PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 1, January 2008, Pages 127–131
S 0002-9939(07)08916-2
Article electronically published on August 30, 2007

A LOWER BOUND FOR THE EQUILATERAL NUMBER OF NORMED SPACES

KONRAD J. SWANEPOEL AND RAFAEL VILLA

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We show that if the Banach-Mazur distance between an n-dimensional normed space X and ℓ_∞^n is at most $3/2$, then there exist $n+1$ equidistant points in X. By a well-known result of Alon and Milman, this implies that an arbitrary n-dimensional normed space admits at least $e^{c\sqrt{\log n}}$ equidistant points, where $c>0$ is an absolute constant. We also show that there exist n equidistant points in spaces sufficiently close to ℓ_p^n, $1<p<\infty$.

1. Notation

Throughout the paper we use the same symbol c for different absolute positive constants. Let X denote a normed space of finite dimension $\dim X = n$. Let $e(X)$ denote the largest size of an equilateral set in X. As usual, the space ℓ_p^n, $1 \le p < \infty$, is defined as $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \dots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, and ℓ_∞^n is $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \dots, x_n)\|_\infty = \max_i |x_i|$. The *Banach-Mazur distance* between two n-dimensional normed spaces is defined as $d(X,Y) = \inf \|T\|\,\|T^{-1}\|$, where the infimum is taken over all linear, invertible operators $T : X \to Y$. We say that X is a $(1+\varepsilon)$-*copy* of Y if $d(X,Y) \le 1+\varepsilon$.

2. The main theorems

It is conjectured [4, 9, 11, 12, 17] that $e(X) \ge n+1$ for all n-dimensional normed spaces X. This is known for $n \le 3$ [12] but open for $n \ge 4$. It is true for spaces sufficiently close to Euclidean:

Theorem 1 (Brass [3] and Dekster [6])**.** *Let X be an n-dimensional normed space with Banach-Mazur distance $d(X, \ell_2^n) \le 1 + \frac{1}{n}$. Then an equilateral set in X of at most n points can be extended to one of $n+1$ points. In particular, $e(X) \ge n+1$.*

Combining this theorem with Schechtman's estimate [13, Theorem 3] in the Dvoretzky theorem [7, §4], the following general lower bound follows:

$$e(X) \ge \frac{c\sqrt{\log n}}{\log \log n}.$$

Received by the editors March 23, 2006 and, in revised form, September 1, 2006.
2000 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B20, 52A21, 52C17.
This material is based upon work supported by the South African National Research Foundation under Grant number 2053752. The second author thanks the DGES grant BFM2003-01297 for financial support. Parts of this paper were written during a visit of the second author to the Department of Mathematical Sciences, University of South Africa, in January 2006.

(The previous best estimate in the Dvoretzky theorem, due to Gordon [8], gave $e(X) \geq c(\log n)^{1/3}$.) We improve this to the following:

Theorem A. *For any n-dimensional normed space X we have $e(X) \geq e^{c\sqrt{\log n}}$, where $c > 0$ is an absolute constant.*

The theoretical limit beyond which Dvoretzky's theorem cannot be improved in general (by considering ℓ_∞^n; see the discussion above Theorem 3 in [13]) would give only $e(X) \geq c \log n / \log\log n$, still worse than Theorem A. Thus something more than Dvoretzky's theorem is needed. Indeed, the proof of Theorem A works by looking for large subspaces close to either ℓ_2^k or ℓ_∞^k. To this end we use the following theorem:

Theorem 2 (Alon and Milman [1])**.** *Let X be an n-dimensional normed space. Then for each $\varepsilon > 0$ there exists $c \in (0,1)$ such that X has a subspace of dimension $m \geq e^{c\sqrt{\log n}}$ that is $(1+\varepsilon)$-isomorphic to either ℓ_2^m or ℓ_∞^m.*

If the second case occurs in the above theorem, i.e., if we find a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^m in X, then we need a result similar to Theorem 1 for spaces near ℓ_∞^n. This we provide as follows:

Theorem B. *Let X be an n-dimensional normed space with Banach–Mazur distance $d(X, \ell_\infty^n) \leq 3/2$. Then $e(X) \geq n+1$.*

