THE COMPLETE ISOMORPHISM CLASS OF AN OPERATOR SPACE

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Abstract. Suppose $X$ is an infinite-dimensional operator space and $n$ is a positive integer. We prove that for every $C > 0$ there exists an operator space $\tilde{X}$ such that the formal identity map $id : X \to \tilde{X}$ is a complete isomorphism, $I_{M_n} \otimes id$ is an isometry, and $d_{cb}(X, \tilde{X}) > C$. This provides a non-commutative counterpart to a recent result of W. Johnson and E. Odell.

1. Introduction

Recently, W. Johnson and E. Odell [2] solved a problem of V. I. Gurarii by showing that the isomorphism class of any separable infinite-dimensional Banach space has infinite diameter with respect to the Banach–Mazur distance. More precisely, they proved that, for every $C > 0$, and for every separable infinite-dimensional Banach space $X$, there exists a Banach space $\tilde{X}$ isomorphic to $X$ such that the Banach–Mazur distance between these two spaces exceeds $C$. In this paper, we prove a non-commutative counterpart of this result:

**Theorem 1.** Suppose $X$ is an infinite-dimensional operator space and $n$ is a positive integer. Then for every $C > 0$ there exists an operator space $\tilde{X}$ such that the formal identity map $id : X \to \tilde{X}$ is a complete isomorphism, $I_{M_n} \otimes id$ is an isometry, and $d_{cb}(X, \tilde{X}) > C$.

Here and below, $E \otimes F$ refers to the minimal (or spatial) tensor product of operator spaces $E$ and $F$. $M_n$ stands for the space of $n \times n$ matrices, with its usual operator space structure. For the sake of brevity, we often use $M_n(E)$ instead of $M_n \otimes E$.

The proof of Theorem 1 (given in Section 3) relies on the properties of the operator space $\text{MIN}_n(X)$, which we explore in Section 2. Throughout the paper, we use freely the standard operator space and Banach space terminology and results. The reader is referred to [1, 8, 9] for operator spaces, and [5] for Banach spaces.
2. The functor MINₙ

Recall that (see [4, 6, 7]), for an operator space E and n ∈ ℕ, we define an operator space MINₙ(E) to be isometric to E on the Banach space level, and

\[\|x\|_{K₀ \otimes \text{MIN}_n(E)} = \sup\{|(I_{K₀} \otimes u)x| \mid u \in CB(E, Mₙ), \|u\|_{cb} \leq 1\}\]

for \(x \in K₀ \otimes E\) (\(K₀\) denotes the space of infinite matrices with finitely many non-zero entries). Letting \(I\) be the closed unit ball of \(CB(E, Mₙ)\), we can view \(\text{MIN}_n(E)\) as the image of the map \(U \in \ell_∞(I, Mₙ)\), where \(U : E \rightarrow \ell_∞(I, Mₙ)\) is defined by \(U(e) = (u(e))_{u \in I}\). By a compactness argument, for any finite-dimensional subspace \(F\) of \(\text{MIN}_n(E)\), and every \(\varepsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(F\) embeds into \(\ell_∞^{k}(Mₙ)\) \((1 + \varepsilon)\)-completely isomorphically. Consequently, \(\text{MIN}_n(E)\) is 1-exact. Recall that an operator space \(X\) is \(c\)-exact if, for every finite-dimensional subspace \(Z \rightarrow X\), and every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) and a \(\tilde{Z} \hookrightarrow M_N\) such that \(d_{cb}(Z, \tilde{Z}) < c + \varepsilon\). The smallest \(c\) satisfying this condition is called the exactness constant of \(X\) and is denoted by \(\text{ex}(X)\).

In addition to \(\text{MIN}_n\), in Section 3 we also use the “dual” functor \(\text{MAX}_n\) (see [4, 7] for more information). One should note that, for any operator space \(E\), \(\text{MIN}_1(E)\) and \(\text{MAX}_1(E)\) are identical to \(\text{MIN}(E)\) and \(\text{MAX}(E)\), respectively (the “minimal” and “maximal” quantizations).

