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SUBMANIFOLDS OF REAL ALGEBRAIC VARIETIES

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Abstract. By the Nash-Tognoli theorem, each compact smooth manifold M is diffeomorphic to a nonsingular real algebraic set, called an algebraic model of M. We construct algebraic models X of M with controlled behavior of the group of cohomology classes represented by algebraic subsets of X.

1. Introduction

A compact smooth (of class C^∞) manifold is said to be a *boundary* if it is diffeomorphic to the boundary of a compact smooth manifold with boundary. By convention, the empty manifold is a boundary. The goal of the present paper is to demonstrate that boundaries play a surprisingly interesting role in real algebraic geometry (cf. [4, Theorem 1.4] for the first result of this kind).

All real algebraic varieties in this paper are assumed to be affine (that is, isomorphic to an algebraic subset of $\mathbb{R}^n$ for some n). For background material on real algebraic geometry the reader may consult [2]. Every real algebraic variety carries also the Euclidean topology, induced by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a compact nonsingular real algebraic variety X, we denote by $H_d^{alg}(X, \mathbb{Z}/2)$ the subgroup of $H_d(X, \mathbb{Z}/2)$ of homology classes represented by d-dimensional algebraic subsets of X; cf. [1, 2, 3, 5]. For technical reasons it is more convenient to work with cohomology groups. We set

$$H_{alg}^c(X, \mathbb{Z}/2) := D_X^{-1}(H_d^{alg}(X, \mathbb{Z}/2)),$$

where $c + d = \dim X$ and $D_X : H^c(X, \mathbb{Z}/2) \to H_d(X, \mathbb{Z}/2)$ is the Poincaré duality isomorphism. The groups $H_{alg}^c(-, \mathbb{Z}/2)$ are of fundamental interest in real algebraic geometry and will also be in the center of our attention here. Their basic properties and applications are surveyed in [3].

Let M be a compact smooth manifold of dimension m. Denote by $[M]$ the fundamental class of M in $H_m(M, \mathbb{Z}/2)$. For any smooth d-dimensional submanifold N of M (submanifolds are always assumed to be closed subsets), we let $[N]_M$ denote the homology class in $H_d(M, \mathbb{Z}/2)$ represented by N. We set $[N]_M = 0$ if N is empty.

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Given a finite collection $\mathcal{F}$ of smooth submanifolds of M, we define

$$G^k(\mathcal{F}) \quad := \quad \{u \in H^k(M, \mathbb{Z}/2) \mid \langle u, [N]_M \rangle = 0 \text{ for all } N \text{ in } \mathcal{F} \text{ with } \dim N = k\}.$$

Here, as usual, $\langle\ ,\ \rangle$ stands for scalar product (Kronecker index). If there is no k-dimensional submanifold in $\mathcal{F}$, then $G^k(\mathcal{F}) := H^k(M, \mathbb{Z}/2)$. Clearly, $G^k(\mathcal{F})$ is a subgroup of $H^k(M, \mathbb{Z}/2)$.

Recall that $\mathcal{F}$ is said to be in *general position* if for each point x in M, the collection $\mathcal{F}_x = \{N \in \mathcal{F} \mid x \in N\}$ is either empty or otherwise $\operatorname{codim}(\bigcap_{N \in \mathcal{F}_x} T_x N) = \sum_{N \in \mathcal{F}_x} \operatorname{codim} T_x N$, where $T_x N$ is regarded as a subspace of the tangent space $T_x M$. If $\mathcal{F}$ is in general position and $N_1, \ldots, N_s$ are in $\mathcal{F}$, then $N_1 \cap \ldots \cap N_s$ is a smooth submanifold of M.

