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UNIFORM PERIODIC POINT GROWTH IN ENTROPY RANK ONE

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Abstract. We show that algebraic dynamical systems with entropy rank one have uniformly exponentially many periodic points in all directions.

1. Introduction

Let α be an action of $\mathbb{Z}^d$ by continuous automorphisms of a compact metrizable abelian group X (such a system is called an algebraic $\mathbb{Z}^d$-action). For a continuous map $\beta : X \to X$ write $h(\beta)$ for the topological entropy and

$$\mathrm{F}(\beta) = \{x \in X \mid \beta x = x\}$$

for the set of fixed points. The action α is said to have *entropy rank one* if, for each $\mathbf{n} \in \mathbb{Z}^d$, $h(\alpha^{\mathbf{n}}) < \infty$.

If α is a mixing entropy rank one action and the topological dimension $\dim(X)$ is finite, $\mathrm{F}(\alpha^{\mathbf{n}})$ is finite for all $\mathbf{n} \in \mathbb{Z}^d$. Our purpose here is to show that under natural conditions $|\mathrm{F}(\alpha^{\mathbf{n}})|$ exhibits uniform exponential growth. Write $\mathbf{n}_j \to \infty$ for a sequence in $\mathbb{Z}^d$ if for any finite set $F \subset \mathbb{Z}^d$ there is some J for which $j > J$ implies that $\mathbf{n}_j \notin F$. Equivalently, this means $\|\mathbf{n}_j\| \to \infty$ as $j \to \infty$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{Z}^d$. The Noetherian condition mentioned in Theorem 1.1 is explained in Section 2.

Theorem 1.1. *Let α be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group X. Then there exist constants $C_1, C_2 \geq 0$ such that*

$$\limsup_{\mathbf{n}\to\infty} \frac{1}{\|\mathbf{n}\|} \log |\mathrm{F}(\alpha^{\mathbf{n}})| = C_1 < \infty$$

and

$$\liminf_{\mathbf{n}\to\infty} \frac{1}{\|\mathbf{n}\|} \log |\mathrm{F}(\alpha^{\mathbf{n}})| = C_2.$$

If $\dim(X) > 0$ *and the action is Noetherian, then* $C_2 > 0$.

An affirmative answer to Lehmer's problem (see Remark 2.3) would render the finite-dimension assumption in Theorem 1.1 redundant. The assumption is also not required for a mixing Noetherian entropy rank one algebraic action; in that setting

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it is a consequence of the Noetherian condition. Essentially, the only non-trivial conclusion in the theorem is that $C_2 > 0$.

The action α is *expansive* if there exists a neighbourhood U of 0_X such that

$$\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0_X\}.$$

For $d = 1$ and α expansive, the growth rate of periodic points exists, so $C_1 = C_2$ in Theorem 1.1. The constant coincides with the entropy and is non-zero provided X is infinite.

The $\mathbb{Z}^2$-action on a zero-dimensional group in Example 1.3 is expansive yet has $C_2 = 0$. Expansive actions on connected groups are more indicative of the import of a positive value for C_2. In terms of expansive subdynamics (see [3] and [6]), there are sequences $\mathbf{n} \to \infty$ converging to non-expansive lines; along such sequences $|\,\mathrm{F}(\alpha^{\mathbf{n}})|$ is much smaller than the same expression with $\mathbf{n}$ of similar Euclidean size and far from non-expansive directions. Nonetheless, there is a uniform exponential growth in all directions.

Example 1.2. Consider the $\mathbb{Z}^2$-action α dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto 2r$ and $r \mapsto 3r$ on $\mathbb{Z}[\frac{1}{6}]$. Table 1 shows the map

$$\mathbf{n} \mapsto |\,\mathrm{F}(\alpha^{\mathbf{n}})| \in \mathbb{N}.$$

Notice that $|\,\mathrm{F}(\alpha^{-\mathbf{n}})| = |\,\mathrm{F}(\alpha^{\mathbf{n}})|$, so only the region $n_2 \geq 0$ is shown, with ∞ denoting the lattice point $(0,0)$ corresponding to the identity map $\alpha^{(0,0)}$. In an expansive direction such as $(1,1)$ we have $|\,\mathrm{F}(\alpha^{n(1,1)})| = 6^n - 1$ and the exponential growth rate along the sequence in italics is clear. For lattice points close to the non-expansive line $2^x 3^y = 1$ we find (for example) that $|\,\mathrm{F}(\alpha^{(-5,3)})| = 5$, and it is not immediately clear that there is exponential growth along the sequence shown in bold; Theorem 1.1 asserts that there is.

