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CHARACTERISTIC CYCLES OF STANDARD SHEAVES ASSOCIATED WITH OPEN ORBITS

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ABSTRACT. Let $G_{\mathbb{R}}$ be a real form of a complex, semisimple Lie group G. We compute the characteristic cycle of a standard sheaf associated with an open $G_{\mathbb{R}}$-orbit on the partial flag variety of G. We apply the result to obtain a Rossmann-type integral formula for elliptic coadjoint orbits. These results were previously obtained by the author under the assumption that the rank of $G_{\mathbb{R}}$ is equal to the rank of a maximal compact subgroup.

Let $G_{\mathbb{R}}$ be a linear, semisimple Lie group. The present paper is a complement to [1], where we obtained a limit formula for elliptic orbital integrals. The proof of the formula was based on the powerful theory of characteristic cycles of equivariant sheaves developed by Schmid and Vilonen in [2], [3] and [4]. An important ingredient in the argument was the formula for the characteristic cycle of a standard sheaf on a generalized flag variety associated to an open $G_{\mathbb{R}}$-orbit, which was proved under the assumption that the rank of $G_{\mathbb{R}}$ is equal to the rank of a maximal compact subgroup. The goal of this paper is to extend the formula for the characteristic cycle to the case of nonequal ranks. Compared to the proof of the corresponding statement in [1], this case does not require essentially new ideas. However, we have to verify that the exhaustion function for an open $G_{\mathbb{R}}$-orbit can be defined analogously as in loc. cit., and we have to discuss in some detail the orientation of the homology cycles appearing in the formula. We also prefer to work with the standard sheaf for proper direct image, because this turns out to be more relevant for the applications. The computation of the characteristic cycle is carried out in Theorem 1, and the main application is recalled in Theorem 5.

Suppose $G_{\mathbb{R}}$ is a real, connected, linear, semisimple Lie group. We embed $G_{\mathbb{R}}$ into the complexification G, and we denote by $\tau : G \longrightarrow G$ a conjugation on G having $G_{\mathbb{R}}$ as the set of fixed points. Next we choose a Cartan involution $\theta : G_{\mathbb{R}} \longrightarrow G_{\mathbb{R}}$, and we denote by $K_{\mathbb{R}}$ the set of fixed points. Extend θ to G, and denote by K the set of fixed points. We denote by $U_{\mathbb{R}}$ the set of fixed points of $\theta\tau$ on G. Write $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{u}_{\mathbb{R}}$ for the Lie algebras of G, K, $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $U_{\mathbb{R}}$, respectively, and we denote the involutions on $\mathfrak{g}$ induced by θ, τ by the same letters. In addition, let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}\,,\ \ \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

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be the Cartan decompositions defined by θ. We fix a θ-stable, fundamental Cartan subalgebra $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$, and we denote by $\mathfrak{h} \subset \mathfrak{g}$ the complexification. Let

$$\mathfrak{h}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}, \ \ \mathfrak{t}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}, \, \mathfrak{a}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}}$$

be the Cartan decomposition. If $\mathfrak{m} \subset \mathfrak{g}$ is an $\operatorname{ad}\mathfrak{h}$-invariant subspace, we denote by $\Delta(\mathfrak{m}, \mathfrak{h})$ the set of $\mathfrak{h}$-weights. Write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta$ denote by $\mathfrak{g}^{\alpha}$ the corresponding root space. Write Δ_I for the set of imaginary and $\Delta_{\mathbb{C}}$ for the set of complex roots in Δ. We further decompose $\Delta_I = \Delta_{I,c} \cup \Delta_{I,nc}$ into compact ($\mathfrak{g}^{\alpha} \subset \mathfrak{k}$) and noncompact ($\mathfrak{g}^{\alpha} \subset \mathfrak{p}$) roots. Since $\mathfrak{h}_{\mathbb{R}}$ is fundamental, $\Delta = \Delta_I \cup \Delta_{\mathbb{C}}$. Choose $h_0 \in i\mathfrak{t}_{\mathbb{R}}$, and define the parabolic subalgebra $\mathfrak{q}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l}$ and $\mathfrak{u}$ are specified by the conditions

$$\Delta(\mathfrak{l}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \alpha(h_0) = 0\}, \ \ \Delta(\mathfrak{u}, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \alpha(h_0) > 0\}. \tag{1}$$

Then we have

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{u} + \tau\mathfrak{u}, \ \ \theta\mathfrak{u} = \mathfrak{u}, \ \text{ and } \ \mathfrak{l} = \mathfrak{q} \cap \tau\mathfrak{q}.$$