In [6] we proved:

**Lemma 2.** Suppose \(X\) and \(Y\) are operator spaces and \(n \in \mathbb{N}\).

1. If \(u \in B(X, Y)\), then
   \[\|u\|_{CB(X, \text{MIN}_n(Y))} = \|I_{Mₙ} \otimes u\|_{B(Mₙ(X), Mₙ(Y))}\]
   In particular,
   \[\|u\|_{CB(\text{MIN}_n(X), \text{MIN}_n(Y))} = \|I_{Mₙ} \otimes u\|_{B(Mₙ(X), Mₙ(Y))}\]

2. If \(X\) is a subspace of \(Y\), then \(\text{MIN}_n(X)\) is a subspace of \(\text{MIN}_n(Y)\).

More results concerning \(\text{MIN}_n\) are needed:

**Lemma 3.** Suppose \(X\) is an operator space and \(n, s \in \mathbb{N}\). Then, for any \(x \in M_s(X)\),

\[\|x\|_{Mₙ(\text{MIN}_n(X))} = \sup \|P \otimes I_X)x(Q \otimes I_X)\|_{Mₙ(X)},\]

where the supremum runs over all orthogonal projections \(P, Q \in M_s\) of rank not exceeding \(n\).

**Corollary 4.** Suppose \(X\) is an operator space, \(n, s \in \mathbb{N}\), and \(s \geq n\). Then, for any \(x \in M_s(X)\),

\[\|x\|_{Mₙ(\text{MIN}_n(X))} \geq \left(\frac{n}{s}\right)^2 \|x\|_{Mₙ(X)}\]

**Proof of Lemma 3.** Clearly, we only need to consider the case of \(s > n\). If \(P\) and \(Q\) are orthogonal projections of rank \(\leq n\), then, by Ruan’s axioms,

\[\|(P \otimes I_X)x(Q \otimes I_X)\|_{Mₙ(\text{MIN}_n(X))} \leq \|x\|_{Mₙ(\text{MIN}_n(X))}\]

Furthermore, \((P \otimes I_X)x(Q \otimes I_X)\) can be thought of as an element of \(Mₙ(X)\). Thus, by [7],

\[\|(P \otimes I_X)x(Q \otimes I_X)\|_{Mₙ(X)} = \|(P \otimes I_X)x(Q \otimes I_X)\|_{Mₙ(\text{MIN}_n(X))}\]

hence \(\|x\|_{Mₙ(\text{MIN}_n(X))} \geq \sup \|(P \otimes I_X)x(Q \otimes I_X)\|_{Mₙ(X)}\).
To prove the reverse inequality, it suffices to prove that, whenever \( \|x\|_{\text{M}_s(\text{MIN}_n(X))} > 1 \), there exist orthogonal projections \( P \) and \( Q \), of rank \( \leq n \), for which \( \|(P \otimes I_X)x(Q \otimes I_X)\|_{\text{M}_s(X)} > 1 \). To do this, find a complete contraction \( u : X \to \text{M}_n \), for which \( \|y\|_{\text{M}_s(\text{MIN}_n(X))} > 1 \), where \( y = (I_{\text{M}_s} \otimes u)x \). Then there exist unit vectors \( \xi, \eta \in \ell^2_1(\ell^2_2) \) such that \( \langle y\xi, \eta \rangle > 1 \). Write \( \xi = (\xi_i)_{i=1}^n \) and \( \eta = (\eta_i)_{i=1}^n \), with \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \ell^2_2 \). Denote by \( P \) and \( Q \) the orthogonal projections onto \( \text{span}[\eta_i] \mid 1 \leq i \leq n \) and \( \text{span}[\xi_i] \mid 1 \leq i \leq n \), respectively. Then \( (Q \otimes I_X)\xi = \xi \), and \( (P \otimes I_X)\eta = \eta \). Therefore,

\[
\|(P \otimes I_X)x(Q \otimes I_X)\|_{\text{M}_s(X)} \geq \|(I_{\text{M}_s} \otimes u)(P \otimes I_X)x(Q \otimes I_X)\|_{\text{M}_s(\text{MIN}_n(X))}
\]

\[
= \|(P \otimes I_{\ell^2_2})y(Q \otimes I_{\ell^2_2})\|
\]

\[
\geq \|\langle (P \otimes I_{\ell^2_2})y(Q \otimes I_{\ell^2_2})\xi, \eta \rangle\| = \langle y\xi, \eta \rangle > 1,
\]

as desired. \( \square \)

