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ON SCH AND THE APPROACHABILITY PROPERTY

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Abstract. We construct a model of $\neg SCH + \neg AP +$ (Very Good Scale). This answers questions of Cummings, Foreman, Magidor and Woodin.

1. Introduction

Notions of Very Good Scale$_\kappa$ (VGS_κ), Weak square $_\kappa$ ($\square^*_\kappa$) and the Approachability Property$_\kappa$ (AP_κ), for a singular κ, play a central role in Singular Cardinals Combinatorics. They were extensively studied by Shelah [9, 10, 11] and by Cummings, Foreman and Magidor [2].

All of these properties break down above a supercompact cardinal as was shown by S. Shelah in [9]. By R. Solovay [12], the Singular Cardinal Hypothesis (SCH) holds above strong compact cardinals. Also by Ben-David and Magidor [1] the Prikry forcing adds $\square^*_\kappa$. Hence it is natural to ask about interconnections between SCH and the above principles. Cummings, Foreman and Magidor [2] asked if VGS_κ implies $\square^*_\kappa$. Woodin previously asked if it is possible to have $\neg SCH_\kappa + \neg \square^*_\kappa$. In [4] the positive answer to the second question was claimed. The second author found a gap in the argument and was able to show that the forcing used there (extender based forcing with long extenders) actually adds a $\square^*_\kappa$- sequence.

Our goal here will be to give a negative answer to the first question and a positive answer to the second. Thus we prove the following:

Theorem 1.1. *Suppose κ is a supercompact cardinal. Then there is a generic extension in which κ is a strong limit singular cardinal of cofinality ω so that*

(a) $2^\kappa > \kappa^+$;
(b) $\neg AP_\kappa$ *(and hence* $\neg \square^*_\kappa$*)*;
(c) VGS_κ.

Using standard methods we can make κ into $\aleph_{\omega^2}$. Namely the following holds:

Theorem 1.2. *Suppose κ is a supercompact cardinal. Then there is a generic extension in which $\kappa = \aleph_{\omega^2}$ is a strong limit cardinal so that*

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(a) $2^{\aleph_{\omega^2}} > \aleph_{\omega^2+1}$;
(b) $\neg AP_{\aleph_{\omega^2}}$;
(c) $VGS_{\aleph_{\omega^2}}$.

2. The main construction

Let us first recall some basic definitions:

Definition 2.1. (S. Shelah [9]) A sequence $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ is called an AP_κ-sequence iff

(a) $\lim(\alpha) \to C_\alpha$ is a club in α and o.t.$(C_\alpha) = cf(\alpha)$.
(b) There is a club subset D of κ^+ such that

$$\forall\alpha \in D\ \forall\beta < \alpha\ \exists\gamma < \alpha\ C_\alpha \cap \beta = C_\gamma.$$

It is not hard to see that $\square^*_\kappa \to AP_\kappa$.

Definition 2.2. (a) Let $\langle \kappa_n \mid n < \omega\rangle$ be a sequence of regular cardinals such that $\bigcup_{n<\omega} \kappa_n = \kappa$. A sequence $\langle f_\alpha \mid \alpha < \kappa^+\rangle \subseteq \prod_{n<\omega} \kappa_n$ is called a *very good scale* on $\prod_{n<\omega}\kappa_n$ iff

(i) $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ is a scale on $\prod_{n<\omega}\kappa_n$, i.e., for every $f \in \prod_{n<\omega}\kappa_n$ there exists $\beta < \kappa^+$ and $n < \omega$ such that $f(m) < f_\beta(m)$ for every $m > n$ and for every $\alpha < \beta < \kappa^+, f_\alpha(m) < f_\beta(m)$ for almost every m;
(ii) for every $\beta < \kappa^+$ such that $\omega < cf(\beta)$ there exists a club C of β and $n < \omega$ such that $f_{\gamma_1}(m) < f_{\gamma_2}(m)$ for every $\gamma_1 < \gamma_2 \in C$ and $m > n$.

(b) VGS_κ holds iff there exists a sequence $\langle \kappa_n \mid n < \omega\rangle$ and $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ such that $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ is a very good scale on $\prod_{n<\omega}\kappa_n$.

Definition 2.3. (S. Shelah [9]) Let κ be an uncountable cardinal such that $cf(\kappa) = \omega$, and $d : [\kappa^+]^2 \to \omega$.

(a) d is called *normal* if $\forall\beta\ \forall n < \omega \mid \{\alpha < \beta | d(\alpha, \beta) \leq n\}| < \kappa$.
(b) d is called *subadditive* if $\forall\alpha < \beta < \gamma < \kappa^+$, $d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.
(c) $S_0(d) = \{\alpha < \kappa^+ \mid \exists A, B \subseteq \alpha$ unbounded in α such that

$$\forall\beta \in B\ \exists n_\beta \in \omega\ \forall\alpha \in A \cap \beta\ d(\alpha, \beta) \leq n_\beta.$$

The next Lemma, which was stated in Shelah [9], shows that such a function always exists. Let us give the proof for the benefit of the reader.

