PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 12, December 2007, Pages 4023–4029
S 0002-9939(07)09265-9
Article electronically published on September 5, 2007

DEGREE ONE MAPS BETWEEN HYPERBOLIC SMALL 3-MANIFOLDS

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(Communicated by Daniel Ruberman)

Abstract. We construct the first examples of degree one maps between non-homeomorphic closed hyperbolic small 3-manifolds.

1. Introduction

All terminologies not defined in this paper are standard; see [Ja], [Ro] and [Th].

A compact 3-manifold M is *small* if it is orientable, irreducible and if any incompressible surface in M is parallel to ∂M. A knot k in a 3-manifold M is small if its exterior $M - \text{int}\mathcal{N}(k)$, denoted by $E(k)$, is a small 3-manifold.

The main theme of this work is to study the existence of degree one maps between closed hyperbolic small 3-manifolds. All known and expected closed small 3-manifolds are either Seifert fibered or hyperbolic. The existence of degree one maps between closed small Seifert manifolds has been studied (see [HWZ] and the references there).

For closed hyperbolic small 3-manifolds, even some general results for degree one maps between them have been established (see [RW] and [BW2]), but to our knowledge, there are no known examples of such degree one maps. Indeed the authors of [RW] wondered how to find such degree one maps when they were working on [RW], which is the first motivation of the present work.

Note that there are many ways of producing degree one maps between closed hyperbolic 3-manifolds (cf. [BW1], [Ka], [Ru]), but none of them insure that both hyperbolic manifolds are small. The following theorem provides the first examples of degree one maps between non-homeomorphic closed small hyperbolic 3-manifolds.

Theorem 1.1. *There are infinitely many pairs of non-homeomorphic, closed, small hyperbolic* 3*-manifolds* (M, N) *such that there is a degree one map* $f : M \to N$.

Remark 1.2. The construction of degree one maps in Theorem 1.1 is based on Thurston's hyperbolic Dehn surgery theorem [Th] and Proposition 2.2, which provides non-trivial proper degree one maps between the exteriors of hyperbolic small knots in the 3-sphere. These are also, to our knowledge, the first such examples. Our examples of degree one maps in Proposition 2.2 are constructed from the exteriors of some n-bridge knots, with $n > 2$, to the exteriors of some 2-bridge knots.

Received by the editors March 16, 2006 and, in revised form, June 21, 2006.
2000 *Mathematics Subject Classification.* Primary 57M50.
The second author is partially supported by MSTC and NSFC.

It would be interesting to find a degree one map between the exteriors of two hyperbolic 2-bridge knots,[1] since such a degree one map will produce degree one maps between closed 3-manifolds obtained by Dehn surgeries on those knots with the same coefficients, and for most given coefficients those two closed 3-manifolds are small and hyperbolic.

In general we wonder if there is a degree one map $f : E(k_1) \to E(k_2)$ for k_1 and k_2 two knots in S^3, is the bridge number of k_1 not smaller than that of k_2 [W]?

2. Examples of degree one maps between small hyperbolic 3-manifolds

This section is devoted to the proof of Theorem 1.1. Our construction uses tangle sum in the sense of Conway.

A *tangle* $T = (B^3, a_1 \cup a_2)$ is a properly embedded disjoint pair of arcs $(a_1 \cup a_2, \partial a_1 \cup \partial a_2) \hookrightarrow (B^3, \partial B^3)$.

Such a tangle is *irreducible* if there is no 2-sphere $S \subset B^3$ meeting transversely an arc a_i in two points such that the intersection of the 3-ball V bounded by S in B^3 and a_i is a knotted arc in V.

We denote by $T_0 = (B^3, b_1 \cup b_2)$ the *trivial tangle.* It is formed by two unknotted arcs separated by a properly embedded disk in B^3 disjoint from them.