Definition 1.1. A finite collection $\mathcal{F}$ of smooth submanifolds of M is said to be admissible if:

(i) $\dim N < \dim M$ for all N in $\mathcal{F}$;
(ii) $\mathcal{F}$ is in general position;
(iii) if $N_1, \ldots, N_s$ are in $\mathcal{F}$, then $N_1 \cap \ldots \cap N_s$ is in $\mathcal{F}$;
(iv) each submanifold in $\mathcal{F}$ has trivial normal vector bundle.

If M' is another smooth manifold and if $\varphi : M' \to M$ is a smooth diffeomorphism, then

$$\varphi^* \mathcal{F} := \{\varphi^{-1}(N) \mid N \in \mathcal{F}\}$$

is a collection of smooth submanifolds of M'. The collection $\varphi^* \mathcal{F}$ is admissible, provided $\mathcal{F}$ is admissible.

Theorem 1.2. *Let X be a compact nonsingular real algebraic variety and let $\mathcal{F}$ be an admissible collection of smooth submanifolds of X. If*

$$H^k_{alg}(X, \mathbb{Z}/2) \subseteq G^k(\mathcal{F})$$

for some nonnegative integer k, then each k-dimensional manifold in $\mathcal{F}$ is a boundary.

Theorem 1.2 is easy to prove. Its purpose is to provide motivation for our main result, Theorem 1.3, below. By Tognoli's theorem [14] (cf. also [2, Theorem 14.1.10] and, for a weaker but influential result, [12]), each compact smooth manifold M is diffeomorphic to a nonsingular real algebraic variety, called an *algebraic model* of M. Constructing algebraic models satisfying some additional desirable conditions is an interesting and active area of research; cf. [1, 2, 3, 8, 9].

Theorem 1.3. *Let M be a compact smooth manifold and let $\mathcal{F}$ be an admissible collection of smooth submanifolds of M. If each manifold in $\mathcal{F}$ is a boundary, then there exist an irreducible algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ such that*

$$H^k_{alg}(X, \mathbb{Z}/2) \subseteq G^k(\varphi^* \mathcal{F})$$

for all nonnegative integers k.

Theorem 1.3 is useful for constructing examples such that $H^k_{alg}(-, \mathbb{Z}/2) \neq H^k(-, \mathbb{Z}/2)$.

Example 1.4. The n-fold product $T^n = S^1 \times \cdots \times S^1$ of the unit circle S^1 has an algebraic model X with $H^k_{alg}(X, \mathbb{Z}/2) \neq H^k(X, \mathbb{Z}/2)$ for $1 \leq k \leq n-1$. Indeed, suppose $n \geq 2$, since for $n = 1$ the assertion is void. Choose $n-1$ distinct points $p_1, \ldots, p_{n-1}$ in S^1 and set

$$N_i = \{(x_1, \ldots, x_n) \in T^n \mid x_1 = \cdots = x_i = p_i\}$$

for $1 \leq i \leq n-1$. Then $\mathcal{F} := \{N_1, \ldots, N_{n-1}\}$ is an admissible collection of smooth submanifolds of T^n. Moreover, $\operatorname{codim}_{T^n} N = i$ and N_i is a boundary for $1 \leq i \leq n-1$. Hence the assertion follows from Theorem 1.3.

Theorems 1.2 and 1.3 will be proved in the next section.

2. Proofs

As usual, the ith Stiefel-Whitney class of a smooth manifold M will be denoted by $w_i(M)$. Given a smooth submanifold N of M, we set

$$[N]^M := D_M^{-1}([N]_M)\,.$$

Henceforth, both the homology class $[N]_M$ and cohomology class $[N]^M$ will be used. Basic properties of the products $\langle\ ,\ \rangle, \cap, \cup$ familiar from algebraic topology [6] will be used without further explanation.