Lemma 2.4. *(S.Shelah [9]) There exists a normal subadditive function $d : [\kappa^+]^2 \to \omega$ for every uncountable cardinal κ such that $cf(\kappa) = \omega$.*

Proof. Fix an increasing sequence $\langle \kappa_n \mid n < \omega\rangle$ of regular cardinals cofinal in κ. For every $d : [\kappa^+]^2 \to \omega$, let $A(\beta, n)$ and $(A(\beta, \leq n))$ denote the set of all $\gamma < \beta$ such that $d(\gamma, \beta) = n$ and $d(\gamma, \beta) \leq n$ respectively. We are going to define the function $d\restriction_{\gamma\times\gamma}$ by induction on γ such that for every $\beta \geq \kappa$ the size of $A(\beta, n)$ is at most κ_n. For every $\gamma < \beta < \kappa$, we define $d(\gamma, \beta)$ to be the least n such that $\gamma < \kappa_n$. Assume that $d\restriction_{\gamma\times\gamma}$ is defined. If $\gamma = \eta+1$ is a successor, then let $d(\alpha, \gamma) = d(\alpha, \eta)$ for every $\alpha < \eta$ and $d(\eta, \gamma) = 0$. It is simple to see that $d\restriction_{\gamma\times\gamma}$ is normal and subadditive. Assume now that γ is a limit ordinal. Let $\langle B_i \mid i < \omega\rangle$ be a $\subseteq$- increasing sequence such that $\bigcup_{i<\omega} B_i = \gamma$ and $|B_i| = \kappa_i$. We define the sets $A(\gamma, n)$ by induction on n as follows: By the induction hypothesis we can find $A(\gamma, 0)$ such that $B_0 \subseteq A(\gamma, 0)$

and for every $\alpha \in A(\gamma,0)$ the set $A(\alpha,0)$ is contained in $A(\gamma,0)$. Assume that $A(\gamma,n-1)$ is defined. Set

$$X_n = \bigcup\{A(\alpha,n) \mid \alpha \in \bigcup_{i<n} A(\gamma,i)\}.$$

Note that by the induction hypothesis $|X_n| \leq \kappa_n$. By another application of the induction hypothesis, it is possible to find $Y_n \supseteq X_n \cup B_n$ of size κ_n such that $A(\alpha,\leq n) \subseteq Y_n$ for every $\alpha \in Y_n$. Let $A(\gamma,n) = Y_n - \bigcup_{i<n} A(\gamma,i)$. Note that the size of $A(\gamma,n)$ is κ_n. Now define

$$d(\alpha,\gamma) = n \text{ iff } \alpha \in A(\gamma,n).$$

Let us show that the function $d\restriction_{\gamma\times\gamma}$ is subadditive: Let $\beta < \alpha < \gamma$. Set $n = d(\alpha,\gamma)$ and $k = d(\beta,\alpha)$. We consider two cases:

Case 1: $n \geq k$. But then by our construction, $\beta \in \bigcup_{i\leq n} A(\gamma,i)$ and so $d(\beta,\gamma) \leq n$.

Case 2: $n < k$. But then $\beta \in X_k$ and so $\beta \in Y_k$ and $d(\beta,\gamma) \leq k$.

This finishes the proof of the lemma. □

Fact 2.5. (S. Shelah [9]) Suppose that κ is a strong limit cardinal of cofinality ω and $d, d' : [\kappa^+]^2 \to \omega$ are two normal functions. Then $S_0(d) \equiv S_0(d') \pmod{\mathcal{D}_{\kappa^+}}$ (where $\mathcal{D}_{\kappa^+}$ is the club filter).

Fact 2.6. (S. Shelah [9]) Let κ be a singular strong limit cardinal of cofinality ω. The statement AP_κ is equivalent to the existence of a normal function $d : [\kappa^+]^2 \to \omega$ such that $S_0(d)$ contains a club.

$S_0(d)$ is in fact the set of all approachable points and AP_κ means that modulo the club filter every point less than κ^+ is approachable.

Let us now prove Theorem 1.1. We start with a model V of $ZFC + GCH$ such that $V \models$ "κ is supercompact". Iterate first in Backward Easton fashion the Cohen forcing $C(\alpha,\alpha^{+\omega+2})$ for each inaccessible $\alpha \leq \kappa$, where $C(\alpha,\alpha^{+\omega+2})$ is defined as the poset consisting of functions f such that $Dom(f)$ is a subset of $\alpha^{+\omega+2}$ of size less than α and for every $\beta \in Dom(f)$, $f(\beta)$ is a partial function from α to α of size less than α.

Let $\mathcal{P}_{<\kappa}$ denote the iteration below κ and $\mathcal{P}_\kappa = \mathcal{P}_{<\kappa} * C(\kappa,\kappa^{+\omega+2})$. Note that the forcing $\mathcal{P}_\kappa$ preserves the cofinality of the ordinals. Let G be a generic subset of $\mathcal{P}_\kappa$. Denote $G_{<\kappa} = \mathcal{P}_{<\kappa} \cap G$. Let for each $\alpha < \kappa^{+\omega+2}$, F_α denote the α-th generic function from κ to 2 in G, i.e. $\bigcup\{f(\alpha) \mid f \in G\}$.