Our construction of non-trivial degree one maps between small closed hyperbolic 3-manifolds relies on Thurston' hyperbolic Dehn surgery theorem and the following two propositions:

Proposition 2.1. *There exists an irreducible, non-trivial tangle $T = (B^3, a_1 \cup a_2)$ with the following properties:*

(1) *The 2-fold covering of B^3 branched along $a_1 \cup a_2$ is the exterior E of a small hyperbolic knot in S^3.*

(2) *There is a proper degree one map $f : T = (B^3, a_1 \cup a_2) \to T_0 = (B^3, b_1 \cup b_2)$ onto the trivial tangle T_0 such that*

- *the restriction $f|_{\partial B^3} : \partial B^3 \to \partial B^3$ is a homeomorphism,*
- *for $i \in \{1, 2\}, f^{-1}(b_i) = a_i$ and the restriction $f| : a_i \to b_i$ is a homeomorphism.*

By using a Conway sum of the tangle T with rational tangles, we deduce from Proposition 2.1 the following result:

Proposition 2.2. *There are infinitely many hyperbolic small knots in S^3 with bridge number ≥ 3 such that their exteriors admit a proper degree one map to the exterior of a hyperbolic 2-bridge knot.*

The three subsections below are devoted to the proofs of Proposition 2.1, Proposition 2.2 and Theorem 1.1, respectively.

2.1. **Construction of the tangle T and proof of Proposition 2.1.** We consider the non-alternating knot $\tilde{k}$ with eight crossings, the knot 8_{21} in Rolfsen's book tabulation [Ro]. It is the Montesinos knot $\mathcal{M}(1; 1/2, 2/3, 2/3)$ with 3-branches in the notation of [BoZ]. By Oertel's work [O1] it is a small hyperbolic knot. It is also a fibred knot with fibre a surface of genus 2; see for example [Ga].

[1] After the acceptance of this paper, T. Ohtsuki, R. Riley and M. Sakuma [ORS] informed us that they have constructed degree one maps between 2-bridge knot complements.

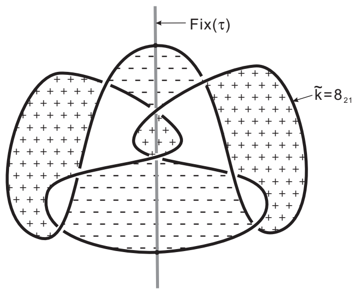

FIGURE 1

Moreover, it is a strongly invertible knot; i.e., there is a smooth involution τ of the pair $(S^3, \tilde{k})$ such that $Fix(\tau)$, the fixed point set of τ, is an unknotted circle and meets $\tilde{k}$ in exactly two points (Figure 1).

After an isotopy of the fibration of the exterior $E = E(\tilde{k})$, we can assume that the restriction of τ on E is fibre preserving. Hence there are two fibres F_1 and F_2 invariant by τ in E. Moreover $Fix(\tau) \cap E = \tilde{a}_1 \cup \tilde{a}_2$, where $\tilde{a}_i$ is a properly embedded arc in F_i which separates F_i into two symmetric parts, $i = 1, 2$ (Figure 1).

The orbifold quotient E/τ has an underlying space $|E/\tau|$ homeomorphic to B^3 and the ramification locus formed by the union of two properly embedded disjoint arcs $a_1 \cup a_2$ in E/τ. That gives naturally a tangle $T = (|E/\tau|, a_1 \cup a_2)$, which by construction verifies property (1) of Proposition 2.1.

To verify property (2), we construct a proper degree one map $\tilde{f} : E \to S^1 \times D^2$, which is equivariant with respect to the involution τ on E and to the involution τ_0 obtained on $S^1 \times D^2$ by extending the involution $\tau|_{\partial E}$ to the solid torus.

We identify ∂E with $S^1 \times \partial D^2$ by choosing a preferred meridian-longitude coordinate system $(\tilde{\mu}, \tilde{\lambda})$ on ∂E and by identifying the meridian $\tilde{\mu}$ with $S^1 \times \{\star\}$ and the longitude $\tilde{\lambda}$, which is the boundary of a fibre of the fibration of E, with $\{\star\} \times \partial D^2$.