We can then choose $\varepsilon = 1/2$ in Theorem 2 to obtain $e(X) \geq e^{c\sqrt{\log n}}$. On the other hand, if we find a $(1+\frac{1}{2})$-isomorphic copy of ℓ_2^m where $m \geq e^{c\sqrt{\log n}}$, we cannot yet apply Theorem 1. We first have to find a k-dimensional subspace of the m-dimensional space that is $(1+\frac{1}{k})$-isomorphic to ℓ_2^k. This is guaranteed by the following:

Theorem 3 (Milman [10])**.** *Let X be an m-dimensional normed space and $0 < \varepsilon < 1$. Then X contains a subspace Y of dimension $k \geq c\varepsilon^2 m / d^2(X, \ell_2^m)$ such that $d(Y, \ell_2^k) \leq 1 + \varepsilon$.*

See also [7, Corollary 4.2.2]. Putting $\varepsilon = 1/k$ and X the m-dimensional subspace that is 3/2-isomorphic to ℓ_2^m into the above theorem, we obtain a subspace Y of dimension $k > cm^{1/3}$, and Theorem 1 then gives $e(X) \geq e(Y) > e^{c\sqrt{\log n}}$. To complete the proof of Theorem A, it only remains to prove Theorem B.

Proof of Theorem B. We use the Brouwer fixed point theorem [5, §14.3], as in Brass' proof of Theorem 1. Without loss of generality we may assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \leq \|x\|_\infty \leq \frac{3}{2}\|x\| \quad \text{for all } x \in X.$$

Let $I = \{(i,j) : 1 \leq i < j \leq n+1\}$, with $|I| = n(n+1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j)\in I} \in [0, \frac{1}{2}]^N$, let

$$\begin{aligned}
p_1(\varepsilon) &= (-1, 0, \dots 0),\\
p_j(\varepsilon) &= (\varepsilon_{1,j}, \dots \varepsilon_{j-1,j}, -1, 0, \dots 0), \qquad 2 \leq j \leq n-1,\\
p_n(\varepsilon) &= (\varepsilon_{1,n}, \dots \varepsilon_{n-1,n}, -1),\\
p_{n+1}(\varepsilon) &= (\varepsilon_{1,n+1}, \dots \varepsilon_{n,n+1}).
\end{aligned}$$

For $1 \le i < j \le n$ we have $\|p_i(\varepsilon) - p_j(\varepsilon)\|_\infty = 1 + \varepsilon_{i,j}$. Define $\varphi : [0, 1/2]^N \to [0, 1/2]^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - \|p_i(\varepsilon) - p_j(\varepsilon)\|$, $1 \le i < j \le n$. Note that

$$\varphi_{i,j}(\varepsilon) \ge 1 + \varepsilon_{i,j} - \|p_i - p_j\|_\infty = 0$$

and

$$\varphi_{i,j}(\varepsilon) \le 1 + \varepsilon_{i,j} - \frac{2}{3}\|p_i - p_j\|_\infty = \frac{1}{3}(1 + \varepsilon_{i,j}) \le \frac{1}{2},$$

so φ is well defined. Brouwer now gives the existence of a point $\varepsilon' = (\varepsilon'_{i,j}) \in [0, 1/2]^N$ with $\varphi(\varepsilon') = \varepsilon'$, which implies that $\|p_i(\varepsilon') - p_j(\varepsilon')\| = 1$ for all $1 \le i < j \le m$. We have obtained $n + 1$ equilateral points. □

3. A generalization to ℓ_p^n

The following theorem partially generalizes Theorem B to all ℓ_p^n spaces with $1 < p < \infty$.