**Proof of Corollary 5.** For \( S \subset \{1, \ldots, s\} \), we denote by \( P_S \) the corresponding basis projection on \( \ell^2_1 \). That is, \( P_S e_i = 0 \) if \( i \notin S \), and \( P_S e_i = e_i \) if \( i \in S \) (\( e_1, \ldots, e_s \) is the canonical basis in \( \ell^2_2 \)). An easy calculation shows that, for any \( x \in \text{M}_s(X) \),

\[
x = \frac{s^2}{n^2} \text{Ave}(P_{S_1} \otimes I_X)x(P_{S_2} \otimes I_X),
\]

where the average is taken over all subsets of \( \{1, \ldots, s\} \) of cardinality \( n \). By Lemma 3,

\[
\|x\|_{\text{M}_s(\text{MIN}_n(X))} \geq \|(P_{S_1} \otimes I_X)x(P_{S_2} \otimes I_X)\|_{\text{M}_s(X)}
\]

for any \( S_1 \) and \( S_2 \) as above. Taken together, the two centered expressions yield the proof. \( \square \)

**3. Proof of the main results**

To prove Theorem 1, we need to introduce some notation. For an operator space \( X \) and \( \lambda > 0 \), we denote by \( \lambda X \) the operator space, isomorphic to \( X \), and equipped with the norm \( \|x\|_{\kappa_0 \otimes \lambda X} = \lambda \|x\|_{\kappa_0 \otimes X} \) (here, \( x \in \kappa_0 \otimes X \)). For \( n \in \mathbb{N} \), set \( \alpha_n(X) = \|\text{id} : \text{MIN}_n(X) \to X\|_{cb} \) (note that \( \alpha_n(X) \in [1, \infty) \)). We have:

**Lemma 5.** Suppose \( n \in \mathbb{N} \), and suppose an operator space \( X \) is \( C \)-completely isomorphic to \( \lambda X \cap \text{MIN}_n(X) \) for every \( \lambda \in (0, 1) \). Then \( \alpha_n(X) \leq C \).

Note that, for a compatible couple \( (Y_0, Y_1) \) of operator spaces, \( Y_0 \cap Y_1 \) is defined by assigning to any \( y \in \kappa_0 \otimes (Y_0 \cap Y_1) \) the norm

\[
\|y\|_{\kappa_0 \otimes (Y_0 \cap Y_1)} = \max\{\|y\|_{\kappa_0 \otimes Y_0}, \|y\|_{\kappa_0 \otimes Y_1}\}
\]

(see Section 2.7 of \( \text{[9]} \) for more information about interpolation).

**Proof of Lemma 5.** Suppose, for the sake of contradiction, that \( \alpha_n(X) > C \). Then there exists \( x \in \text{M}_s(X) \) (\( s > n \)) such that \( \|x\|_{\text{M}_s(\text{MIN}_n(X))} \leq 1 \) and \( \|x\|_{\text{M}_s(X)} > C \). Pick \( \lambda \in (0, n^2/s^2) \). We shall show that, for any complete contraction \( T : X \to \bar{X} \) (here, \( \bar{X} = \lambda X \cap \text{MIN}_n(X) \)), we have

\[
\|(I_{\text{M}_s} \otimes T)x\|_{\text{M}_s(\bar{X})} \leq 1,
\]