The involution τ_0 preserves two meridian disks D_1 and D_2, and $Fix(\tau_0) \cap D_i = \tilde{b}_i$ is a properly embedded arc in D_i for $i \in \{1, 2\}$. In particular the orbifold quotient $S^1 \times D^2/\tau_0$ has B^3 as its underlying space and the union of the two disjoint properly embedded arcs $b_1 \cup b_2$ as its ramification locus. That gives a trivial tangle $T_0 = (|S^1 \times D^2/\tau_0|, b_1 \cup b_2)$.

The construction of the equivariant degree one map $\tilde{f}$ is done in three steps:

Step 1. *By the choice of τ_0 on $S^1 \times D^2$ and the identification of ∂E with $S^1 \times \partial D^2$, we can take $\tilde{f} : \partial E \to \partial(S^1 \times D^2)$ to be the identity.*

Step 2. *We extend $\tilde{f}$ equivariantly to the two fibres F_1 and F_2, so that for $i \in \{1, 2\}$, $\tilde{f}(F_i) = D_i$, $\tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i$ and $\tilde{f}|_{\tilde{a}_i} : \tilde{a}_i \to \tilde{b}_i$ is a homeomorphism.*

Since the properly embedded arc $\tilde{a}_i \subset F_i$ is separating, we have $F_i = F_i' \cup \tau(F_i')$ for $i \in \{1, 2\}$, where F_i' is a genus 1 surface. In the same way, we have $D_i = D_i' \cup \tau(D_i')$ for $i \in \{1, 2\}$, where D_i' is a disk.

We consider the pinch $p_i : F_i' \to D_i'$ which is the identity on a collar neighborhood of $\partial F_i'$, for $i \in \{1, 2\}$. Then we extend $\tilde{f}$ equivariantly on F_i by taking $\tilde{f}|_{F_i'} = p_i$ and $\tilde{f}|_{\tau(F_i')} = \tau_0 \circ p_i \circ \tau$, for $i \in \{1, 2\}$.

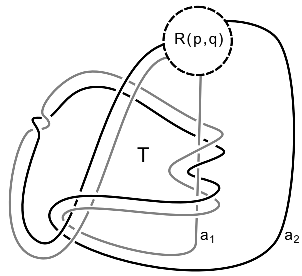

FIGURE 2

At this point, for $i \in \{1,2\}, \tilde{f}(F_i) = D_i, \tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i$ and $\tilde{f}| : \tilde{a}_i \to \tilde{b}_i$ is a homeomorphism.

Step 3. *We extend $\tilde{f}$ equivariantly to the remaining part of E.*

By cutting E along the two fibres $F_1 \cup F_2$, we get $E = F \times [1,2] \cup_{F_1 \cup F_2} \tau(F \times [1,2])$. In the same way $S^1 \times D^2 = D^2 \times [1,2] \cup_{D_1 \cup D_2} \tau_0(D^2 \times [1,2])$.

From Steps 1 and 2 we have defined a degree one map $\tilde{f} : \partial(F \times [1,2]) \to \partial(D^2 \times [1,2])$. Since $D^2 \times [1,2]$ is a 3-ball, we can extend this map to a degree one map $\tilde{f} : F \times [1,2] \to D^2 \times [1,2]$.

Since $\tilde{f} : F_1 \cup F_2 \cup \partial E \to D_1 \cup D_2 \cup \partial(S^1 \times D^2)$ is already equivariant with respect to τ and τ_0, we can finally define the map $\tau_0 \circ \tilde{f} \circ \tau : \tau(F \times [1,2]) \to \tau_0(D^2 \times [1,2])$ to get the desired proper degree one map $\tilde{f} : E \to S^1 \times D^2$ with the following properties:

(a) $\tau_0 \circ \tilde{f} = \tilde{f} \circ \tau$.

