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MOD 4 GALOIS REPRESENTATIONS AND ELLIPTIC CURVES

CHRISTOPHER HOLDEN

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ABSTRACT. Galois representations $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}/n)$ with cyclotomic determinant all arise from the n-torsion of elliptic curves for $n = 2, 3, 5$. For $n = 4$, we show the existence of more than a million such representations which are surjective and do not arise from any elliptic curve.

1. INTRODUCTION

Given an elliptic curve E defined over $\mathbb{Q}$, its n-torsion subgroup $E[n]$ is isomorphic to $(\mathbb{Z}/n)^2$ as an abstract group. The Galois group of $\mathbb{Q}(E[n])$ over $\mathbb{Q}$ embeds into $GL_2(\mathbb{Z}/n)$. We then have a continuous Galois representation

$$\overline{\rho}_{E,n} : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[n]) \hookrightarrow GL_2(\mathbb{Z}/n).$$

The composition of $\overline{\rho}_{E,n}$ with the determinant map is the mod n cyclotomic character. Thus $\overline{\rho}_{E,n}$ is odd, that is, $\det(\overline{\rho}_{E,n}(\text{complex conjugation})) = -1$. In this way, a continuous mod n Galois representation with cyclotomic determinant is created from the elliptic curve. Naturally, one asks if every such representation corresponds to the n-torsion of *some* elliptic curve.

For $n = 2$, there is an elementary verification of this question (see below), and for $n = 3,\ 5$, an affirmative answer is given in [7] and [8]. For all primes $n \geq 7$ it has been shown that there exist continuous, irreducible mod n Galois representations with cyclotomic determinant that do not come from the n-torsion of any elliptic curve. See [2] and [5].

When n is not prime, less is known. [7] gives a mod 4 representation with image isomorphic to $\mathbb{Z}/2$ which does not correspond to any elliptic curve's 4-torsion, but the approach does not carry over to representations having larger images. In this paper, we show the existence of many surjective mod 4 representations with cyclotomic determinant which do not arise from the 4-torsion of any elliptic curve.

Theorem 1. *There exist continuous, surjective mod* 4 *Galois representations having cyclotomic determinant which do not arise from the* 4*-torsion of any elliptic curve.*

Equivalently, we prove the existence of a Galois extension $W/\mathbb{Q}$ such that

(1) $\mathrm{Gal}(W/\mathbb{Q}) \simeq GL_2(\mathbb{Z}/4)$;
(2) $\mathbb{Q}(i) \subset W$ is the fixed field of $SL_2(\mathbb{Z}/4)$;
(3) There is no elliptic curve E such that $W = \mathbb{Q}(E[4])$.

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Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}/4)$ be a continuous and surjective homomorphism having cyclotomic determinant. The fixed field of the kernel of $\overline{\rho}$ is a finite extension $W/\mathbb{Q}$ such that $\mathrm{Im}(\overline{\rho}) \simeq \mathrm{Gal}(W/\mathbb{Q}) \simeq GL_2(\mathbb{Z}/4)$. Likewise, such an extension determines a mapping of $G_{\mathbb{Q}}$ into $GL_2(\mathbb{Z}/4)$. (1) guarantees that $\overline{\rho}$ is surjective and continuous, whilst (2) is equivalent to $\overline{\rho}$ having cyclotomic determinant.

In the following sections we describe $\mathcal{G} = GL_2(\mathbb{Z}/4)$ using two of its normal subgroups $\mathcal{M}$ and $\mathcal{N}$ where $\mathcal{M} \cap \mathcal{N} = 1$ and $[\mathcal{G} : \mathcal{M}\mathcal{N}] = 2$ (see Theorem 2). Then a field $W/\mathbb{Q}$ satisfying (1) is the product of two quartic subfields M and N where $\mathrm{Gal}(M/\mathbb{Q}) \simeq \mathcal{G}/\mathcal{M}$ and $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathcal{G}/\mathcal{N}$. A criterion developed by Cremona in [4] allows us to find fields $M/\mathbb{Q}$ which do not come from the 4-torsion of any elliptic curve. It is then a simple matter to construct N so that $W = MN$ satisfies (2) and (3). In fact, we are able to find 1,628,554 such counterexamples. We expect, but are unable to show, that there are infinitely many.