Fix in V a normal ultrafilter U over $P_\kappa(\kappa^{+\omega+2})$. Let $j : V \to M \simeq Ult(V,U)$ be the corresponding elementary embedding. Then $crit(j) = \kappa$ and ${}^{\kappa^{+\omega+2}}M \subseteq M$.

By standard arguments (see [6]) j extends in $V[G]$ to an elementary embedding $j^* : V[G] \to M[G^*]$, where $G^* \cap \mathcal{P}_\kappa = G_\kappa$ and G^* above κ is constructed in $V[G]$ using closure of the forcing and the fact that the number of dense sets we need to meet is small. Also, over $j(\kappa)$, we need to start with the condition $\{\langle j(\alpha), F_\alpha\rangle \mid \alpha < \kappa^{+\omega+2}\}$ in order to satisfy j"$G \subseteq G^*$. This means that for each $\alpha < \kappa^{+\omega+2}$ the function $F_{j(\alpha)}$ (i.e. the one G^* defines to be $j(\alpha)$-th function from $j(\kappa)$ to $j(\kappa)$) should extend F_α.

Note that above κ we are free in choosing values of $F_{j(\alpha)}$. Let us require $F_{j(\alpha)}(\kappa) = \alpha$ for each $\alpha < \kappa^{+\omega+2}$ and then continue to build G^*.

Let $U^* = \{X \subseteq P_\kappa(\kappa^{+\omega+2}) \mid j$"$\kappa^{+\omega+2} \in j^*(X)\}$. Then $U^* \supseteq U$ and it is a normal ultrafilter over $P_\kappa(\kappa^{+\omega+2})$ in $V[G]$.

Lemma 2.7. (1) *For every* $\xi < \rho < \kappa^{+\omega+2}$ $\{P \in P_\kappa(\kappa^{+\omega+2}) \mid F_\xi(P\cap\kappa) < F_\rho(P\cap\kappa)\} \in U^*$.

(2) *For each* $f \in \prod\{\delta^{+\omega+1} \mid \delta < \kappa,\ \delta$ *is an inaccessible*$\}$ *there is* $\xi < \kappa^{+\omega+1}$ *such that*

$$\{P \in P_\kappa(\kappa^{+\omega+1}) \mid f(P\cap\kappa) = F_\xi(P\cap\kappa)\} \in U^*.$$

Proof. (1) In $M[G^*]$, we have $j^*(F_\xi)(\kappa) = \xi < j^*(F_\rho)(\kappa) = \rho$. Hence the conclusion follows from the definition of U^*.

(2) Again, in $M[G^*]$, we have $j^*(f)(\kappa) < \kappa^{+\omega+1}$. Let $\xi = j^*(f)(\kappa)$. It is simple to see that ξ satisfies the desired property.

□

For every $n \in \omega$ let U_n be the projection of U^* on $P_\kappa(\kappa^{+n})$, i.e., $X \in U_n$ iff $\{P \in P_\kappa(\kappa^{+\omega+2}) \mid P\cap\kappa^{+n} \in X\} \in U^*$. Clearly U_n is a normal ultrafilter on $\mathcal{P}_\kappa(\kappa^{+n})$.

Let $a, b \in P_\kappa(\kappa^{+n})$ and $b\cap\kappa \in \kappa$. Set

$$a \underset{\sim}{\subsetneq} b \leftrightarrow (a \subseteq b) \wedge o.t.(p(a)) < b\cap\kappa\ .$$

Lemma 2.8. [7]

(a) $\forall a \in P_\kappa(\kappa^{+n})$ $\{b \in P_\kappa(\kappa^{+n}) \mid a \underset{\sim}{\subsetneq} b\} \in U_n$.

(b) $\{a \in P_\kappa(\kappa^{+n}) \mid a\cap\kappa$ *is inaccessible and* $a\cap\kappa \in \kappa\} \in U_n$.

(c) *Let* $\vec{X} = \langle X_a \mid a \in P_\kappa(\kappa^{+n})\rangle$ *be a sequence of sets from* U_n. *Then* $\Delta\vec{X} = \{b \in P_\kappa(\kappa^{+n}) \mid \forall a \in P_\kappa(\kappa^{+n})\ a \underset{\sim}{\subsetneq} b \to b \in X_a\} \in U_n$. *(*$\Delta\vec{X}$ *is called the diagonal intersection of* $\vec{X}$.)

We now define a version of the diagonal supercompact Prikry forcing.

Definition 2.9. $p \in Q$ iff $p = \langle a_0^p, a_1^p, \ldots, a_{n-1}^p, X_n^p, X_{n+1}^p, \ldots\rangle$ where

(i) $\forall \ell < n$ $a_\ell^p \in P_\kappa(\kappa^{+\ell})$ and $a_\ell^p\cap\kappa$ is an inaccessible cardinal;
(ii) $\forall m \geq n$ $X_m^p \in U_m$;
(iii) $\forall m \geq n$ $\forall b \in X_m^p \forall \ell < n$ $a_\ell^p \underset{\sim}{\subsetneq} b$;
(iv) $\forall i < j < n$ $a_i^p \underset{\sim}{\subsetneq} a_j^p$.

n is called the length of p and will be denoted by $\ell(p)$.