(b) The restriction $\tilde{f}|_{\partial E} : \partial E \to S^1 \times \partial D^2$ is the identity with respect to the chosen parametrization of ∂E.

(c) For $i \in \{1,2\}, \tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i$ and the restriction $\tilde{f}|_{\tilde{a}_i} : \tilde{a}_i \to \tilde{b}_i$ is a homeomorphism.

Now this equivariant proper degree one map $\tilde{f}$ covers through the involutions τ and τ_0 a proper degree one map: $f : T = (|E/\tau|, a_1 \cup a_2) \to T_0 = (|S^1 \times D^2/\tau_0|, b_1 \cup b_2)$, between the tangle T and the trivial tangle T_0. This degree one map verifies the properties of (2) of Proposition 2.1 because of properties (b) and (c) of $\tilde{f}$. That finishes the proof of Proposition 2.1.

2.2. **Proof of Proposition 2.2.** In the parametrization of ∂E by the preferred meridian-longitude pair $(\tilde{\mu}, \tilde{\lambda})$, any simple closed curve on ∂E is determined by a unique slope (p,q), where $p \geq 0$ and $q \in \mathbb{Z}$ are coprime. We denote by $E(p,q)$ the closed orientable 3-manifold obtained from E by Dehn filling ∂E along the slope (p,q).

The involution τ on the knot exterior E can be extended to the glued solid torus to get an involution still called τ on the closed 3-manifold $E(p,q)$. According to Montesinos's construction [Mo], the quotient of $E(p,q)$ by τ is S^3 and the branching locus $k(p,q)$ is a knot or a link with two components, according to whether p is odd or even.

This knot or link $k(p,q)$ is obtained by a Conway sum of the tangle T with the rational tangle $R(p,q)$ of type (p,q) (see Figure 2), where the outside tangle T is drawn by using Montesinos's technique.

Now, the degree one map $f : T \to T_0$ can be extended trivially by a homeomorphism to a degree one map of pairs $g : (S^3, k(p,q)) \to (S^3, b(p,q))$, such that

- $b(p,q)$ is the 2-bridge knot or link, obtained by a Conway sum of the trivial tangle T_0 with the rational tangle $R(p,q)$,
- $g^{-1}(b(p,q)) = k(p,q)$ and the restriction $\bar{g}| : k(p,q) \to b(p,q)$ is a homeomorphism.

This last property follows immediately from the properties of (2) of f in Proposition 2.1.

By considering the restriction of g to the exterior of $k(p,q)$, Proposition 2.2 follows now from the following lemma:

Lemma 2.3. *For* $p = 2p' + 1\,, p' > 1$ *and* $q \neq np \pm 1\,, n \in \mathbb{Z}$, $k(p,q)$ *and* $b(p,q)$ *are small hyperbolic knots in* S^3*. Moreover* $k(p,q)$ *has bridge number* ≥ 3*.*

Proof. By the classification of 2-bridge knots or links (cf. [BuZ]), $b(p,q)$ is a hyperbolic knot iff $p = 2p' + 1, p' > 1$ and $q \neq np \pm 1, n \in \mathbb{Z}$. By [HT] it is a small knot.

By Oertel [O2] (see also [Dun1]) the boundary slopes of the knot 8_{21} are the following integral slopes:

$$\{(12,-1),(6,-1),(2,-1),(0,1),(1,1)\}.$$

In particular all odd $p > 1$ are not in this list. So for the slopes (p,q) given in Lemma 2.3, the closed 3-manifolds $E(p,q)$ are small.

Since $E(p,q)$ is the 2-fold branched covering of the knot $k(p,q)$, it follows from the equivariant Dehn lemma that $k(p,q)$ is a small knot in S^3 (cf. [GL]). Hence it is either a hyperbolic or a torus knot.