In particular, $\mathfrak{l}$ is the complexification of a reductive, θ-stable subalgebra $\mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$. Let $Q \subset G$ be the parabolic subgroup corresponding to $\mathfrak{q}$. Set $Y = G/\tau Q$, and denote by $y_0 \in Y$ the point determined by τQ. Let $S = G_{\mathbb{R}}.y_0 \subset Y$. A short computation shows that $\dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}}) = \dim_{\mathbb{R}}(\mathfrak{g}/\tau\mathfrak{q})$. Hence the orbit S is open in Y. In particular, it inherits a complex structure from Y. Denote by $\mathfrak{c}$ the center of $\mathfrak{l}$, and let $C_{\mathfrak{u}} \subset \mathfrak{c} \cap i\mathfrak{t}_{\mathbb{R}}$ be the chamber defined by $\mathfrak{u}$. In other words

$$C_{\mathfrak{u}} = \{h \in \mathfrak{c} \cap i\mathfrak{t}_{\mathbb{R}} : \alpha(h) > 0, \ \alpha \in \Delta(\mathfrak{u}, \mathfrak{h})\}. \tag{2}$$

Set $2\rho_{\mathfrak{u}} = \sum_{\alpha \in \Delta(\mathfrak{u})} \alpha$. Then $2\rho_{\mathfrak{u}} \in C_{\mathfrak{u}}$ and in particular $C_{\mathfrak{u}} \neq \emptyset$. Denote by T^*Y the cotangent bundle of the variety Y and by T^*_yY its fiber at $y \in Y$. Suppose $\lambda \in \mathfrak{h}^*$ is singular for the roots in $\Delta(\mathfrak{l}, \mathfrak{h})$ and regular for the roots in $\Delta(\mathfrak{u}, \mathfrak{h})$. Then we define the twisted moment map $\mu_\lambda : T^*Y \longrightarrow G \cdot \lambda$ by the formula

$$\mu_\lambda(u \cdot (y_0, \nu)) = \operatorname{Ad}(u)(\lambda + \nu), \ \ u \in U_{\mathbb{R}}, \, \nu \in T^*_{y_0}Y.$$

One can show that μ_λ is a $U_{\mathbb{R}}$-equivariant, real algebraic isomorphism.

Let $j : S \hookrightarrow Y$ be a natural embedding. Denote by $\mathbb{C}_S$ a constant sheaf on S and by $Rj_!\mathbb{C}_S$ (proper direct image) the standard sheaf associated to the pair $(S, \mathbb{C}_S)$. Our goal is to compute the characteristic cycle $CC(Rj_!\mathbb{C}_S)$. We begin by recalling the setting from [1, Sec. 3]. Let $V = V_{-2\rho_{\mathfrak{u}}}$ be the space of algebraic functions $F : G \longrightarrow \mathbb{C}$ satisfying

$$F(gp) = e^{2\rho_{\mathfrak{u}}}(p)F(g), \ \ g \in G, \, p \in Q.$$

Define an action of the Cartan involution θ on V by the formula $\theta F(g) = F(\theta g)$, $F \in V$, $g \in G$. Choose a $U_{\mathbb{R}}$-invariant, positive definite, hermitian form h_u on V. In addition, we may assume that the form h_u is θ-invariant. Another (indefinite) form on V is defined by

$$h_r(v_1, v_2) = h_u(v_1, \theta v_2).$$

One can check that h_r is hermitian and $G_{\mathbb{R}}$-invariant. If $y = g.y_0 \in Y$, $g \in G$, then $\mathfrak{u}_y = Ad(g).\tau\mathfrak{u}$ is the unipotent radical of the parabolic subalgebra $\mathfrak{q}_y$ determined by y. By Kostant's Borel-Weil theorem the space $V^{\mathfrak{u}_y}$ of $\mathfrak{u}_y$ invariant vectors in V is 1-dimensional. This implies that the formula

$$f(y) = \frac{h_r(v, v)}{h_u(v, v)}, \ \ v \in V^{\mathfrak{u}_y},$$

defines a real algebraic function on Y. If necessary, we multiply h_r by -1 to make f positive on S.