2. 2-Torsion and S_3-extensions

When $\overline{\rho}$ is a surjective and continuous mod 2 representation ($\det(\overline{\rho})$ is necessarily trivial), it has image $GL_2(\mathbb{Z}/2) \simeq S_3$, the symmetric group on 3 letters. Given a finite Galois extension $L/\mathbb{Q}$ with Galois group isomorphic to S_3, there exists an irreducible cubic polynomial $f \in \mathbb{Q}[x]$ such that L is the splitting field of f. Then the nontrivial 2-torsion points of the elliptic curve $E : y^2 = f$ are $(\phi_i, 0)$, where the ϕ_i are the roots of f.

$$\begin{array}{cc} & L = \mathbb{Q}(E[2]) \\ & \Big| \; 2 \\ S_3 \left\{ \begin{array}{c} \\ \\ \end{array} \right. & \mathbb{Q}(\phi_i) \\ & \Big| \; 3 \\ & \mathbb{Q} \end{array}$$

Thus $L = \mathbb{Q}(E[2])$ and $E[2] \simeq V_{\overline{\rho}}$, the Galois module corresponding to $\overline{\rho}$. Any other irreducible cubic polynomial g with coefficients in $\mathbb{Q}$ and roots generating L gives an elliptic curve $E' : y^2 = g$ for which $E'[2] \simeq E[2] \simeq V_{\overline{\rho}}$. Furthermore, every elliptic curve having this property arises in this manner.

In addition to supplying concrete motivation for the other cases, this is directly relevant to the present discussion since $E[2] \subset E[4]$ for any elliptic curve E defined over $\mathbb{Q}$. As with the above case, we aim to understand surjective mod 4 representations by considering extensions with Galois group $GL_2(\mathbb{Z}/4)$.

3. $GL_2(\mathbb{Z}/4)$ as a Galois group

Let S_4 be the symmetric group on 4 letters, A_4 its alternating subgroup; let D_4 denote the dihedral group of order 8 and let V_4 denote the Klein four-group.

Theorem 2. *$\mathcal{G} = GL_2(\mathbb{Z}/4)$ has unique normal subgroups $\mathcal{M}$ and $\mathcal{N}$ so that*

(1) *$\mathcal{M} \simeq V_4$ and $\mathcal{G}/\mathcal{M} \simeq S_4$;*
(2) *$\mathcal{N} \simeq A_4$ and $\mathcal{G}/\mathcal{N} \simeq D_4$;*
(3) *$\mathcal{M} \cap \mathcal{N}$ is trivial;*
(4) *$[\mathcal{G} : \mathcal{M}\mathcal{N}] = 2$.*

Also, $\mathcal{G}$ is the only group (up to isomorphism) satisfying all of the above properties.

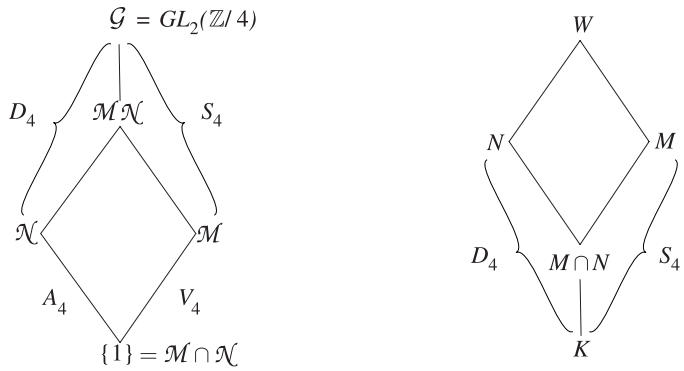


FIGURE 1. $GL_2(\mathbb{Z}/4)$ as a Galois group

Proof. This follows from a more general set of theorems concerning the structure of $GL_2(\mathbb{Z}/n)$ as well as basic group theory. See [1, section 5.1]. We discovered these facts via MAGMA by constructing this group, looking for relations among normal subgroups and inspecting other groups of the same order. □

Remark. While $\mathcal{M}\mathcal{N}$ and $SL_2(\mathbb{Z}/4)$ are both index-2 subgroups of G, it is not true in general that $\mathcal{M}\mathcal{N} \not\simeq SL_2(\mathbb{Z}/4)$.