Definition 2.10. Let $p, q \in Q$. Then $p \leq^* q$ iff

(i) $\ell(p) = \ell(q)$;
(ii) $\forall \ell < \ell(p)$ $a_\ell^p = a_\ell^q$;
(iii) $\forall m \geq \ell(p)$ $X_m^q \subseteq X_m^p$.

Definition 2.11. Suppose that $p \in Q$ and $\vec{a} = \langle \vec{a}(\ell(p)), \cdots, \vec{a}(m)\rangle$ where $\vec{a}(i) \in X_i^p$ for every $\ell(p) \leq i \leq m$. We denote by $p^\frown\langle\vec{a}\rangle$ the sequence

$$\langle a_1^p, \ldots, a_{\ell(p)-1}^p, \vec{a}(\ell(p)), \ldots, \vec{a}(m), Y_{m+1}, Y_{m+2}, \ldots\rangle,$$

where

$$Y_n = \{b \in X_n^p \mid \forall \ell(p) \leq i \leq m\ \vec{a}(i) \underset{\sim}{\subsetneq} b\}$$

for every $n \geq m+1$.

By Lemma 2.8(a) it is easy to see that $Y_k \in U_k$, for each $k > m$ and $p^\frown\langle\vec{a}\rangle \in Q$.

Definition 2.12. Let $p, q \in Q$. $p \le q$ iff there exists $\vec{a}$ such that $p^\frown\langle\vec{a}\rangle \le^* q$.

The proof of the next two claims is quite standard, and it uses the same arguments as in the case of the ordinary diagonal Prikry forcing notion; see [5].

Lemma 2.13. (a) *$\langle Q \ \le, \le^* \ \rangle$ is a Prikry type forcing notion, i.e., if σ is a statement in the forcing language, then for every $p \in P$ there exists $p \le^* q \in P$ such that q forces σ or $\neg\sigma$.*
(b) *$\langle Q, \le^* \rangle$ is κ-closed.* □

Proof. (a) Assume for simplicity that $\ell(p) = 0$. Let σ be a statement in the forcing language. Since any two conditions of length 0 are compatible, it is sufficient to find a condition q such that $\ell(q) = 0$ and q decides σ. Let $\vec{a} = \langle a_0, ..., a_n\rangle$ be such that $a_i \in P_\kappa(\kappa^i)$ for every $i \le n$ and $a_i \underset{\sim}{\subsetneq} a_{i+1}$ for every $i < n$. Define a sequence $X_{\vec{a}}$ as follows: If there exists a sequence $\vec{X} = \langle X_m \mid m \ge n+1\rangle$ such that $\vec{a}^\frown\vec{X}$ is in Q and decides σ, then let $X_{\vec{a}}$ be such a sequence. Otherwise let $X_{\vec{a}}(m) = P_\kappa(\kappa^{+m})$ for every $m \ge n+1$. Using Lemma 2.8 (c), we can find $Y_n \in U_n$ such that for every $\underset{\sim}{\subsetneq}$ increasing sequence $\vec{a} = \langle a_0, ..., a_n\rangle$ and for every $m \ge n+1$,

$$\{b \in Y_m \mid a_n \underset{\sim}{\subsetneq} b\} \subseteq X_{\vec{a}}(m).$$

Using Lemma 2.8 again, we can find a condition $q = \langle Y'_0, Y'_1, ..., \rangle$ such that $Y'_i \subseteq Y_i$ with the following property: if there exists $\vec{a} \in \prod_{i \le n} Y'_i$ such that $q^\frown\langle\vec{a}\rangle$ decides σ, then $q^\frown\langle\vec{a}\rangle$ decides σ for every $\vec{a} \in \prod_{i \le n} Y'_i$ (in the same way). Now it is easy to see that q decides σ and is of length 0.

(b) This is an immediate consequence of the κ completeness of the ultrafilters. □

Lemma 2.14. *Let G_Q be Q generic over $V[G]$.*
(a) *$\langle Q, \le \rangle$ does not add any new bounded subsets to κ.*
(b) *$\forall n \ cf^{V[G][G_Q]}(\kappa^{+n}) = \omega$ (in fact for every $\kappa \le \delta < \kappa^{+\omega}$ such that $cf^{V[G]}(\delta) \ge \kappa$ we have $cf^{V[G][G_Q]}(\delta) = \omega$).*

Proof. (a) This is a consequence of Lemma 2.13.

(b) Let $\langle a_0, a_1, ...\rangle$ be the generic sequence added by G_Q. Let $\delta < \kappa^{+\omega}$ be such that $cf^{V[G]}(\delta) \ge k$. A simple density argument shows that the sequence $\gamma_m = \sup(a_m \cap \delta)$ is cofinal in δ. □

The next lemma is crucial for the construction.

Lemma 2.15. *Q_3 is $\kappa^{+\omega+1}$- c.c.*

Proof. Just note that the total number of finite sequences used in the conditions is $\kappa^{+\omega}$. □

The next lemma now follows easily.