Example 1.3. Consider the expansive $\mathbb{Z}^2$-action α on X dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto u_1 r$ and $r \mapsto u_2 r$ on

$$\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]/\langle 2, 1 + u_1 + u_2\rangle$$

(this is an example of the type introduced by Ledrappier [12]; X is a zero-dimensional group). As shown in [4], $|\,\mathrm{F}(\alpha^{(n,0)})| = 2^{n-2^{\mathrm{ord}_2(n)}}$. In particular,

$$\lim_{n\to\infty} \frac{1}{\|(2^n, 0)\|} \log |\,\mathrm{F}(\alpha^{(2^n,0)})| = 0,$$

showing that some assumption on the topological dimension of X is needed to have $C_2 > 0$ in Theorem 1.1. If the map dual to $r \mapsto u_2 r$ is ignored, then this

TABLE 1. Periodic point counts for $\times 2, \times 3$.

211	227	235	239	241	121	485	971	1943	3887	*7775*
49	65	73	77	79	5	161	323	647	*1295*	2591
5	11	19	23	25	13	53	107	*215*	431	863
23	**7**	**1**	5	7	1	17	*35*	71	143	287
29	13	**5**	1	1	1	*5*	11	23	47	95
31	5	7	**1**	**1**	∞	1	1	7	5	31

example is a non-expansive (and non-Noetherian) $\mathbb{Z}$-action on a zero-dimensional group with $C_2 = 0$.

Example 1.4. The Noetherian condition is needed to have $C_2 > 0$ even for $d = 1$ on connected groups. For example, the automorphism α dual to the map $r \mapsto 2r$ on $\mathbb{Q}$ has $|\operatorname{F}(\alpha^n)| = 1$ for all n. For rings between $\mathbb{Z}$ and $\mathbb{Q}$ a variety of exotic periodic point behavior is possible (see [20] or [21] for the details).

The full statement of Theorem 1.1 does however apply to Noetherian non-expansive systems. Thus, for example, the genuinely partially hyperbolic systems such as that described by Damjanović and Katok [5, Ex. 7.3] satisfy the hypotheses.

2. Proof of Theorem 1.1

The case $d = 1$ is covered by [4] and [16], so we may assume $d \geq 2$. Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [11].

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \dots, u_d$ with integer coefficients. If α is an algebraic $\mathbb{Z}^d$-action of a compact abelian group X, the character group $M = \widehat{X}$ has the structure of a discrete countable R_d-module, obtained by first identifying the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by the monomial $u^{\mathbf{n}} = u_1^{n_1} \dots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, any countable R_d-module M has an associated algebraic $\mathbb{Z}^d$-action obtained by dualizing the action induced by multiplying by monomials on M. The action $\alpha = \alpha_M$ is described as *Noetherian* if the module M is Noetherian. A full account of this correspondence and the resulting theory is given in Schmidt's monograph [19].

Entropy rank one actions are described in [7] and developments concerning their periodic points may be found in the papers [16] and [18]. By a slight abuse of notation, write $h(\cdot)$ for the topological entropy of maps and for the extension of the entropy function to all of $\mathbb{R}^d$ in the sense explained below.

Proposition 2.1. *Let α_M be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group. Then*

(1) *The set of associated primes of the associated R_d-module, $\operatorname{Asc}(M)$, is finite. For each $\mathfrak{p} \in \operatorname{Asc}(M)$, the domain $R_d/\mathfrak{p}$ has Krull dimension 1 and its field of fractions $\mathbb{K}(\mathfrak{p})$ is a global field.*

(2) *There exist positive integers $m(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Asc}(M)$, such that for every non-zero $\mathbf{n} \in \mathbb{Z}^d$,*

$$h(\alpha_M^{\mathbf{n}}) = \sum_{\mathfrak{p} \in \operatorname{Asc}(M)} m(\mathfrak{p}) h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) \tag{2.1}$$

and

$$|\operatorname{F}(\alpha_M^{\mathbf{n}})| \leqslant \prod_{\mathfrak{p} \in \operatorname{Asc}(M)} |\operatorname{F}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})|^{m(\mathfrak{p})}, \tag{2.2}$$

with equality if M is Noetherian.