Lemma 2.1. *Let M be a compact smooth manifold and let N be a k-dimensional smooth submanifold of M. If the normal vector bundle of N in M is trivial, then the following conditions are equivalent:*

(a) *N is a boundary,*
(b) *$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [N]_M\rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$,*
(c) *$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup [N]^M, [M]\rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$.*

Proof. Let $e : N \hookrightarrow M$ be the inclusion map. Then

$$[N]^M \cap [M] = D_M([N]^M) = [N]_M = e_*([N]).$$

Setting $w = w_{i_1}(M) \cup \ldots \cup w_{i_r}(M)$, we have

$$\begin{aligned}
\langle w \cup [N]^M, [M]\rangle &= \langle w, [N]^M \cap [M]\rangle \\
&= \langle w, [N]_M\rangle \\
&= \langle w, e_*([N])\rangle \\
&= \langle e^*(w), [N]\rangle \\
&= \langle e^*(w_{i_1}(M)) \cup \ldots \cup e^*(w_{i_r}(M)), [N]\rangle \\
&= \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle,
\end{aligned}$$

where the last equality holds since the triviality of the normal vector bundle of N implies that $e^*(w_i(M)) = w_i(N)$ for all $i \geq 0$. The proof is complete since, by [13, Théorèmes IV.3, IV.10], N is a boundary if and only if $\langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N]\rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$. □

Proof of Theorem 1.2. Every cup product $w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)$ is in $H^k_{alg}(X, \mathbb{Z}/2)$, where $k = i_1 + \cdots + i_r$; cf. [1, 2, 3, 5]. By assumption, for any k-dimensional submanifold N in $\mathcal{F}$, we have

$$\langle w_{i_1}(X) \cup \ldots \cup w_{i_r}(X), [N]_X\rangle = 0,$$

which completes the proof in view of Lemma 2.1. □

The proof of Theorem 1.3 requires further preparations. Given a smooth manifold P, let $\mathfrak{N}_*(P)$ denote the unoriented bordism group of P; cf. [7]. The following fundamental result will play a crucial role.

Theorem 2.2. *Let P be a smooth manifold. Two smooth maps $f : M \to P$ and $g : N \to P$, where M and N are compact smooth manifolds of dimension d, represent the same bordism class in $\mathfrak{N}_*(P)$ if and only if for every nonnegative integer q and every cohomology class v in $H^q(P, \mathbb{Z}/2)$, one has*

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(v), [M]\rangle = \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N) \cup g^*(v), [N]\rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = d - q$.

Reference for the proof. [7, (17.3)]. □

If W is a nonsingular real algebraic variety, then a bordism class in $\mathfrak{N}_*(W)$ is said to be *algebraic* if it can be represented by a regular map $h : Y \to W$, where Y is a compact nonsingular real algebraic variety.

We will also make use of a certain construction from real algebraic geometry. Let X be a compact nonsingular real algebraic variety. Define $Alg^l(X)$ to be the subset of $H^l(X, \mathbb{Z}/2)$ that consists of all elements v for which there exist a compact irreducible nonsingular real algebraic variety T (depending on v), two points t_0 and t_1 in T, and a cohomology class z in $H^l_{alg}(X \times T, \mathbb{Z}/2)$ such that

$$v = i^*_{t_1}(z) - i^*_{t_0}(z)\,.$$

Here given t in T, we let $i_t : X \to X \times T$ denote the map defined by $i_t(x) = (x, t)$ for all x in X. For properties and an alternative definition of $Alg^l(-)$, the reader may refer to [10, 11]. Below we will need the following facts. The set $Alg^l(X)$ is a subgroup of $H^l_{alg}(X, \mathbb{Z}/2)$, which has the expected functorial property: if $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^* : H^l(Y, \mathbb{Z}/2) \to H^l(X, \mathbb{Z}/2)$ satisfies

$$f^*(Alg^l(Y)) \subseteq Alg^l(X);$$

cf. [1, p. 114].