Lemma 2.16. (a) *$V[G][G_Q] \models$ " κ is strong limit, $2^\kappa = \kappa^{+2} = (\kappa^{+\omega+2})^{V[G]}$ and $cf(\kappa) = \omega$".*
(b) *If $V[G][G_Q] \models$ "$\omega < \mu = cf(\mu) < \kappa$ and $f : \mu \to V[G]$", then there is $X \in V[G]$ unbounded in μ such that $f \restriction X \in V[G]$.*

Proof. (b): Let $\dot{f}$ be a Q name for f. Let D be the set of all conditions p in P such that for every $\underset{\sim}{\subsetneq}$ increasing sequence $\vec{a}$ in $X^p_{\ell(p)} \times ... \times X^p_m$ and for every $i < \eta$, if there exists $p^\frown\langle\vec{a}\rangle \leq^* q \in P$ such that q decides the value of $\dot{f}(i)$, then $p^\frown\langle\vec{a}\rangle$ already decides the value of $\dot{f}(i)$. Let us show that D is dense. Let p be a condition in P. Assume for simplicity that $\ell(p) = 0$. Using the fact that the ultrafilters U_n are κ closed, pick for every $\underset{\sim}{\subsetneq}$ increasing element $\vec{a}$ from $P_\kappa(\kappa) \times P_\kappa(\kappa^+) \times ... \times P_\kappa(\kappa^{+n})$ a condition $p_{\vec{a}}$ with initial segment $\vec{a}$ such that for every $i < \eta$ if there is a direct extension of $p_{\vec{a}}$ which decides the value of $\dot{f}(i)$, then $p_{\vec{a}}$ already decides this value. Using Lemma 2.8(c), find a condition q such that $\ell(q) = 0$ and $q^\frown\vec{a} \geq^* p_{\vec{a}}$ for every $\underset{\sim}{\subsetneq}$ increasing sequence $\vec{a}$ in $X^p_0 \times ... \times X^p_m$. Since every two conditions of length 0 are compatible, we can assume that $q \geq^* p$. But q is in D and so D is dense in P.

Pick $p \in D \cap G_Q$ and let $p \restriction_{\ell(p)}^\frown \vec{a}$ be the Prikry sequence added by G_Q. For every $i < \eta$ we can find $m(i) < \omega$ and $q \in G$ such that q is a direct extension of $p^\frown\vec{a} \restriction_{m(i)}$ and q decides the value of $\dot{f}(i)$. But then $p^\frown\vec{a} \restriction_{m(i)}$ already decides the value of $\dot{f}(i)$. Since $cf(\eta) > \omega$, we can find a stationary set $X' \subseteq \eta$ and m such that $m = m(i)$ for every i in X'. In $V[G]$ let

$$X = \{i < \eta \mid p^\frown\vec{a} \restriction_m \text{ decides the value of } \dot{f}(i)\}.$$

Then X is as required. $\square$

Definition 2.17. A submodel N of $H_{\kappa^{+\omega+1}}$ is called a *supercompact submodel* iff

(1) $|N| < \kappa$ and $N \cap \kappa$ is a cardinal less than κ;
(2) $cf(\sup(N \cap \kappa^{+\omega+1}) = (N \cap \kappa)^{+\omega+1}$;
(3) for every $A \subseteq \kappa^{+\omega+1}$ there exists $B \in N$ such that

$$A \cap N = B \cap N.$$

It is simple to see that if κ is $\kappa^{+\omega+2}$ supercompact, then the collection of all supercompact submodels is stationary. The following lemma was proved by Shelah in [9]:

Lemma 2.18 ([9])**.** *Suppose that κ is $\kappa^{+\omega+2}$ supercompact and $d : [\kappa^{+\omega+1}]^2 \to \omega$ is normal and subadditive. Let S be the set of $\delta < \kappa^{+\omega+1}$ such that $\delta = \sup(N \cap \kappa^{+\omega+1})$ for some supercompact submodel. Then*

$$S \subseteq \kappa^{+\omega+1} \cap cf(< \kappa) \text{ is stationary}$$

and

$$S \subseteq \kappa^{+\omega+1} - S_0(d).$$

$\square$

Let G_Q be a generic subset of Q over $V[G]$.

Proposition 2.19. $V[G][G_Q] \models \neg AP_\kappa$.

Proof. The idea is to try to find a normal function d such that $\kappa^+ - S_0(d)$ is stationary. The next lemma shows that it is sufficient to find any two-place function d with this property.

Lemma 2.20. *Let κ be a cardinal such that $cf(\kappa) = \omega$. If there is $d : [\kappa^+]^2 \to \omega$ such that $\kappa^+ - S_0(d)$ is stationary, then there is a normal $\overline{d}$ such that $\kappa^+ - S_0(\overline{d})$ is stationary.*

Proof. Let $d_0 : [\kappa^+]^2 \to \omega$ be any normal function. Set $\overline{d} = d + d_0$. We need to show that $\overline{d}$ is normal and that $\kappa^+ - S_0(\overline{d})$ is stationary.

(i) $\overline{d}$ is normal: pick $\beta < \kappa^+$. Since $\overline{d}(\alpha, \beta) \geq d_0(\alpha, \beta)$, we see that $\{\alpha < \beta \mid \overline{d}(\alpha, \beta) \leq n\} \subseteq \{\alpha < \beta \mid d_0(\alpha, \beta) \leq n\}$ and the conclusion follows from the normality of d_0.