Theorem C. *For each $n > 2$ and $p \in (1, \infty)$ there exists $R(p, n) > 1$ such that for any n-dimensional normed space X with Banach-Mazur distance $d(X, \ell_p^n) \le R(p, n)$ we have $e(X) \ge n$. In fact,*

$$\begin{aligned} R(p, n) &= \max_{\theta > 0} \left(\frac{1 + (1 + \theta)^p}{2 + (n - 2)\theta^p} \right)^{1/p} \\ &\sim 1 + \frac{p - 1}{2p} n^{-\frac{1}{p-1}} \text{ as } n \to \infty \text{ with } p \text{ fixed.} \end{aligned}$$

Proof. We follow the proof of Theorem B. Assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \le \|x\|_p \le R\|x\| \quad \text{for all } x \in X.$$

Fix $\beta, \gamma > 0$. Let $I = \{(i, j) : 1 \le i < j \le n\}$, with $|I| = n(n - 1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j) \in I} \in [0, \beta]^N$, let

$$\begin{aligned} p_1(\varepsilon) &= (-\gamma, 0, \dots 0), \\ p_j(\varepsilon) &= (\varepsilon_{1,j}, \dots \varepsilon_{j-1,j}, -\gamma, 0, \dots 0), \qquad 2 \le j \le n - 1, \\ p_n(\varepsilon) &= (\varepsilon_{1,n}, \dots \varepsilon_{n-1,n}, -\gamma). \end{aligned}$$

For $1 \le i < j \le n$ we have

$$\|p_j - p_i\|_p^p = \sum_{k=1}^{i-1} |\varepsilon_{k,j} - \varepsilon_{k,i}|^p + (\varepsilon_{i,j} + \gamma)^p + \sum_{k=i+1}^{j} \varepsilon_{k,j}^p + \gamma^p.$$

Define $\varphi : [0, \beta]^N \to [0, \beta]^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - \|p_i - p_j\|$ for $1 \le i < j \le n$. On the one hand,

$$\begin{aligned} \varphi_{i,j}(\varepsilon) &\le 1 + \varepsilon_{i,j} - R^{-1}\|p_i - p_j\|_p \\ &\le 1 + \varepsilon_{i,j} - R^{-1}\left[(\gamma + \varepsilon_{i,j})^p + \gamma^p\right]^{1/p}. \end{aligned}$$

Taking into account that the latter is increasing with respect to $\varepsilon_{i,j}$, the inequality $\varepsilon_{i,j} \le \beta$ implies that

$$\varphi_{i,j}(\varepsilon) \le 1 + \beta - R^{-1}\left[(\gamma + \beta)^p + \gamma^p\right]^{1/p}.$$

Therefore, if $(\gamma+\beta)^p+\gamma^p \geq R^p$, then $\varphi_{i,j}(\varepsilon) \leq \beta$. On the other hand,

$$\begin{aligned}\varphi_{i,j}(\varepsilon) &\geq 1+\varepsilon_{i,j}-\|p_i-p_j\|_p \\ &\geq 1+\varepsilon_{i,j}-\left[(n-2)\beta^p+(\gamma+\varepsilon_{i,j})^p+\gamma^p\right]^{1/p}.\end{aligned}$$

Again the latter is increasing with respect to $\varepsilon_{i,j}$, so using $\varepsilon_{i,j} \geq 0$ we have

$$\varphi_{i,j}(\varepsilon) \geq 1-\left[(n-2)\beta^p+2\gamma^p\right]^{1/p}.$$

Then $\varphi_{i,j}(\varepsilon_1,\dots\varepsilon_m) \geq 0$ would follow if $(n-2)\beta^p+2\gamma^p \leq 1$. Subsequently, if

$$(*) \qquad (\gamma+\beta)^p+\gamma^p \geq R^p \quad \text{and} \quad (n-2)\beta^p+2\gamma^p \leq 1,$$

then φ is well defined. Brouwer now gives a point $\varepsilon'=(\varepsilon'_{i,j}) \in [0,\beta]^N$ such that $\varphi(\varepsilon')=\varepsilon'$, implying that the points $p_1(\varepsilon'),\dots p_n(\varepsilon')$ are equilateral.