Proposition 2.3. *Let X be a compact nonsingular real algebraic variety. If u is in $H^k_{alg}(X, \mathbb{Z}/2)$ and v is in $Alg^l(X)$, where $k + l = \dim X$, then $\langle u \cup v, [X]\rangle = 0$.*

References for the proof. [10, Theorem 2.1] (cf. also [11, Theorem 4.4]). □

Example 2.4. Given a positive integer l let B^l be an irreducible nonsingular real algebraic variety with precisely two connected components B^l_0 and B^l_1, each diffeomorphic to the unit l-sphere S^l (for example $B^l = \{(x_0, \ldots, x_l) \in \mathbb{R}^{l+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_l^2 = 0\}$). Let $B = B^{l_1} \times \cdots \times B^{l_s}$ and $B_0 = B^{l_1}_0 \times \cdots \times B^{l_s}_0$. Then

$$H^q(B_0, \mathbb{Z}/2) = \delta^*(H^q(B, \mathbb{Z}/2)) = \delta^*(Alg^q(B))$$

for all $q \geq 0$, where $\delta : B_0 \hookrightarrow B$ is the inclusion map. This assertion is a minor generalization of [11, Example 4.5].

For convenience we introduce the following notation. For any finite collection $\mathcal{F}$ of smooth submanifolds of a compact smooth manifold M, denote by $A^l(\mathcal{F})$ the subgroup of $H^l(M, \mathbb{Z}/2)$ generated by $\{[N]^M \mid N \in \mathcal{F},\ \operatorname{codim}_M N = l\}$. If $k + l = \dim M$, then

$$G^k(\mathcal{F}) = \{u \in H^k(M, \mathbb{Z}/2) \mid \langle u \cup v, [M]\rangle = 0 \text{ for all } v \text{ in } A^l(\mathcal{F})\}.$$

This follows immediately from the equality $\langle u \cup v, [M]\rangle = \langle u, D_M(v)\rangle$.

Proof of Theorem 1.3. Observe that $G^0(\mathcal{F})$ contains the subgroup of $H^0(M, \mathbb{Z}/2)$ generated by 1. On the other hand, if Y is a compact irreducible nonsingular real algebraic variety, then $H^0_{alg}(Y, \mathbb{Z}/2)$ is the subgroup of $H^0(Y, \mathbb{Z}/2)$ generated by 1. Hence, without loss of generality, we can enlarge the collection $\mathcal{F}$ by adding to it finitely many 0-dimensional submanifolds of M, each consisting of an even number of points. Thus we can assume that $\mathcal{F}$ has the following property: if M_1 and M_2 are distinct connected components of M, then $\{x_1, x_2\}$ is in $\mathcal{F}$ for some points x_1 in M_1 and x_2 in M_2.

Let $\mathcal{F} = \{N_1, \ldots, N_s\}$ and $l_i = \operatorname{codim}_M N_i$ for $1 \le i \le s$. Note that $l_i \ge 1$. We will use notation introduced in Example 2.4. There is a smooth map $f_i : M \to B^{l_i}$ such that $f_i(M) \subseteq B_0^{l_i}$ and $f_i^*(H^{l_i}(B^{l_i}, \mathbb{Z}/2))$ is the subgroup of $H^{l_i}(M, \mathbb{Z}/2)$ generated by $[N_i]^M$. This assertion follows from a well-known fact that if the normal vector bundle of a smooth submanifold N of M is trivial and $\operatorname{codim}_M N = l \ge 1$, then one can find a smooth map $h : M \to S^l$ with $h^*(\lambda) = [N]^M$, where λ is the unique generator of $H^l(S^l, \mathbb{Z}/2) \cong \mathbb{Z}/2$; cf. [13, Théorème II.2]. Setting $f = (f_1, \ldots, f_s) : M \to B = B^{l_1} \times \cdots \times B^{l_s}$, we have

$$f^*(H^q(B, \mathbb{Z}/2)) = A^q(\mathcal{F}) \qquad \text{for all } q \ge 1. \tag{1}$$