An immediate consequence of this theorem is an analogous statement concerning field extensions (see also Figure 1):

Corollary 3. *Let K be a field. Any extension W/K with Galois group $\mathcal{G}$ has 2 unique normal subextensions, M/K and N/K, such that*

(1) $\mathrm{Gal}(W/M) = \mathcal{M}$ *and* $\mathrm{Gal}(M/K) \simeq S_4$*;*
(2) $\mathrm{Gal}(W/N) = \mathcal{N}$ *and* $\mathrm{Gal}(N/K) \simeq D_4$*;*
(3) $MN = W$*;*
(4) $[M \cap N : K] = 2$.

4. 4-Torsion and $GL_2(\mathbb{Z}/4)$

Let $W/\mathbb{Q}$ have Galois group $GL_2(\mathbb{Z}/4)$ and subextensions $M/\mathbb{Q}$ and $N/\mathbb{Q}$ as above. If $W = \mathbb{Q}(E[4])$ for some elliptic curve E, then $N = \mathbb{Q}(i, \sqrt[4]{\Delta})$ and $M \cap N = \mathbb{Q}(\sqrt{\Delta})$, where Δ is the discriminant of E. See pages 80, 81 in [1]. Also, if $W = \mathbb{Q}(E[4])$, then $\mathbb{Q}(i)$ is the subfield of W fixed by $SL_2(\mathbb{Z}/4)$ ($\mathcal{M}\mathcal{N} \not\simeq SL_2(\mathbb{Z}/4)$). See pages 69, 72 in [1]. To find a $W/\mathbb{Q}$ that does not come from an elliptic curve, it suffices then to construct an appropriate S_4-extension $M/\mathbb{Q}$:

Lemma 4. *Let $M/\mathbb{Q}$ have Galois group S_4 and let D be the discriminant of M such that $\mathbb{Q}(\sqrt{D})$, the quadratic normal subfield of M, is not $\mathbb{Q}(i)$. If $M \not\subset \mathbb{Q}(E[4])$ for any elliptic curve E, then $W = M\mathbb{Q}(i, \sqrt[4]{D})$ satisfies the three properties following Theorem 1.*

Proof. This is clear because W has the correct Galois group over $\mathbb{Q}$, $\mathbb{Q}(i) \subset W$ is the fixed field of $SL_2(\mathbb{Z}/4)$, and W is uniquely determined by its subfields M and N. □

Given an elliptic curve E, we can determine the polynomial which generates the S_4-extension contained within $\mathbb{Q}(E[4])$.

Theorem 5. *Let $E : y^2 = x^3 + a_4x + a_6$ be an elliptic curve defined over $\mathbb{Q}$ with discriminant Δ such that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) \simeq GL_2(\mathbb{Z}/4)$. Then the unique S_4-subextension of $\mathbb{Q}(E[4])$ is the splitting field of the following quartic:*

$$B_E(x) = x^4 - 4\Delta x - 12a_4\Delta.$$

Proof. See page 83 in [1] and page 9 in [6] (slightly different form). □

Every S_4-extension M contains a unique S_3-subextension L. If M were to come from the 4-torsion of an elliptic curve E, then $L = \mathbb{Q}(E[2])$ as in Section 2. Also, $\mathbb{Q}(\sqrt{\Delta}) = M \cap N \subset L \subset M$.

If we replace E with another curve E' such that $E'[2] \simeq E[2]$ (as Galois modules), then Δ and a_4 may both change. In this case, $B_{E'} \neq B_E$, and the two polynomials may determine distinct S_4-extensions of $\mathbb{Q}$. However, the S_3-field $L = \mathbb{Q}(E[2]) = \mathbb{Q}(E'[2])$ remains constant. Therefore, in order to construct an S_4-extension that does not come from an elliptic curve, we need to understand better the connection between S_4-extensions of $\mathbb{Q}$ that share a common S_3-subfield and the quartic polynomials that generate them.

