \partial\Omega$, $Q(z) := \left\{\xi \in A : \rho(z,\xi) \geq \frac{C}{2\sqrt{c_1 N}}\right\}$, *then* $\sum_{\xi\in Q(z)} |\langle z, \nu_\xi\rangle|^m < a$.
2. *If* $z \in \partial\Omega$, *then* $Q(z) \setminus A$ *has at most one element.*
3. *If* $\xi_0 \in A$, $z \in \partial\Omega$ *are such that* $\rho(z,\xi_0) \leq \frac{a}{\sqrt{c_2 N}}$, *then*
   (a) $Q(z) = A \setminus \{\xi_0\}$,
   (b) $|\langle z, \nu_{\xi_0}\rangle|^m > 1 - 2a^2$,
   (c) $|p_m(z)| > 1 - 2a^2 - a$.
4. $|p_m(z)| \leq \sum_{\xi\in A} |\langle z, \nu_\xi\rangle|^m < 1 + a$ *for all* $z \in \partial\Omega$.

Proof. There exists a constant $C > 2$ large enough that for $k \in \mathbb{N}_+$ we have

$$\sum_{k=1}^{\infty} \frac{q_1(k+2)^{2d-1}}{\exp\left(\frac{k^2C^2}{4}\right)} < a. \tag{7}$$

Due to Proposition 2.4 we can estimate:

$$\lim_{N\to\infty} \exp\left(\frac{-2a^2}{1-a^2N^{-1}}\right) = \exp\left(-2a^2\right) > 1 - 2a^2.$$

In particular we can choose $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

$$\exp\left(\frac{-2a^2}{1-a^2N^{-1}}\right) > 1 - 2a^2. \tag{8}$$

Let $z \in \partial\Omega$, $Q(z) := \left\{\xi \in A : \rho(z,\xi) \geq \frac{C}{2\sqrt{c_1 N}}\right\}$ and

$$A_k(z) = \left\{\xi \in A : \frac{kC}{2\sqrt{c_1 N}} \leq \rho(z,\xi) < \frac{(k+1)C}{2\sqrt{c_1 N}}\right\}.$$

Due to Lemma 2.2 the set $A_0(z)$ has at most 1 element and

$$\#A_k(z) \leq q_1(k+2)^{2d-1}. \tag{9}$$

Since $Q(z) \setminus A = A_0(z)$ we have the property (2). Due to Lemma 2.1 for $\xi \in A_k(z)$ we have

$$|\langle z, \nu_\xi\rangle| \overset{(3)}{\leq} 1 - c_1\rho^2(z,\xi) \leq 1 - \frac{k^2C^2}{4N}. \tag{10}$$

Now we may obtain the property (1):

$$\begin{aligned}\sum_{\xi\in Q(z)} |\langle z,\nu_\xi\rangle|^m &\leq \sum_{k=1}^{\infty}\sum_{\xi\in A_k(z)} |\langle z,\nu_\xi\rangle|^m \overset{(10)}{\leq} \sum_{k=1}^{\infty}\sum_{\xi\in A_k(z)} \left(1-\frac{k^2C^2}{4N}\right)^N \\ &\overset{(6)}{\leq} \sum_{k=1}^{\infty} \#A_k(z)\exp\left(-\frac{k^2C^2}{4}\right) \overset{(9)}{\leq} \sum_{k=1}^{\infty}\frac{q_1(k+2)^{2d-1}}{\exp\left(\frac{k^2C^2}{4}\right)} \overset{(7)}{<} a.\end{aligned}$$

Since $A_0(z)$ has at most one element we obtain the property (4):

$$|p_m(z)| \leq \sum_{\xi\in A}|\langle z,\nu_\xi\rangle|^m \leq 1 + \sum_{\xi\in Q(z)}|\langle z,\nu_\xi\rangle|^m < 1+a.$$

Now let $\xi_0 \in A, z \in \partial\Omega$ be such that $\rho(z,\xi_0) \leq \frac{a}{\sqrt{c_2 N}} < \frac{C}{2\sqrt{c_1 N}}$. Since $A_0(z)$ has at most 1 element we have $A_0(z) = \{\xi_0\}$, which gives the property (3a). Moreover we have:

$$|\langle z,\nu_{\xi_0}\rangle| \overset{(3)}{\geq} 1 - c_2\rho^2(z,\xi_0) \geq 1 - \frac{a^2}{N}. \tag{11}$$

Now we observe the property (3b) for $N \geq N_0$:

$$|\langle z,\nu_{\xi_0}\rangle|^m \overset{(11)}{\geq} \left(1-\frac{a^2}{N}\right)^{2N} \overset{(6)}{>} \exp\left(\frac{-a^2N^{-1}2N}{1-a^2N^{-1}}\right) \overset{(8)}{>} 1-2a^2.$$

Moreover we may conclude the property (3c):

$$|p_m(z)| \geq |\langle z,\nu_{\xi_0}\rangle|^m - \sum_{\xi\in Q(z)}|\langle z,\nu_\xi\rangle|^m > 1-2a^2-a,$$

which finishes the proof. □

We are ready for main result of this paper.