Write $Y^{\mathbb{R}}$ for the underlying real analytic structure on Y, and identify the real cotangent bundle $T^*Y^{\mathbb{R}}$ and the holomorphic cotangent bundle T^*Y via the pairing $(v_y, \xi_y) \mapsto 2\mathrm{Re}\, \xi_y(v_y)$, $v_y \in T_yY$, $\xi_y \in T_y^*Y$. View the differential $d\log f$ as a section of $T^*S^{\mathbb{R}}$, and orient it via the isomorphism $d\log f \cong S$ and the complex structure on S.

On the other hand, for $x \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$ write $x \cdot \lambda$ for the coadjoint action. Then the formula

$$\sigma_\lambda|_\xi(a \cdot \xi, b \cdot \xi) = \xi([a, b]), \ \ a,\, b \in \mathfrak{g},$$

defines a symplectic form on $T_\xi(G \cdot \lambda)$. If $\lambda \in i\mathfrak{g}_{\mathbb{R}}^*$, the restriction of the form $-i\sigma_\lambda$ to the coadjoint orbit $G_{\mathbb{R}} \cdot \lambda$ is real valued and

$$\xi \mapsto (-i\sigma_\lambda|_\xi)^m, \ \ 2m = \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot \lambda$$

defines a volume form on $G_{\mathbb{R}} \cdot \lambda$. We use this form to orient the coadjoint orbit $G_{\mathbb{R}} \cdot \lambda$. Set $p = \dim_{\mathbb{C}} \mathfrak{u} \cap \mathfrak{p}$. The properties of the function f are given by the following theorem.

Theorem 1. *The function f is positive on S and vanishes on the boundary of S. The twisted moment map $\mu_{2\rho_{\mathfrak{u}}}$ restricted to $-d\log f$ defines a real algebraic isomorphism*

$$\mu_{2\rho_{\mathfrak{u}}} : -d\log f \longrightarrow G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{u}}).$$

The map $\mu_{2\rho_{\mathfrak{u}}}$ preserves (resp. reverses) orientation if p is even (resp. odd). In particular,

$$CC(Rj_!\mathbb{C}_S) = (-1)^p \lim_{t\to 0^+} \mu_{2\rho_{\mathfrak{u}}}^{-1}(G_{\mathbb{R}} \cdot 2\rho_{\mathfrak{u}}).$$

Proof. We shall check that the argument from [1, 3.2] applies also in the present situation. Let $y \in Cl(S) \setminus S$. We may assume that $\mathfrak{q}_y$ contains a Cartan subalgebra $\mathfrak{h}_1$ stable for τ and θ. For $\lambda \in \mathfrak{h}_1^*$ denote by $V(\lambda)$ the corresponding weight space. As in loc. cit. we show $h(V(\lambda_1), V(\lambda_2))$ if $\lambda_1 \neq \lambda_2$. Observe that $V^{\mathfrak{u}_y} = V(-2\rho_{\mathfrak{u}_y})$ and $\theta V^{\mathfrak{u}_y} = V(-2\theta\rho_{\mathfrak{u}_y})$, where $2\rho_y$ is the sum of roots from $\mathfrak{u}_y$. To prove $h_r(v, v) = 0$ for $v \in V^{\mathfrak{u}_y}$, it suffices to show $\rho_y \neq \theta\rho_y$. Otherwise, $\rho_y = \theta\rho_y$ would imply $\mathfrak{q}_y \cap \tau\mathfrak{q}_y = \mathfrak{l}_y$. In particular, $2\dim_{\mathbb{R}}(\mathfrak{q}_y \cap \mathfrak{g}_{\mathbb{R}}) = \dim_{\mathbb{R}} \mathfrak{l}_y$; hence $\dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}/\mathfrak{q}_y \cap \mathfrak{g}_{\mathbb{R}}) = \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{q}_y)$. This would imply that $G_{\mathbb{R}}.\mathfrak{q}_y$ is open in Y, contrary to the assumption. We conclude that f vanishes on the boundary of S.

We can prove that $\mu_{2\rho_{\mathfrak{u}}}$ induces a real algebraic isomorphism between $-d\log f$ and $G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{u}})$ in the same way as in [1, 3.2]. Now we turn to the orientation statement. It suffices to compare the orientations only at the points y_0 and $2\rho_{\mathfrak{u}}$. The map $I : S \longrightarrow G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{u}})$, $I(g.y_0) = \mathrm{Ad}(g)(2\rho_{\mathfrak{u}})$, $g \in G_{\mathbb{R}}$, is a real algebraic isomorphism, and the differential of I at y_0 is a natural map

$$dI_{y_0} : \mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}} \longrightarrow \mathfrak{g}_{\mathbb{R}} \cdot (2\rho_{\mathfrak{u}}), \ \ \xi + \mathfrak{l}_{\mathbb{R}} \mapsto \xi \cdot (2\rho_{\mathfrak{u}}), \ \xi \in \mathfrak{g}_{\mathbb{R}}.$$