5. S_4-Extensions and semi-invariants

In [4], invariants and semi-invariants of binary quartic forms are used to develop a better 2-descent for elliptic curves. Here we use them to show that the class of S_4-extensions properly contains those that come from the 4-torsion point fields of elliptic curves.

5.1. **Definitions.** Suppose K is a field, not having characteristic 2 or 3. To a quartic polynomial

$$g = ax^4 + bx^3 + cx^2 + dx + e \in K[x]$$

we associate a binary form $g(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$. $GL_2(K)$ acts on the set of these forms by

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : g(x, y) \mapsto g(\alpha x + \beta y, \gamma x + \delta y).$$

Two quartics g and h are considered *equivalent* under this action if they lie in the same $GL_2(K)$ orbit. In this case we write $g \sim h$.

We define the two basic invariants of g:

$$I = 12ae - 3bd + c^2 \qquad J = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3.$$

I and J are invariants under the subgroup of matrices $\mathbf{A}$ such that $\det \mathbf{A} = \pm 1$ (not all of $GL_2(K)$). They are thus invariant up to multiplication by fourth and sixth powers of $\det \mathbf{A}$, respectively. We then think of I and J as being defined up to multiplication by fourth and sixth powers in K and note that two equivalent quartics have the same I and J. We will also follow [4] and define two semi-invariants of g:

$$p = 3b^2 - 8ac \qquad r = b^3 + 8a^2d - 4abc,$$

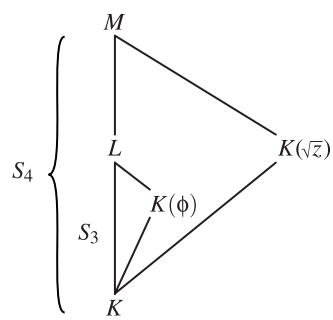
which are invariant under action by a Borel subgroup in $GL_2(K)$, i.e., the subgroup of upper triangular matrices. Using I and J, we then define the resolvent polynomial

$$F(x) = x^3 - 3Ix + J.$$

This definition of F does not become problematic due to our notion that I and J are only defined up to multiplication by fourth and sixth powers, respectively. Suppose $F'(x) = x^3 + 3\alpha^4 Ix + \alpha^6 J$, where α lies in K. Then $F'(\alpha^2 x) = \alpha^6 F(x)$. The splitting field of F is thus invariant under $GL_2(K)$. Finally, let ϕ be any root of F. This gives us a third semi-invariant of g,

$$z = \frac{4a\phi - p}{3}.$$

5.2. **Classification.** These invariants and semi-invariants then allow us to describe the field extensions associated with quartic polynomials. Suppose a quartic $g \in K[x]$ is irreducible with splitting field M/K and $\mathrm{Gal}(M/K) \simeq S_4$. The splitting field of F, L, is the unique S_3-extension contained within M. By choosing ϕ to be any root of F, we also define a cubic extension $K(\phi)/K$. Additionally, M is the Galois closure of $K(\sqrt{z})$ and $N_{K(\phi)/K}(z) = r^2$.



Conversely, if we have any v in $K(\phi) - K$ with square norm over $K(\phi)$, then the Galois closure of $K(\sqrt{v})$ is an S_4-extension containing L. Ostensibly, $v = \lambda_2\phi^2 + \lambda_1\phi + \lambda_0$ ($\lambda_i \in \mathbb{Q}$), but if v is linear in ϕ, then the quartic generating this extension has the same I and J invariants as g.

In section 3 of [4] it is shown that the set of S_4-extensions M/K containing a given S_3-extension L is in one-to-one correspondence with nontrivial elements of $H^1(G_K, V_4)$, where G_K acts via the quotient $\mathrm{Gal}(L/K)$ on V_4 as an automorphism group. This bijection is given by identifying $H^1(G_K, V_4)$ with $H = \ker(\mathrm{Norm} : \big(K(\phi)^\times/(K(\phi)^\times)^2 \to K^\times/(K^\times)^2\big))$ in a way that is explicitly compatible with the construction of M via z. One then obtains, as a subset, a one-to-one correspondence between quartics g with invariants I and J up to $GL_2(K)$-equivalence and nonzero elements z of $K(\phi)$ that are linear in ϕ and have square norm, up to equivalence mod $(K(\phi)^\times)^2$.