Finally, to take the best choice for the parameters in $(*)$ we have to maximize the expression $(\gamma+\beta)^p+\gamma^p$ under the constraints $(n-2)\beta^p+2\gamma^p \leq 1$ and $\beta,\gamma \geq 0$. Setting $\theta=\beta/\gamma$, we obtain

$$R^p=\max_{\theta>0}\frac{1+(1+\theta)^p}{2+(n-2)\theta^p}.$$

It is not difficult to see that for θ close to $n^{-1/(p-1)}$, the right-hand side is >1 and $R-1 \sim \frac{p-1}{2p}n^{-\frac{1}{p-1}}$. □

4. Concluding remarks

For $p=2$ the estimate in the above theorem is $d(X,\ell_2^n) \lesssim 1+\frac{1}{4n}$, slightly worse than Theorem 1. However, we don't know how to obtain $n+1$ equidistant points as in Theorem B. It would also be interesting to know whether arbitrary equilateral sets of at most n points in spaces near ℓ_p^n can be extended as in Theorem 1. A different idea will be needed to extend the above theorem to the case $p=1$. See [16] for a survey on equilateral sets, as well as [2, 14, 15] for further results on equilateral sets in ℓ_p^n.

Acknowledgment

We thank the referee for helpful remarks leading to an improved paper.

References

[1] N. Alon and V. D. Milman, *Embedding of l_∞^k in finite-dimensional Banach spaces*, Israel J. Math. **45** (1983), 265–280. MR720303 (85f:46027)

[2] N. Alon and P. Pudlák, *Equilateral sets in l_p^n*, Geom. Funct. Anal. **13** (2003), no. 3, 467–482. MR1995795 (2004h:46011)

[3] P. Brass, *On equilateral simplices in normed spaces*, Beiträge Algebra Geom. **40** (1999), no. 2, 303–307. MR1720106 (2000i:52012)

[4] P. Brass, W. Moser, and J. Pach, *Research problems in discrete geometry*, Springer, New York, 2005. MR2163782 (2006i:52001)

[5] A. Browder, *Mathematical analysis: An introduction*, Springer-Verlag New York, 1996. MR1411675 (97g:00001)

[6] B. V. Dekster, *Simplexes with prescribed edge lengths in Minkowski and Banach spaces*, Acta Math. Hungar. **86** (2000), 343–358. MR1756257 (2001b:52001)

[7] A. A. Giannopoulos and V. D. Milman, *Euclidean structure in finite dimensional normed spaces*, Handbook of the Geometry of Banach spaces (eds. W. B. Johnson and J. Lindenstrauss), Vol. 1, Elsevier, 2001, pp. 707–779. MR1863705 (2003b:46008)

[8] Y. Gordon, *Some inequalities for Gaussian processes and applications*, Israel J. Math. **50** (1985), 265–289. MR800188 (87f:60058)

[9] B. Grünbaum, *On a conjecture of H. Hadwiger*, Pacific J. Math. **11** (1961), 215–219. MR0138044 (25:1492)
[10] V. D. Milman, *New proof of the theorem of Dvoretzky on sections of convex bodies*, Funct. Anal. Appl. **5** (1971), 28–37. MR0293374 (45:2451)
[11] F. Morgan, *Minimal surfaces, crystals, shortest networks, and undergraduate research*, Math. Intelligencer **14** (1992), no. 3, 37–44. MR1184317 (93h:53012)
[12] C. M. Petty, *Equilateral sets in Minkowski spaces*, Proc. Amer. Math. Soc. **29** (1971), 369–374. MR0275294 (43:1051)
[13] G. Schechtman, *Two observations regarding embedding subsets of Euclidean spaces in normed spaces*, Adv. Math. **200** (2006), 125–135. MR2199631 (2006j:46015)
[14] Cliff Smyth, *Equilateral or* 1*-distance sets and Kusner's conjecture*, manuscript, 2002.
[15] K. J. Swanepoel, *A problem of Kusner on equilateral sets*, Arch. Math. **83** (2004), 164–170. MR2104945 (2005i:52024)
[16] K. J. Swanepoel, *Equilateral sets in finite-dimensional normed spaces*, In: Seminar of Mathematical Analysis, eds. Daniel Girela Álvarez, Genaro López Acedo, Rafael Villa Caro, Secretariado de Publicationes, Universidad de Sevilla, Seville, 2004, pp. 195–237. MR2117069 (2005j:46009)
[17] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996. MR1406315 (97f:52001)

Department of Mathematical Sciences, University of South Africa, PO Box 392, Pretoria 0003, South Africa
E-mail address: swanekj@unisa.ac.za

Departamento Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, c/Tarfia, S/N, 41012 Sevilla, Spain
E-mail address: villa@us.es