It cannot be a torus knot since its exterior admits a proper degree one map onto the exterior of a hyperbolic 2-bridge knot. This would contradict the fact that the simplicial volume of a torus knot exterior vanishes, while it is always non-zero for a hyperbolic knot exterior.

The knot $k(p,q)$ has bridge number ≥ 3. Otherwise its 2-fold branched covering would be a lens space and by the cyclic surgery theorem [CGLS] q would be equal to ± 1, since $\tilde{k}$ is a hyperbolic knot. This contradicts our choice for q.

That finishes the proof of Lemma 2.3 and hence of Proposition 2.2. □

2.3. **Degree one map between closed small hyperbolic 3-manifolds and proof of Theorem 1.1.** In Proposition 2.2 we constructed a small hyperbolic knot $k_1 \subset S^3$ with bridge number ≥ 3 and a hyperbolic 2-bridge knot $k_2 \subset S^3$ such that there is a degree one map $g : (S^3, k_1) \to (S^3, k_2)$ such that $g^{-1}(k_2) = k_1$ and such that the restriction $\bar{g}| : k_1 \to k_2$ is a homeomorphism. Let E_1 and E_2 be the exteriors of k_1 and k_2, respectively. As before we choose for $i \in \{1,2\}$ a trivialization of ∂E_i by a preferred meridian-longitude pair (μ_i, λ_i). Then (after possibly some isotopy on the boundary) g induces a proper degree one map $h : E_1 \to E_2$ such that

- the restriction $h| : \partial E_1 \to \partial E_2$ is a homeomorphism,
- $h(\mu_1) = \mu_2$ and $h(\lambda_1) = \lambda_2$.

For any slope (r,s) on ∂E_i, $i = 1,2$, this degree one map h extends trivially by a homeomorphism to a degree one map $h_{r,s} : E_1(r,s) \to E_2(r,s)$. Now Theorem 1.1 is a consequence of the following lemma:

Lemma 2.4. *For almost all slopes (r, s) (i.e. except finitely many), the two closed orientable 3-manifolds $E_1(r, s)$ and $E_2(r, s)$ are small, hyperbolic and not homeomorphic.*

Proof. By [Hat] there are only finitely many slopes (p, q) that can be boundary slopes on either ∂E_1 or ∂E_2. Since k_1 and k_2 are small knots in S^3, if (r, s) avoids this finite set of slopes, then $E_1(r, s)$ and $E_2(r, s)$ are small, closed 3-manifolds.

Let $v_i = vol(E_i)$ be the hyperbolic volume of $E_i, i \in \{1, 2\}$. Since k_1 is not a 2-bridge knot, E_1 is not homeomorphic to E_2, because knots are determined by their complement in S^3. Since there is a proper degree one $h : E_1 \to E_2$, Gromov-Thurston's strict rigidity theorem ([Th, Chap. 6], [Dun2]) implies that $v_1 > v_2$.

By Thurston's hyperbolic Dehn surgery theorem ([Th, Chap. 4]; see also [BP, Appendix B]) and Schläfli's formula (cf. [Mi]), there is a constant $c > 0$ (depending only on k_1 and k_2) such that for $r^2 + s^2 \geq c^2$ the following happens:

- both $E_1(r, s)$ and $E_2(r, s)$ are hyperbolic,
- $v_1 > vol(E_1(r, s)) > v_2 > vol(E_2(r, s))$.

Therefore if (r, s) avoids the finite set of slopes $r^2 + s^2 < c^2$, then $E_1(r, s)$ and $E_2(r, s)$ are both hyperbolic and not mutually homeomorphic.

This finishes the proof of Lemma 2.4 and of Theorem 1.1. □

Remark 2.5. With further effort, one can show that the bridge number of the knots $k(p, q)$ in Proposition 2.2 is at most 4 and the Heegaard genus of the 3-manifolds $E_1(r, s)$ in Lemma 2.3 is at most 3.

Acknowledgments

We wish to thank A. Kawauchi and A. Reid for helpful conversations.

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