Theorem 6. *Let g and h be two quartics defined over K whose splitting fields, M_g and M_h, are S_4-extensions of K. Then,*

$$M_g = M_h \Longleftrightarrow g \sim h.$$

Proof. Suppose $g \sim h$. Then they share the same I and J and we have L, the splitting field of $F = x^3 - 3Ix + J$, contained in both M_g and M_h. Under the

$$M/K \mid L \subset M,\ Gal(M/\mathrm{K}) \simeq S_4\}$$
$$\updownarrow$$
$$H^1(G_K, V_4) \quad H$$

$$\{z = \alpha\phi + \beta \mid \alpha, \beta \in K,\ N_{K(\phi)/K}(z) \in K^2\}/K(\phi)^2$$
$$\updownarrow$$
$$\text{quartics } g \in K[x] \text{ with invariants } I,\ J\}/\sim$$

correspondences above, g and h then generate the same V_4-extension of L; thus $M_g = M_h$.

Now suppose $M_g = M_h = M$. M contains a unique field L with Galois group S_3 over K. Clearly, if g and h share the same I and J, then the above correspondences indicate that they would generate the same extension only if they were *equivalent*. So suppose g and h have distinct I- and J-invariants. Then they correspond to different elements of H and thus distinct V_4-extensions of L. But this cannot be. Hence $g \sim h$. □

There is also a simple criterion to determine when two quartics are equivalent.

Theorem 7 (Prop. 3.2(2) in [4])**.** *Let g and h be two quartics having the same invariants I, J and semi-invariants z_g, z_h. Then,*

$$g \sim h \Longleftrightarrow z_g z_h \in (K(\phi)^\times)^2.$$

This is precisely what we need in order to prove that a particular S_4-extension $M/\mathbb{Q}$ cannot sit inside $\mathbb{Q}(E[4])$ for any E.

6. The result

Theorem 8. *Let g be any quartic polynomial defined over $\mathbb{Q}$ such that its splitting field M has $\mathrm{Gal}(M/\mathbb{Q}) \simeq S_4$. Let L be the unique S_3-subextension, choose ϕ, and let g have semi-invariants p and a.*

$M \subset \mathbb{Q}(E[4])$ for some elliptic curve $E \implies \phi(4a\phi - p) \in (K(\phi)^\times)^2$.

Proof. Let z_E, a_E and p_E be the semi-invariants associated to B_E, and let z be the z-semi-invariant of g. Suppose M is contained within $\mathbb{Q}(E[4])$ for some elliptic curve E. Then $g \sim B_E$. In particular, g and B_E share the same I and J (a more stringent criterion than $\mathbb{Q}(E[2]) = L$), and z_E is defined to be linear in ϕ. I and J determine F (i.e. ϕ), and hence $z_E = \frac{4a_E\phi - p_E}{3}$. B_E is monic, so $a_E = 0$. Also B_E has no quadratic or cubic terms, so p_E is zero. Thus $z_E = \frac{4\phi}{3}$.

Now we have $zz_E = \frac{4}{9}(\phi)(4a\phi - p)$. By Theorem 7, this is a square and we are done. □

Tables of quartic extensions of $\mathbb{Q}$ have been constructed by [3], ordered by absolute value of their discriminants (and signature), up to 10^7. There are 1,635,308 S_4-fields in these tables, each given by a defining polynomial. Of these polynomials, all but 6,755 have $\phi(4a\phi - p) \notin (K(\phi)^\times)^2$ and so do not come from the 4-torsion of

Table 1.