We shall use dI_{y_0} to compare the orientations. For any $\alpha \in \Delta$ we can find $X_\alpha \in \mathfrak{g}^\alpha$ such that

$$e_\alpha = X_\alpha - X_{-\alpha} \in \mathfrak{u}_{\mathbb{R}}, \ f_\alpha = i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}_{\mathbb{R}}, \ [X_\alpha, X_{-\alpha}] = H_\alpha,$$

where $H_\alpha \in \mathfrak{h}$ is a unique element such that $\alpha(H_\alpha) = 2$. Recall that $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + i\mathfrak{p}_{\mathbb{R}}$, so we further write

$$e_\alpha = e_{\alpha,1} + ie_{\alpha,2} \ \text{ and } \ f_\alpha = f_{\alpha,1} + if_{\alpha,2},$$

where $e_{\alpha,1},\ f_{\alpha,1} \in \mathfrak{k}_{\mathbb{R}},\ e_{\alpha,2},\ f_{\alpha,2} \in \mathfrak{p}_{\mathbb{R}}$. Write $\Delta(\mathfrak{u}) = \Delta(\mathfrak{u}, \mathfrak{h})$, and set

$$\Delta(\mathfrak{u})_I = \Delta(\mathfrak{u}) \cap \Delta_I = \Delta(\mathfrak{u})_{I,c} \cup \Delta(\mathfrak{u})_{I,nc} \text{ and } \Delta(\mathfrak{u})_{\mathbb{C}} = \Delta(\mathfrak{u}) \cap \Delta_{\mathbb{C}}.$$

Since $\Delta(\mathfrak{u})$ is θ-stable, we may choose $A \subset \Delta(\mathfrak{u})_{\mathbb{C}}$ such that $\Delta(\mathfrak{u})_{\mathbb{C}} = A \cup \theta A$. For $\alpha \in \Delta(\mathfrak{u})$ we define a subspace of $\mathfrak{g}$ by $\mathfrak{g}(\alpha) = \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}$ if $\alpha \in \Delta_I$ and $\mathfrak{g}(\alpha) = \mathfrak{g}^{\alpha} + \mathfrak{g}^{\theta\alpha} + \mathfrak{g}^{-\alpha} + \mathfrak{g}^{-\theta\alpha}$ if $\alpha \in \Delta_{\mathbb{C}}$. Then $\mathfrak{g}(\alpha)$ is θ- and τ-stable. Write $\mathfrak{g}(\alpha)_{\mathbb{R}} = \mathfrak{g}(\alpha) \cap \mathfrak{g}_{\mathbb{R}}$. The following lemma can be deduced by a straightforward computation.

Lemma 2. (1) *If $\alpha \in \Delta(\mathfrak{u})_{I,c}$, then $\mathfrak{g}(\alpha)_{\mathbb{R}} = \mathbb{R} \cdot e_{\alpha} + \mathbb{R} \cdot f_{\alpha}$.*
(2) *If $\alpha \in \Delta(\mathfrak{u})_{I,nc}$, then $\mathfrak{g}(\alpha)_{\mathbb{R}} = \mathbb{R} \cdot ie_{\alpha} + \mathbb{R} \cdot if_{\alpha}$.*
(3) *If $\alpha \in \Delta(\mathfrak{u})_{\mathbb{C}}$, then $\mathfrak{g}(\alpha)_{\mathbb{R}} = \mathbb{R} \cdot e_{\alpha,1} + \mathbb{R} \cdot e_{\alpha,2} + \mathbb{R} \cdot f_{\alpha,1} + \mathbb{R} \cdot f_{\alpha,2}$.*

To compute the orientation on S, we observe that the holomorphic cotangent space to Y at y_0 is isomorphic to $(\mathfrak{g}/\tau\mathfrak{q})^* \cong \tau\mathfrak{u}$, where we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form.