Theorem 2.6. *There exists $K \in \mathbb{N}$ such that for $0 < \epsilon < 1$ and for each pair of compact, circular and disjoint sets D, T such that $T \subset X$, $D \subset \partial\Omega$, we can choose $m_0 = m_0(D,T,\epsilon) \in \mathbb{N}$ and a sequence p_m of homogeneous polynomials of degree m which satisfy*

1. $|p_m(z)| \leq 2$ *for all* $z \in \partial\Omega$, $m > m_0$,
2. $\sum_{k=Km}^{K(m+1)-1}|p_k(z)|^2 \geq 0.25$ *for all* $z \in T$, $m > m_0$,
3. $\sum_{k=Km}^{K(m+1)-1}|p_k(z)|^2 \leq 2^{-(Km)^{1-\epsilon}}$ *for all* $z \in D$, $m > m_0$.

Proof. Let $0 < c_1 < c_2$ be from Lemma 2.1. For $a = \frac{1}{4}$ we can choose C from Lemma 2.5. Let $K = K(\alpha, \beta)$ be from Lemma 2.3 for $\alpha = \frac{1}{4\sqrt{c_2}}$ and $\beta = \frac{C}{\sqrt{c_1}}$. For $N = Km$ fix a maximal $1/(4\sqrt{c_2 N})$-separated subset $A \subset T$. Using Lemma 2.3 we can divide A into at most K disjoint $C/\sqrt{c_1 N}$-separated subsets $A_0, A_1, ..., A_{K-1}$. We define

$$p_{Km+j}(z) := \sum_{\xi \in A_j} \langle z, \nu_\xi \rangle^{Km+j}$$

for $j = 0, 1, ..., K-1$. From Lemma 2.5 we infer that there exists m_0 so high that for $m > m_0$ we have $|p_{Km+j}(z)| < 1 + a = \frac{5}{4} < 2$ for all $z \in \partial\Omega$ and $|p_{Km+j}(z)| > 1 - 2a^2 - a = \frac{6}{16} > 0.5$ for

$$z \in \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2 N}}\right).$$

Since $A = \bigcup_{j=0}^{K-1} A_j$ is a maximal $1/(4\sqrt{c_2 N})$-separated subset of T we conclude that

$$\bigcup_{j=0}^{K-1} \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2 N}}\right) = \bigcup_{\xi \in A} B\left(\xi; \frac{1}{4\sqrt{c_2 N}}\right) \supset T,$$

and from this follows that

$$\sum_{j=Km}^{K(m+1)-1} |p_j(z)|^2 \geq 0.25 \text{ for all } z \in T,\ m > m_0.$$

Without loss of generality we can assume that m_0 is so large that $\rho(z, w) > \sqrt{1/c_1 N^\epsilon}$ for all $z \in D$ and $w \in T$. Due to Lemma 2.2 we have

$$\#A_j \leq q_1 \left(\frac{\sqrt{c_1 N}}{C}\right)^{2d-1}.$$

If $\xi \in A$ and $z \in D$, then on the basis of Lemma 2.1 we have

$$|\langle z, \nu_\xi \rangle| \leq 1 - c_1 \rho^2(z, \xi) \leq 1 - \frac{1}{N^\epsilon}.$$

Now for m_0 large enough, $m > m_0$, $N = Km$ and $z \in D$ we may estimate

$$\begin{aligned}
\sum_{j=0}^{K-1} |p_{Km+j}(z)|^2 &\leq \sum_{j=0}^{K-1} \sum_{\xi \in A_j} |\langle z, \nu_\xi \rangle|^{2Km+j} \leq \sum_{\xi \in A} |\langle z, \nu_\xi \rangle|^N \\
&\leq \sum_{\xi \in A} \left(1 - \frac{1}{N^\epsilon}\right)^N \leq q_1 K \left(\frac{\sqrt{c_1 N}}{C}\right)^{2d-1} \left(1 - \frac{1}{N^\epsilon}\right)^{N^\epsilon N^{1-\epsilon}} \\
&\leq \frac{1}{2^{N^{1-\epsilon}}}.
\end{aligned}$$

□

As an application we can present the following result:

Theorem 2.7. *Assume that Ω is a circular, bounded and strictly convex domain with the boundary of class C^2. Then for any circular subset $E \subset \partial\Omega$ of type G_δ there exists a holomorphic function f which is square integrable on $\Omega \setminus \mathbb{D}E$ and such that $E = E^2_\Omega(f) := \left\{z \in \partial\Omega : \int_{\mathbb{D}z} |f|^2 \, d\mathfrak{L}^2_{\mathbb{D}z} = \infty\right\}$.*

Proof. Let σ be the natural measure on $\partial\Omega$. On the basis of [8, Theorem 2.6] there exist sequences $\{D_i\}_{i\in\mathbb{N}}$, $\{T_i\}_{i\in\mathbb{N}}$ of compact, circular sets in $\partial\Omega$ such that:

(1) $\bigcup_{i\in\mathbb{N}} D_i = \partial\Omega \setminus E$ and $D_j \subset D_{j+1}$ for $j \in \mathbb{N}$,
(2) $T_j \cap D_j = \emptyset$ for $j \in \mathbb{N}$,
(3) $E = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} T_i$,
(4) $\sigma\left(\partial\Omega \setminus (E \cup D_j)\right) \leq 2^{-j}$.