thus obtaining a contradiction. Indeed, the last inequality implies that

\[
\|T^{-1}\|_{cb} \geq \frac{\|x\|_{\text{M}_s(X)}}{\|(I_{\text{M}_s} \otimes T)x\|_{\text{M}_s(\bar{X})}} > C.
\]
Let \( id : X \to \tilde{X} \) be the formal identity map. Since \( \lambda \in (0,1) \), \( I_{M_n} \otimes id : M_n(X) \to M_n(\tilde{X}) \) is an isometry. Therefore, by Lemma 2 \( id : MIN_n(X) \to MIN_n(\tilde{X}) \) is a complete isometry, and
\[
\|T\|_{CB(MIN_n(X))} = \|T\|_{CB(MIN_n(\tilde{X}))} = \|I_{M_n} \otimes T\| \leq \|T\|_{CB(X,\tilde{X})} \leq 1,
\]
and \( \|(I_{M_n} \otimes T)x\|_{M_n(MIN_n(X))} \leq 1 \).

Now recall that
\[
\|(I_{M_n} \otimes T)x\|_{M_n(\tilde{X})} = \max\{\lambda\|M_n(x)\|_{M_n(X)}, \|(I_{M_n} \otimes T)x\|_{M_n(MIN_n(X))}\}.
\]
By (3.2), \( \|(I_{M_n} \otimes T)x\|_{M_n(MIN_n(X))} \leq 1 \). Moreover, by Corollary 4
\[
\|(I_{M_n} \otimes T)x\|_{M_n(X)} \leq \frac{s^2}{n^2}\|(I_{M_n} \otimes T)x\|_{M_n(MIN_n(X))} \leq \frac{s^2}{n^2}.
\]
Therefore, by (3.3), and by the choice of \( \lambda \),
\[
\|(I_{M_n} \otimes T)x\|_{M_n(\tilde{X})} \leq \max\left\{\frac{\lambda s^2}{n^2}, 1\right\} \leq 1,
\]
which yields (3.4). \( \Box \)

Now, we are ready to establish the main result.

**Proof of Theorem 1.** Suppose, for the sake of contradiction, that \( X \) is an infinite-dimensional operator space, and there exists \( C > 0 \) such that \( d_{cb}(X,\tilde{X}) \leq C \) whenever \( \tilde{X} \) is completely isomorphic to \( X \), and \( I_{M_n} \otimes id : M_n(X) \to M_n(\tilde{X}) \) is an isometry. As noted in the proof of Lemma 5 the formal identity \( id : X \to \lambda X \cap MIN_n(X) \) is an isometry, and moreover, \( I_{M_n} \otimes id \) is an isometry. Furthermore,
\[
\|id\|_{CB(X,\lambda X \cap MIN_n(X))} = \max\{\|id\|_{CB(X,\lambda X)}, \|id\|_{CB(X,MIN_n(X))}\} \leq 1
\]
and
\[
\|id^{-1}\|_{CB(\lambda X \cap MIN_n(X),X)} \leq \|id^{-1}\|_{CB(\lambda X,X)} = \lambda^{-1}.
\]
Thus, \( X \) is completely isomorphic to \( \lambda X \cap MIN_n(X) \). If \( d_{cb}(X,\lambda X \cap MIN_n(X)) \leq C \) for any \( \lambda \in (0,1) \), then, by Lemma 5 \( a_n(X) \leq C \). Therefore, \( \text{ex}(X) \leq C \).

Now consider the space \( \tilde{X} = \lambda^{-1}X + \text{MAX}_n(X) \) (as before, \( \lambda \in (0,1) \)). Denoting, once again, by \( id \) the formal identity map from \( X \) to \( \tilde{X} \), we see that \( I_{M_n} \otimes id \) is an isometry (that is, \( \|x\|_{M_n(\tilde{X})} = \|x\|_{M_n(X)} \) for any \( x \in M_n(X) \)), \( id^{-1} \) is a complete contraction, and \( \|id\| \leq \lambda^{-1} \). If \( d_{cb}(\tilde{X},X) \leq C \), then \( d_{cb}(\tilde{X}^*,X^*) \leq C \). However, by [7],
\[
\tilde{X}^* = (\lambda^{-1}X)^* \cap (\text{MAX}_n(X))^* = \lambda X^* \cap MIN_n(X^*).
\]
Thus, by Lemma 5 \( a_n(X^*) \leq C \); hence \( \text{ex}(X^*) \leq C \).