Let $c : M \to B$ be a constant map, whose single value is a point in B_0. Since M has an algebraic model, the bordism class of c in $\mathfrak{N}_*(B)$ is algebraic. We claim that for every nonnegative integer q and every cohomology class b in $H^q(B, \mathbb{Z}/2)$,

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(b), [M]\rangle = \langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup c^*(b), [M]\rangle \tag{2}$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = \dim M - q$. The claim is obvious if $q = 0$. For $q \ge 1$ condition (2) is equivalent to

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(b), [M]\rangle = 0.$$

The last equality holds in view of (1) and Lemma 2.1, and hence (2) is proved.

It follows from (2) and Theorem 2.2 that f and c represent the same bordism class in $\mathfrak{N}_*(B)$. In particular, the bordism class of f in $\mathfrak{N}_*(B)$ is algebraic. By [1, Theorem 2.8.4], there exist an algebraic model X of M, a smooth diffeomorphism $\varphi : X \to M$, and a regular map $g : X \to B$ such that g is homotopic to $f \circ \varphi$. It remains to show that X and φ have the properties stated in the theorem. This can be done as follows. Since g is homotopic to $f \circ \varphi$, we get $\varphi^* \circ f^* = (f \circ \varphi)^* = g^*$ in cohomology, and hence for all $q \ge 0$,

$$\varphi^*(f^*(H^q(B, \mathbb{Z}/2))) = g^*(H^q(B, \mathbb{Z}/2)) = g^*(Alg^q(B)), \tag{3}$$

where the last equality follows from the inclusion $g(X) \subseteq B_0$ and Example 2.4. We also have

$$g^*(Alg^q(B)) \subseteq Alg^q(X), \tag{4}$$

the map $g : X \to B$ being regular. Conditions (1), (3), and (4) combined yield

$$\varphi^*(A^q(\mathcal{F})) \subseteq Alg^q(X) \text{ for all } q \ge 1. \tag{5}$$

Let k and l be integers satisfying $k \ge 0$, $l \ge 0$, and $k + l = m$, where $m = \dim M = \dim X$. Since $\varphi^*(A^l(\mathcal{F})) = A^l(\varphi^*\mathcal{F})$ and

$$G^k(\varphi^*\mathcal{F}) = \{u \in H^k(X, \mathbb{Z}/2) \mid \langle u \cup v, [X]\rangle = 0 \text{ for all } v \in A^l(\varphi^*\mathcal{F})\},$$

it follows from (5) and Proposition 2.3 that

$$H^k_{alg}(X, \mathbb{Z}/2) \subseteq G^k(\varphi^*\mathcal{F}) \tag{6}$$

for $0 \leq k \leq m-1$. In fact (6) holds for all $k \geq 0$, the equality $G^k(\varphi^*\mathcal{F}) = H^k(X, \mathbb{Z}/2)$ being automatically satisfied if $k \geq m$.

In order to complete the proof, we need to demonstrate that X is irreducible. This is obvious if M is connected. Suppose then that M is disconnected and X is reducible. Let X_1 and X_2 be distinct irreducible components of X. There are points p_1 in X_1 and p_2 in X_2 such that $P = \{p_1, p_2\}$ is in $\varphi^*\mathcal{F}$ (see the beginning of the proof). In particular, $[P]^X$ belongs to $A^m(\varphi^*\mathcal{F}) = \varphi^*(A^m(\mathcal{F}))$, which in view of (5) implies that $[P]^X$ is in $Alg^m(X)$. If $e : X_1 \hookrightarrow X$ is the inclusion map, then $e^*(Alg^m(X)) \subseteq Alg^m(X_1)$, and hence $e^*([P]^X) = [\{p_1\}]^{X_1}$ is in $Alg^m(X_1)$. However, $\langle [\{p_1\}]^{X_1}, [X_1] \rangle = 1$, which contradicts Proposition 2.3. Thus X is irreducible, as required. □

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