disc.	generating polynomial	I, J	p	Is $\phi(4a\phi - p)$ a square?
229	$x^4 - x + 1$	12, -27	0	yes, $4\phi^2$
257	$x^4 + x^2 - x + 1$	13, 43	-8	no
-283	$x^4 - x - 1$	-12, -27	0	yes, $4\phi^2$
-331	$x^4 - x^3 + x^2 + x - 1$	-8, -83	-5	no
-491	$x^4 - x^3 - x^2 + 3x - 1$	-2, -115	11	no
-563	$x^4 - x^3 + x^2 - x - 1$	-14, -65	-5	no
592	$x^4 + 2x^2 - 2x + 1$	16, 20	-16	no
-643	$x^4 - x^3 - 2x + 1$	6, -135	3	no
-688	$x^4 - 2x - 1$	-12, -108	0	yes, $4\phi^2$
697	$x^4 - x^3 + 2x^2 - x + 2$	25, 209	-13	no
-731	$x^4 - x^3 + 2x^2 - 1$	-8, -133	-13	no
-751	$x^4 - 2x^3 + x^2 - x - 1$	-17, 25	4	no
761	$x^4 - x^3 + x^2 + 2x + 1$	19, -83	-5	no
788	$x^4 - x^3 + 2x^2 - 2x + 2$	22, 146	-13	no
-848	$x^4 - x^2 - 2x + 1$	13, -178	8	no
892	$x^4 - x^3 - x^2 + 2$	25, -196	11	no
-976	$x^4 - 2x^3 + 3x^2 - 1$	-3, -162	-12	no
985	$x^4 - x^3 + 2x^2 - 3x + 2$	19, 29	-13	no

any elliptic curve. In Table 1, we list all 18 such extensions with discriminant less than 1000 in absolute value. Of these, only 3 could (and do) lie inside the 4-torsion of an elliptic curve.

We thus have many S_4-extensions of $\mathbb{Q}$ which do not arise from the 4-torsion of any elliptic curve as well as an easy computation which usually tells us whether a given quartic generates such an extension. We close with a few remarks.

7. Concluding remarks

7.1. The converse of Theorem 8 is not true; given a quartic g for which $\phi(4a\phi - p)$ is a square and $\mathrm{Gal}(M_g/\mathbb{Q}) \simeq S_4$, there is not necessarily an elliptic curve $E : y^2 = x^3 + a_4x + a_6$ whose 4-torsion field contains M_g.

For example, consider $g = x^4 + 3x^2 - 2x + 1$, with discriminant 1264, $I = 21$, $J = 54$ and $p = -24$. It is the quartic of smallest discriminant for which $\phi(4a\phi - p)$ is square but $p \neq 0$. Let α be a root of g, and suppose there is an $h = x^4 - 4\Delta x - 12a_4\Delta$ with root $\beta = b_3a^3 + b_2a^2 + b_1a + b_0$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. β must be in the kernel of the map tr:$\mathbb{Q}(\alpha) \longrightarrow \mathbb{Q}$ given by $\alpha \mapsto \mathrm{trace}(\alpha)$, because the x^3-coefficient of h is $\mathrm{tr}(\beta)$. In fact, ker(tr)= $\langle c_1, c_2, c_3 \rangle$ where

$$c_1 = 2\alpha^3 + 5\alpha - 3, \qquad c_2 = 2\alpha^3 + 4\alpha - 3, \qquad c_3 = 21\alpha^3 - \alpha^2 + 48\alpha - 33.$$

So $\beta = uc_1 + vc_2 + wc_3$ for some rationals u, v and w.

Now, $\mathrm{tr}(\beta^2) - \mathrm{tr}(\beta)^2$ is twice the coefficient of x^2 in h. But $\mathrm{tr}(\beta^2)$ and $\mathrm{tr}(c_ic_j)$ $\{i \neq j\}$ are all 0 whilst $\mathrm{tr}(c_1^2) = -2$, $\mathrm{tr}(c_2^2) = -4$ and $\mathrm{tr}(c_3^2) = 158$. Hence, h has zero x^2-coefficient if and only if

$$u^2 + 2v^2 - 79w^2 = 0.$$

However, this quadratic form does not vanish for any nontrivial rationals u, v and w. So there is no elliptic curve $E : y^2 = x^3 + a_4x + a_6$ whose 4-torsion field contains M_g. (We would like to thank the referee for suggesting this argument.)