(ii) $\kappa^+ - S_0(\overline{d})$ is stationary: for every $\beta \in S_0(\overline{d})$, there are $A, B \subseteq \beta$ unbounded in β which satisfy Definition 2.3(c). Since $\forall \alpha < \beta\ d(\alpha, \beta) \leq \overline{d}(\alpha, \beta)$ we get $\beta \in S_0(d)$. We proved that $S_0(\overline{d}) \subseteq S_0(d)$ or equivalently $\kappa^+ - S_0(\overline{d}) \supseteq \kappa^+ - S_0(d)$. But $\kappa^+ - S_0(d)$ is stationary and so $\kappa^+ - S_0(\overline{d})$ is also stationary. □

Work in $V[G]$ and pick any normal subadditive function $d : [\kappa^{+\omega+1}]^2 \to \omega$. Set $S = \kappa^+ - (S_0(d))^{V[G]}$. Since κ is $\kappa^{+\omega+2}$ supercompact, we can apply Lemma 2.18 and conclude that S is stationary. In $V[G, G_Q]$, d is a function from $[\kappa^+]^2$ to ω, but d is no longer normal. Let us prove that $V[G, G_Q] \models S \subseteq \kappa^+ - S_0(d)$. Otherwise there exists $\delta \in S \cap S_0(d)$. Pick $A, B \in V[G, G_Q]$ unbounded in δ such that

$$\forall \beta \in B\ \exists n_\beta\ \forall \alpha < \beta\ \alpha \in A \to d(\alpha, \beta) \leq n_\beta.$$

Since $\omega < cf^{V[G,G_Q]}(\delta) < \kappa$, we can use Lemma 2.16(b) to find $\overline{A}, \overline{B} \in V[G]$ unbounded in δ such that $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$. We have that for every β in $\overline{B}$ there exists n_β such that $d(\alpha, \beta) \leq n_\beta$ for every $\alpha < \beta$ in $\overline{A}$. Thus $V[G] \models \delta \in S_0(d)$. This contradicts Lemma 2.18. By Lemma 2.18 and the fact that Q is $\kappa^{+\omega+1}$- c.c., we get that S is stationary in $V[G, G_Q]$, and therefore $\kappa^+ - S_0(d)$ is stationary. By Fact 2.5 and Lemma 2.20 we get $V[G, G_Q] \models \neg AP_\kappa$ as required. □

Proposition 2.21. *$V[G, G_Q] \models VGS_\kappa$.*

Proof. Let $\langle P_n \mid n < \omega \rangle$ be the supercompact Prikry sequence defined from G_Q, i.e., for each $m < \omega$, there is $p \in G_Q$ such that

$$\langle P_n \mid n < m \rangle = \langle a_0^p, \ldots, a_{m-1}^p \rangle\ .$$

Let $\kappa_n = P_n \cap \kappa$ for each $n < \omega$. Then $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of inaccessible cardinals cofinal in κ. Consider $\prod_{n<\omega} \kappa_n^{+\omega+1}$. For each $\alpha < (\kappa^{+\omega+1})^V = \kappa^+$ and $n < \omega$ let $t_\alpha(n) = F_\alpha(\kappa_n)$ if $F_\alpha(\kappa_n) < \kappa_n^{+\omega+1}$ and $t_\alpha(n) = 0$ otherwise. Clearly $\{t_\alpha \mid \alpha < \kappa^+\} \subseteq \prod_{n<\omega} \kappa_n^{+\omega+1}$. We show below that it is a scale and a very good one.

Claim 2.22. For each $\alpha < \beta < \kappa^+$ we have $t_\alpha(n) < t_\beta(n)$ for all but finitely many n's.

Proof. Note that the set $Y = \{P \in P_\kappa(\kappa^{+\omega+2}) \mid F_\alpha(P \cap \kappa) < F_\beta(P \cap \kappa) < (P \cap \kappa)^{+\omega+1}\} \in U^*$. Hence for each $n < \omega$ the projection Y_n of Y to $P_\kappa(\kappa^{+n})$ belongs to U_n, i.e., the set $Y_n = \{P \cap \kappa^{+n} \mid P \in Y\} \in U_n$. By a simple density argument, we can find $q \in G_Q$ such that $X_n^q \subseteq Y_n$ for every $n \geq \ell(q)$. But by the choice of Y_n, q forces that $t_\alpha(n) < t_\beta(n) < \kappa_n^{+\omega+1}$ for every $n \geq \ell(q)$ as required. □

Claim 2.23. For each $t \in \prod_{n<\omega} \kappa_n^{+\omega+1}$ there exists α such that $t_\alpha(n) > t(n)$ for all but finitely many n's.