The exactness of both \( X \) and \( X^* \) implies, by [11] (see also [10]), that \( X \) is completely isomorphic to \( H^* \oplus K^c \), where \( H^* \) and \( K^c \) denote the Hilbert spaces \( H \) and \( K \) equipped with their column, resp. row, operator space structures. In particular, there exists a constant \( c > 0 \) such that (1) \( \text{ex}(X) \leq c \), and (2) \( X \) contains \( c \)-completely complemented subspaces of arbitrarily large dimension (in particular, those arising from \( H^*_0 \oplus K^c_0 \), where \( H_0 \) and \( K_0 \) are subspaces of \( H \) and \( K \), respectively).
It remains to construct an operator space $\hat{X}$, completely isomorphic to $X$, and such that $I_{M_n} \otimes id$ is an isometry, yet $\text{ex}(\hat{X}) > Cc$. Then the inequality $d_{cb}(X, \hat{X}) \geq \text{ex}(\hat{X})/\text{ex}(X) > C$ will deliver the desired contradiction.

To this end, pick a subspace $E$ of $X$ such that $(4C^2/n^2)^2 < N = \dim E < \infty$ and such that there exists a projection $P$ from $X$ onto $E$ with $\|P\|_{cb} \leq c$. Let $j : E \to \text{MAX}(E)$ be the formal identity map, and consider $u = (cn^2)^{-1} jP \in CB(X, \text{MAX}(E))$. Define the operator space $\hat{X}$ by setting, for $x \in K_0 \otimes X$,

$$\|x\|_{K_0 \otimes \hat{X}} = \max\{\|x\|_{K_0 \otimes X}, \|(I_{K_0} \otimes u)x\|_{K_0 \otimes \text{MAX}(E)}\}.$$ 

Denoting by $id$ the formal identity map from $X$ to $\hat{X}$, we observe that

$$\|id\|_{cb} \leq \|u\|_{cb} \leq (cn^2)^{-1}\|j\|_{cb}\|P\|_{cb} \leq N/n^2 < \infty$$

(here, we use the fact that $\|j\|_{cb} \leq N$: see, e.g., Chapter 3 of [9], or [12] for a better estimate). Moreover, $\|u\| \leq (cn^2)^{-1}\|P\| \approx n^{-2}$. By Corollary 4 (and the fact that, for every operator space $Y$, $\text{MIN}(Y) = \text{MIN}_1(Y)$), $\|e\|_{M_n(\text{MAX}(E))} \leq n^2\|e\|_{M_n(\text{MIN}(E))}$ for any $e \in M_n(Y)$. Therefore,

$$\|I_{M_n} \otimes u\| \leq n^2\|I_{M_n} \otimes u\|_{B(M_n(X), M_n(\text{MIN}(E)))} = n^2\|u\| = 1.$$

Thus, $I_{M_n} \otimes id$ is an isometry.

It remains to estimate $\text{ex}(\hat{X})$ from below. Denote by $\hat{E}$ the image of $E$ in $\hat{X}$, and by $\hat{j}$ the formal identity map from $\hat{E}$ to $\text{MAX}(E)$. Then $\|\hat{j}^{-1}\|_{cb} = \|j^{-1}\| \leq 1$. Moreover,

$$\|e\|_{M_n(\hat{E})} \geq \|(I_{M_n} \otimes u)e\|_{M_n(\text{MAX}(E))} = (cn^2)^{-1}\|(I_{M_n} \otimes j)e\|_{M_n(\text{MAX}(E))}$$

for any $e \in M_n(\hat{E})$: hence $\|\hat{j}\|_{cb} \leq cn^2$. Therefore,

$$\text{ex}(\hat{X}) \geq \text{ex}(\hat{E}) \geq \frac{\text{ex}(\text{MAX}(E))}{\|\hat{j}\|_{cb}\|j^{-1}\|_{cb}} \geq \frac{\sqrt{N}}{4cn^2} > Cc$$

(by [3], $\text{ex}(\text{MAX}(E)) \geq \sqrt{N}/4$).

\[\Box\]

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**References**


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