Since Ω is a strictly convex domain then $X = \partial\Omega$. Let K be a natural number from Theorem 2.6. We can use Theorem 2.6 once again and conclude that there exist a sequence of natural numbers $\{m_j\}_{j\in\mathbb{N}}$ and a sequence of homogeneous polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that

(1) $m_j < m_{j+1}$ for $j \in \mathbb{N}$,
(2) $|p_m(z)| \leq 2$ for all $z \in \partial\Omega$, $m > m_0$,
(3) $\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \geq 0.25$ for all $z \in T_j$,
(4) $\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \leq 2^{-j}$ for all $z \in D_j$.

Now we can define

$$f = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} \sqrt{k+1}\, p_k.$$

Observe that for $z \in \partial\Omega$ we have

$$\int_{\mathbb{D}z} |f|^2 d\mathfrak{L}^2_{\mathbb{D}z} = \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2.$$

In particular for $z \in E$ we have

$$\int_{\mathbb{D}z} |f|^2 d\mathfrak{L}^2_{\mathbb{D}z} \geq \sum_{j:z\in T_j} 0.25 = \infty.$$

If $z \in \partial\Omega \setminus E$, then there exists j_0 such that $z \in D_j$ for $j \geq j_0$. In particular

$$\int_{\mathbb{D}z} |f|^2 d\mathfrak{L}^2_{\mathbb{D}z} \leq \sum_{j=1}^{j_0-1} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 + \sum_{j=j_0}^{\infty} 2^{-j} < \infty.$$

Now we prove that f is square integrable on $\Omega \setminus \mathbb{D}E$. There exists $M > 0$ such that $\int_{\Omega\setminus\mathbb{D}E} |f|^2 d\mathfrak{L}^{2d} \leq M \int_{\partial\Omega\setminus E} \int_{\mathbb{D}z} |f|^2 d\mathfrak{L}^2_{\mathbb{D}z} d\sigma(z)$. In particular we may estimate

$$\begin{aligned}
\int_{\Omega\setminus\mathbb{D}E} |f|^2 d\mathfrak{L}^{2d} &\leq M \sum_{j=1}^{\infty} \int_{\partial\Omega\setminus E} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k|^2 d\sigma \\
&\leq M \sum_{j=1}^{\infty} 2^{-j} \sigma(D_j) + M \sum_{j=1}^{\infty} 4K\sigma(\partial\Omega \setminus (E \cup D_j)) \\
&\leq M\sigma(\partial\Omega) + 4KM < \infty.
\end{aligned}$$

□

References

[1] J. Globevnik, E. L. Stout, Highly noncontinuable functions on convex domains, Bull. Sci. Math. 104 (1980), 417-439. MR602409 (82c:32013)

[2] J. Globevnik, E. L. Stout, Holomorphic functions with highly noncontinuable boundary behavior, J. d'Analyse Math. 41 (1982), 211-216. MR687952 (84h:32016)

[3] J. Globevink, Holomorphic functions which are highly nonintegrable at the boundary, Israel J. Math. 115 (2000), 195-203. MR1749678 (2001c:32005)
[4] P. Jakóbczak, Description of exceptional sets in the circles for functions from the Bergman space, Czechoslovak Journal of Mathematics no. 47, (1997), 633-649. MR1479310 (98h:32042)
[5] P. Jakóbczak, Highly non-integrable functions in the unit ball. Israel J. Math 97 (1997), 175-181. MR1441246 (98b:32002)
[6] P. Jakóbczak, Exceptional sets of slices for functions from the Bergman Space in the ball, Canad. Math. Bull. 44(2), (2001), 150-159 MR1827853 (2002c:32007)
[7] P. Kot, Description of simple exceptional sets in the unit ball, Czechoslovak Mathematical Journal 54 (129) (2004), 55-63. MR2040218 (2004k:32008)
[8] P. Kot, Maximum sets of semicontinuous functions, Potential Analysis (2005) 23, 323-356. MR2139570 (2005m:31004)
[9] S. G. Krantz, Function theory of several complex variables, PWN Warsaw 1991. MR1162310 (93c:32001)
[10] J. Ryll and P. Wojtaszczyk, On homogeneous polynomials on a complex ball, Trans. Amer. Math. Soc. 276 (1983), 107-116. MR684495 (84f:32004)
[11] P. Wojtaszczyk, On values of homogeneous polynomials in discrete sets of points, Studia Math. 84 (1986), 97-104. MR871849 (88e:32014)
[12] P. Wojtaszczyk, On highly nonintegrable functions and homogeneous polynomials, Annales Polonici Mathematici no. 65, (1997), 245-251. MR1441179 (98a:32007)

Politechnika Krakowska, Instytut Matematyki, ul. Warszawska 24, 31-155 Kraków, Poland

E-mail address: pkot@usk.pk.edu.pl