Proof. Let $\dot{t}$ be a name for t and assume that $\Vdash \dot{t} \in \prod_{n<\omega} \kappa_n^{+\omega+1}$. Let us show that for every q there is $q \leq^* p$ and $\alpha < \kappa^{+\omega+1}$ such that

$$(*) \qquad \Vdash t_\alpha(n) > t(n) \text{ for almost every } n.$$

Assume for simplicity that $\ell(q) = 0$. Let $\vec{a}$ be as in Definition 2.11. Since $q^\frown\langle\vec{a}\rangle$ forces that $t(m) < (\vec{a}(m) \cap \kappa)^{+\omega+1} < \kappa$, we can use the Prikry condition and the fact that $\leq^*$ is κ closed to find $r \geq^* q^\frown\langle\vec{a}\rangle$, which determines the value of $\dot{t}(m)$. Using the same arguments as in the proof of the Prikry property, we can find $p' \geq^* q$ such that for every $\vec{a}$ as in Definition 2.11 there exists $\beta_{\vec{a}}$ such that $p'^\frown\langle\vec{a}\rangle$ forces that $\dot{t}(m) = \beta_{\vec{a}}$. Let $h(\vec{a}) = \beta_{\vec{a}}$. Note that for each n we have

$$(**) \qquad j^*(h)(\kappa, j^{''}(\kappa^+), ..., j^{''}(\kappa^{+n})) = \alpha_n < \kappa^{+\omega+1}.$$

Let $\alpha = \sup\{\alpha_n \mid n < \omega\} + 1$. By the construction of F_α, we know that $j^*(F_\alpha)(\kappa) = \alpha$, and so using (**), we can shrink the sets of measure one of p' to form a condition p so that for every $\vec{a}$, $\beta_{\vec{a}} < F_\alpha(\vec{a}(m) \cap \kappa)$. It is simple to see that α and p satisfy (*). ☐

Claim 2.24. $\langle t_\alpha \mid \alpha < \kappa^+\rangle$ is a very good scale.

Proof. Let $\alpha < \kappa^+$ be of uncountable cofinality below κ. Then $(cf\alpha)^{V[G,G_Q]} = (cf\alpha)^{V[G]} = (cf(\alpha))^V$. Then pick a club $C \subseteq \alpha$ in V with $o.t.(p(C)) = cf\alpha$. Now by the choice of U^* we have

$$A = \{P \in \mathcal{P}_\kappa(\kappa^{+\omega+2}) \mid \forall\gamma, \beta \in C(\gamma < \beta \to F_\gamma(P \cap \kappa) < F_\beta(P \cap \kappa)\} \in U^*$$

since $j^*(F_\gamma)(\kappa) = \gamma < \beta = j^*(F_\beta)(\kappa)$ in $M[G^*]$ for each $\gamma < \beta < (\kappa^{+\omega+1})^V$ and $|C| = cf\alpha < \kappa$.

Let A_n be the projection of A to $P_\kappa(\kappa^{+n})$. The set of q such that $X_n^q \subseteq A_n$ for every $n \geq \ell(q)$ is dense in Q and so we can find such a condition q in G_Q. Now it is simple to see that q forces that $t_\gamma(m) < t_\beta(m)$ for every $m \geq \ell(q)$, and we are done. ☐

Remark 2.25.
 (a) The same argument shows that $\langle t_\alpha \mid \alpha < \kappa^{++} = (\kappa^{+\omega+2})^V\rangle$ is a very good scale in $\prod_{n<\omega} \kappa_n^{+\omega+2}$.
 (b) It is possible instead of using the explicit construction producing the scale just to start with an indestructible under κ-directed closed forcing supercompact cardinal κ. Then set $2^\kappa = \kappa^{+\omega+2}$. Any functions H_α such that $[H_\alpha]_{\mathcal{V}} = \alpha$ ($\alpha < \kappa^{+\omega+2}$) with $\mathcal{V}$ being the projection of a supercompact measure from $\mathcal{P}_\kappa(\kappa^{+\omega+2})$ to κ can be used instead of the F_α's.
 (c) Cummings and Foreman have shown in an unpublished work that in V^Q there is a scale on $\prod_n \kappa_n^{n+1}$ which is not good. This gives an alternative argument for the failure of AP_κ in V^Q.

Our next task will be to push everything down to $\aleph_{\omega^2}$. The argument is quite standard, so let us only concentrate on the main points.

Let $j : V \to M$ be a $\kappa^{+\omega+1}$ supercompact embedding. We would like to find an extension j^* of j to $V[G]$ such that all the ordinals $\alpha < j(\kappa)$ will be of the form $j^*(g)(\kappa)$ for some $g : \kappa \to \kappa$.

Work in $V[G]$. Since ${}^{\kappa^{+\omega+1}}M[G] \subseteq M[G]$ and the number of antichains of $j(P_{<\kappa})/G$ in $M[G]$ is $\kappa^{+\omega+2}$, we can find a generic subset H of $j(P_{<\kappa})/G$ over $M[G]$. Set $M^* = M[G * H]$ and let $\langle x_\alpha \mid \alpha < \kappa^{+\omega+2}\rangle$ be an enumeration of $j(\kappa)$.