7.2. When a quartic polynomial does correspond to an elliptic curve, it can be difficult to find the appropriate E. Since Δ appears in the coefficients of B_E (giving us $I = -3^2\Delta a_4$ and $J = -2^4 3^3 \Delta^2$) and the polynomials given in the tables have minimized coefficients (giving us relatively small I and J), one cannot usually read the elliptic curve from the coefficients of the original quartic.

For example, the first extension listed, given by $x^4 - x + 1$, does not correspond to the 4-torsion field of the unique elliptic curve of conductor 229. Instead, the elliptic curve $E : y^2 = x^3 - 395712x - 906180048$ of conductor $2^2 229^2$ has $B_E \sim x^4 - x + 1$. The S_4-extension coming from the curve of conductor 229 occurs much further down the list.

To find E, we employed the following method: Since I and J are determined only up to 4th and 6th powers respectively, we look for an E defined by the minimal polynomial of a nonmaximal order of L of conductor divisible by $\mathrm{disc}(g)$ (and possibly other primes). Then $\Delta = k^2(\mathrm{disc}(g))^3$ and the a_4 from E often is divisible by $\mathrm{disc}(g)$. We can hope in this way to obtain an E such that B_E has the correct I and J.

This procedure was also successful for the other two quartics on the short list above, but it does not tell us whether or not such a curve exists. The order corresponding to E could lie arbitrarily deep in $\mathcal{O}_L$ or not be there at all, as is the case for the quartic in Remark 7.1.

7.3. If g does come from the 4-torsion of some elliptic curve, it is never the case that E is given by $y^2 = F$, the resolvent polynomial of g. This follows from Prop. 4.3 in [4].

7.4. The I and J invariants also determine isomorphism classes of 4-torsion modules. If E and E' are elliptic curves such that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(E'[4])/\mathbb{Q}) \simeq GL_2(\mathbb{Z}/4)$ and B_E and $B_{E'}$ share I and J invariants, then $B_E \sim B_{E'}$ and so they generate the same S_4-extension. I and J also determine the discriminant of this extension (D in Section 4) up to a twelfth power. In particular, the fourth roots of the two elliptic curves' discriminants are the same; both curves generate the same D_4-extension. Thus, the two 4-torsion fields of E and E' are equal and they define the same mod 4 Galois representation.

References

1. C. Adelmann, *The decomposition of primes in torsion point fields*, Lecture Notes in Mathematics, Vol. 1761, Springer-Verlag, 2001. MR1836119 (2002g:11151)
2. F. Calegari, *Mod p representations on elliptic curves*, Pacific J. Math. **225** (2006), 1–11, preprint available at http://front.math.ucdavis.edu/math.NT/0406244. MR2233722 (2007c:11068)
3. H. Cohen, F. Diaz y Diaz, and Michel Olivier, *Constructing complete tables of quartic fields using Kummer theory*, Math. Comp. **72** (2003), 941–951. MR1954977 (2004a:11136)
4. J. Cremona, *Classical invariants and 2-descent on elliptic curves*, Journal of Symbolic Computation (Proceedings of the Second MAGMA Conference, Milwaukee, May 1996), 71–87. MR1806207 (2002a:11055)
5. L. Dieulefait, *Existence of non-elliptic mod l Galois representations for every $l > 5$*, Experimental Mathematics **13**, no. 3, available at http://www.math.ethz.ch/EMIS/journals/EM/expmath/volumes/13/13.3/Dieulefait.pdf. MR2103330 (2005g:11087)

6. E. Goins, *Elliptic curves and icosahedral Galois representations*, Stanford University Thesis, 1999.
7. K. Rubin, *Modularity of mod* 5 *representations*, Modular Forms and Fermat's Last Theorem, Boston, MA, 1995, pp. 463–474. MR1638489
8. N. Shepherd-Barron and R. Taylor, *Mod* 2 *and mod* 5 *Icosahedral representations*, J. Amer. Math. Soc. **10** (1997), 283–298. MR1415322 (97h:11060)

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin 53706-1388

E-mail address: holden@math.wisc.edu