Lemma 3. *The orientation on S defined by the complex structure on Y at the point y_0 is given by the form*

$$\prod_{\alpha \in \Delta(\mathfrak{u})_{I,c}} \frac{c_{\alpha}^2}{4}(e_{\alpha} \wedge f_{\alpha}) \bigwedge \prod_{\alpha \in \Delta(\mathfrak{u})_{I,nc}} \frac{c_{\alpha}^2}{4}(ie_{\alpha} \wedge if_{\alpha}) \bigwedge \prod_{\alpha \in A} \frac{c_{\alpha}^4}{4}(e_{\alpha,1} \wedge f_{\alpha,1} \wedge e_{\alpha,2} \wedge f_{\alpha,2}),$$

where $c_{\alpha} = \frac{1}{2}(\alpha, \alpha)$.

Proof. In view of the identifications made above, the basis $(c_{\alpha}X_{-\alpha},\ \alpha \in \Delta(\mathfrak{u}))$ is dual to the basis $(X_{\alpha},\ \alpha \in \Delta(\mathfrak{u}))$. It follows that the orientation of S at y_0 is defined by the form

$$(\frac{i}{2})^m \prod_{\alpha \in \Delta(\mathfrak{u})} c_{\alpha}^2 (X_{-\alpha} \wedge \tau X_{-\alpha}), \ \ m = \dim_{\mathbb{C}} Y.$$

Hence we have to examine separately the cases when $\alpha \in \Delta(\mathfrak{u})$ is compact, noncompact and complex. If $\alpha \in \Delta(\mathfrak{u})_{I,c}$ ($\alpha \in \Delta(\mathfrak{u})_{I,nc}$), then $e_{\alpha}, f_{\alpha} \in \mathfrak{k}_{\mathbb{R}}$ ($e_{\alpha}, f_{\alpha} \in i\mathfrak{p}_{\mathbb{R}}$) and we have

$$\frac{i}{2} X_{-\alpha} \wedge \tau X_{-\alpha} = \frac{1}{4} e_{\alpha} \wedge f_{\alpha} \ \ (\frac{i}{2} X_{-\alpha} \wedge \tau X_{-\alpha} = \frac{1}{4} ie_{\alpha} \wedge if_{\alpha}).$$

If $\alpha \in A$, then $X_{-\alpha} = -\frac{1}{2}\left[(e_{\alpha,1} - f_{\alpha,2}) + i(e_{\alpha,2} + f_{\alpha,1})\right]$, so we obtain

$$(\frac{i}{2})^2 X_{-\alpha} \wedge \tau X_{-\alpha} \wedge \theta X_{-\alpha} \wedge \tau\theta X_{-\alpha} = \frac{1}{4} e_{\alpha,1} \wedge f_{\alpha,1} \wedge e_{\alpha,2} \wedge f_{\alpha,2}. \qquad \square$$

Next we compute the Liouville form $(-i\sigma_{\lambda})^m$ at $\lambda \in C_{\mathfrak{u}}$. Write $(x_j,\ j = 1, \ldots, 2m)$ for the basis

$$\bigcup_{\alpha \in \Delta(\mathfrak{u})_{I,c}} (e_{\alpha}, f_{\alpha}) \cup \bigcup_{\alpha \in \Delta(\mathfrak{u})_{I,nc}} (ie_{\alpha}, if_{\alpha}) \cup \bigcup_{\alpha \in A} (e_{\alpha,1}, f_{\alpha,1}, e_{\alpha,2}, f_{\alpha,2})$$

of $\mathfrak{g}_{\mathbb{R}}$ modulo $\mathfrak{l}_{\mathbb{R}}$. Then $(x_j \cdot \lambda,\ j = 1, \cdots, 2m)$ form a basis of the real tangent space of $G_{\mathbb{R}} \cdot \lambda$ at λ. Let $A = [a_{jk}]$, where $a_{jk} = -i\lambda([x_j, x_k])$, be a $2m \times 2m$-matrix of $-i\sigma_{\lambda}$ in the basis $(x_j \cdot \lambda)$. Let $(z_j) = (v_{\alpha}, w_{\alpha},\ \alpha \in \Delta(\mathfrak{u})_I; v_{\alpha,1}, w_{\alpha,1}, v_{\alpha,2}, w_{\alpha,2},\ \alpha \in A)$ be the basis dual to $(x_j \cdot (-i\lambda))$. Then a short computation gives the formula

$$(-i\sigma_{\lambda})^m = \mathrm{Pf}(A) z_1 \wedge \cdots \wedge z_{2m} = \mathrm{Pf}(A)^{-1} x_1 \wedge \cdots \wedge x_{2m},$$