Lemma 2.26. *There exists a generic subset K of $C := (C(j(\kappa), j(\kappa^{+\omega+1})))^{M[G*H]}$ with the following properties:*
 (a) $j''(G_\kappa) \subseteq K$;
 (b) $j(F_\alpha)(\kappa) = x_\alpha$, *where F_α is the α-th Cohen function.*

Proof. Let $\langle A_i \mid i < \kappa^{+\omega+2}\rangle$ be an enumeration of the antichains of C in M^*. Since C is $\kappa^{+\omega+1}$ closed in $V[G]$, we can find a C generic subset K^* over M^*. For each $\alpha < j(\kappa^{+\omega+1})$, set $K^* \restriction_\alpha = \{p \restriction_\alpha \mid p \in K^*\}$. Set $F = \bigcup j''(G_\kappa)$. Note that $F \subseteq j''(\kappa^{+\omega+2}) \times \kappa \times \kappa$. For each $\alpha < j(\kappa^{+\omega+1})$, we let $K \restriction_\alpha$ be the set of all conditions p such that for every $\delta < \kappa^{+\omega+2}$, if $j(\delta) < \alpha$, then $p(j(\delta)) \supseteq j(F(\delta)) = F(\delta)$ and $p(j(\delta))(\kappa) = x_\delta$. Note that since $\sup(j''(\kappa^{+\omega+2})) = j(\kappa^{+\omega+2})$, we need to change only $\kappa^{+\omega+1}$ many coordinates and so p is in M^*. Since $K^* \restriction_\alpha$ is $C \restriction_\alpha := (C(j(\kappa), \alpha))^{M^*}$ generic over M^*, and the number of changes is small (that is, $\kappa^{+\omega+1} < j(\kappa)$), we conclude that $K \restriction_\alpha$ is also $(C(j(\kappa), \alpha))^{M^*}$ generic over M^*. Let $K = \bigcup_{\alpha < j(\kappa^{+\omega+2})} K \restriction_\alpha$. Since every antichain in C is an antichain of $C \restriction_\alpha$ for some $\alpha < j(\kappa^{+\omega+2})$, we get that K is C generic over M^*. Also by our construction, K satisfies (a) and (b) and we are done. $\square$

Let $j^* : V[G] \to M^*[K]$ be the extension of j to $V[G]$. Let U_n^* be the κ^{+n} ultrafilter derived from j^*, i.e.,

$$X \in U_n^* \ \text{ iff } j''(\kappa^{+n}) \in j^*(X).$$

Let $i_n^* : V[G] \to Ult(V[G], U_n^*) \cong N_n$ and $k_n : N_n \to M^*[K]$. By standard arguments we can find an $M^*[K]$ generic subset H^* of $Col(\kappa^{+\omega+2}, j(\kappa))$. Now by our construction, the range of k_n contains $\{j^*(F_\alpha)(\kappa) \mid \alpha < \kappa^{+\omega+2}\} \cup \{k_n(i_n(\kappa))\} = j(\kappa)+1$ and so $crit(k_n) > i_n(\kappa)$. But since $(Col(\kappa^{+\omega+2}, i_n(\kappa)))^{N_n}$ satisfies $i_n(\kappa^+)$-c.c, the filter generated by $k_n^{-1}(H^*)$ is $(Col(\kappa^{+\omega+2}, i_n(\kappa)))^{N_n}$ generic over N_n. Denote this filter by H_n.

Now we are ready to define a new forcing Q.

Definition 2.27. $p \in Q$ iff

$$p = \langle a_0^p, f_0^p, a_1^p, f_1^p, \dots, a_{n-1}^p, f_{n-1}^p X_n^p, F_n^p, X_{n+1}^p, F_{n+1}^p, \dots\rangle$$

so that the following holds:

(1) $\langle a_0^p, a_1^p, \dots, a_{n-1}^p, X_n^p, X_{n+1}^p, \dots\rangle$ is as in Definition 2.9 with the U_n^*'s replacing the U_n's.
(2) $\forall \ell < n-1$ $f_\ell^p \in Col((a_\ell^p \cap \kappa)^{+\omega+2}, a_{\ell+1}^p \cap \kappa)$.
(3) $f_{n-1}^p \in Col((a_{n-1}^p \cap \kappa)^{+\omega+2}, \kappa)$.
(4) $\forall \ell \geq n$ F_n is a function on X_n^p such that
  (a) $F_n(P) \in Col((P \cap \kappa)^{+\omega+2}, \kappa)$.
  (b) $j_n^*(F_n)(j_n''\kappa^+) \in H_n$.

All the previous claims remain valid here. Only in Lemma 2.16(b) do we restrict ourselves to μ's of the form $\kappa_n^{+\omega+1}$ or $\kappa_n^{+\omega+2}$ for the Prikry sequence $\langle \kappa_n \mid n < \omega\rangle$.

Let us conclude with two questions.

Question 1. Is it consistent that $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$ and $\neg\square^*_{\aleph_\omega}$ (or $\neg AP_{\aleph_\omega}$)?

Question 2. Is it is consistent that GCH holds below κ, $2^\kappa > \kappa^+$ and $\neg\square^*_\kappa$ (or $\neg AP_\kappa$) for a singular cardinal κ?

Question 3 (Cummings)**.** Is it consistent that there is a very good scale on every increasing sequence $\langle \kappa_n \mid n < \omega\rangle$ of regular cardinals such that $\bigcup_{n<\omega} \kappa_n = \kappa$ and $\neg AP_\kappa$?

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