where $\mathrm{Pf}(A)$ denotes the Pfaffian of A. It follows that we have to determine the sign of $\mathrm{Pf}(A)$. Write $\sigma_\lambda(\alpha)$ for the restriction of σ_λ to $\mathfrak{g}(\alpha)_\mathbb{R} \cdot \lambda$. Let $n_\alpha = 1$ if $\alpha \in \Delta(\mathfrak{u})_I$, and let $n_\alpha = 2$ if $\alpha \in A$. The spaces $\mathfrak{g}(\alpha) \cdot \lambda$, $\alpha \in \Delta(\mathfrak{u})_I \cup A$, are orthogonal for σ_λ. Hence

$$(-i\sigma_\lambda)^m = c \cdot \prod_{\alpha \in \Delta(\mathfrak{u})_I \cup A} (-i\sigma_\lambda(\alpha))^{n_\alpha}, \tag{3}$$

where $c > 0$. Hence, to determine the sign of $\mathrm{Pf}(A)$, we have to compute $\sigma_\lambda(\alpha)$.

Lemma 4. (1) *If $\alpha \in \Delta(\mathfrak{u})_{I,c}$, then $-i\sigma_\lambda(\alpha) = 2\lambda(H_\alpha) v_\alpha \wedge w_\alpha$.*
(2) *If $\alpha \in \Delta(\mathfrak{u})_{I,nc}$, then $-i\sigma_\lambda(\alpha) = -2\lambda(H_\alpha) v_\alpha \wedge w_\alpha$.*
(3) *If $\alpha \in A$, then $(-i\sigma_\lambda(\alpha))^2 = -2\lambda(H_\alpha)^2 v_{\alpha,1} \wedge w_{\alpha,1} \wedge v_{\alpha,2} \wedge w_{\alpha,2}$.*

Proof. The first two cases follow by applying the formula $[e_\alpha, f_\alpha] = 2iH_\alpha$. To prove the third formula, we observe that

$$\lambda([e_{\alpha,1}, e_{\alpha,2}]) = \lambda([e_{\alpha,1}, f_{\alpha,2}]) = \lambda([e_{\alpha,2}, f_{\alpha,1}]) = \lambda([f_{\alpha,1}, f_{\alpha,2}]) = 0,$$

and then we use the fact that α, $\theta\alpha \in \Delta(\mathfrak{u})_\mathbb{C}$ to show

$$-i\lambda([e_{\alpha,1}, f_{\alpha,1}]) = \lambda(H_\alpha),\ -i\lambda([e_{\alpha,2}, f_{\alpha,2}]) = -\lambda(H_\alpha).$$

Now we conclude that

$$-i\sigma_\lambda(\alpha) = \lambda(H_\alpha) v_{\alpha,1} \wedge w_{\alpha,1} - \lambda(H_\alpha) v_{\alpha,2} \wedge w_{\alpha,2},$$

so by taking the square of $-i\sigma_\lambda(\alpha)$, we obtain the desired formula. □

Theorem 1 follows now from Lemmas 3 and 4, equation (3) and [2, 4.2]. □

Finally, we recall the main application of the above theorem. It can be deduced from Theorem 1 analogously as Theorem 3.4 from Lemmma 3.2 and Theorem 3.3 in [1].

Theorem 5. *Let $G_\mathbb{R}$ be a connected, linear, semisimple Lie group. Define a θ-stable parabolic subalgebra $\mathfrak{q}$ by (1), and consider the corresponding generalized flag variety Y and the open orbit $S = Ad(G_\mathbb{R})\tau\mathfrak{q}$. Let $\phi \in C_c^\infty(\mathfrak{g}_\mathbb{R})$. Denote by $\hat{\phi}$ the Fourier transform of ϕ. Then for $\lambda \in C_\mathfrak{u}$ (2) we have*

$$\int_{CC(Rj_!\mathbb{C}_S)} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m) = (-1)^p \int_{G_\mathbb{R}\cdot\lambda} \hat{\phi}\sigma_\lambda^m.$$

References

1. Božičević, M., *A limit formula for elliptic orbital integrals*, Duke Math. J. **113** (2002), 331–353. MR1909221 (2003k:22014)
2. Schmid, W., Vilonen, K., *Characteristic cycles of constructible sheaves*, Invent. Math. **124**(1996), 451–502. MR1369425 (96k:32016)
3. Schmid, W., Vilonen, K., *Two geometric character formulas for reductive Lie groups*, J. Amer. Math. Soc. **11**(1998), 799–867. MR1612634 (2000g:22020)
4. Schmid, W., Vilonen, K., *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Ann. of Math. (2) **151**(2001), 1071–1118. MR1779564 (2001j:22017)

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