(3) *For each $\mathfrak{p} \in \operatorname{Asc}(M)$, there is a finite set of places $\mathcal{S}(\mathfrak{p})$ of $\mathbb{K}(\mathfrak{p})$ such that*

$$h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) = \sum_{v \in \mathcal{S}(\mathfrak{p})} \max\{\mathbf{l}_v \cdot \mathbf{n}, 0\} > 0 \tag{2.3}$$

and

$$|\,\mathrm{F}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})| = \prod_{v\in\mathcal{S}(\mathfrak{p})} |\xi^{\mathbf{n}}_{\mathfrak{p}} - 1|_v, \tag{2.4}$$

where $\xi_{\mathfrak{p}} = (\overline{u}_1, \ldots, \overline{u}_d)$ *denotes the image of* u *in* $(R_d/\mathfrak{p})^d$ *and* $\mathbf{l}_v = (\log|\overline{u}_1|_v, \ldots, \log|\overline{u}_d|_v)$.

Proof. Let $\mathbf{n} \in \mathbb{Z}^d$ be non-zero. Since α_M is mixing, [19, Prop. 6.6] shows that for each $\mathfrak{p} \in \mathrm{Asc}(M)$, $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is ergodic, so $h(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) > 0$. It follows from [7] that for each $\mathfrak{p} \in \mathrm{Asc}(M)$, $R_d/\mathfrak{p}$ has Krull dimension 1 and $\mathbb{K}(\mathfrak{p})$ is a global field. If $\mathrm{char}(R_d/\mathfrak{p}) > 0$, then $h(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) \geq \log 2$. Via Yuzvinskiĭ's formula (see [19, Th. 14.1] or [23]), this implies there are only finitely many such $\mathfrak{p} \in \mathrm{Asc}(M)$, as $h(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) < \infty$.

Also, there can be only finitely many $\mathfrak{p} \in \mathrm{Asc}(M)$ with $\mathrm{char}(R_d/\mathfrak{p}) = 0$, as

$$\dim(X) \geqslant |\{\mathfrak{p} \in \mathrm{Asc}(M) : \mathrm{char}(R_d/\mathfrak{p}) = 0\}|.$$

This establishes (1).

The method of [7, Lem. 8.2] shows that in any prime filtration of a Noetherian submodule of M, each prime $\mathfrak{p} \in \mathrm{Asc}(M)$ appears with a maximum multiplicity

$$m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})}(M \otimes_{R_d} \mathbb{K}(\mathfrak{p})),$$

which is finite by similar reasoning to the proof of (1). By adopting an algorithm for obtaining a prime filtration that selects the associated primes of M first, one obtains a Noetherian submodule $L \subset M$ such that each prime $\mathfrak{p} \in \mathrm{Asc}(M)$ appears with multiplicity $m(\mathfrak{p})$ in a filtration of L. Furthermore, if $L \neq M$, then each $\mathfrak{q} \in \mathrm{Asc}(M/L)$ is maximal and $R_d/\mathfrak{q}$ is a finite field. Hence, Yuzvinskiĭ's formula shows that $h(\alpha^{\mathbf{n}}_{M/L}) = 0$ and $h(\alpha^{\mathbf{n}}_M) = h(\alpha^{\mathbf{n}}_L)$; the formula (2.1) then follows from [15, Lem. 4.3].

If M is Noetherian, equality in (2.2) is given by [16, Th. 3.2]. If M is not Noetherian, (2.2) follows from [16, Th. 3.2] applied to L, together with the inequality

$$|\,\mathrm{F}(\alpha^{\mathbf{n}}_M)| \leqslant |\,\mathrm{F}(\alpha^{\mathbf{n}}_L)|,$$

which is established using a similar method to the proof of [17, Lem. 2.6].

Finally, the entropy formula (2.3) follows from [7, Prop. 8.5] and the periodic point counting formula (2.4) is [16, Lem 3.1]. □

Proof of Theorem 1.1. Let $M = \widehat{X}$ denote the dual R_d-module and let $\mathfrak{p} \in \mathrm{Asc}(M)$. For a fixed $\mathfrak{p} \in \mathrm{Asc}(M)$ and any non-zero $\mathbf{n} \in \mathbb{Z}^d$, set

$$f(\mathbf{n}) = \frac{1}{\|\mathbf{n}\|} \log|\,\mathrm{F}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})|. \tag{2.5}$$

Let $h : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ denote the directional entropy function for the action $\alpha_{R_d/\mathfrak{p}}$. This is the function obtained by extending the entropy formula (2.3) to values of $\mathbb{R}^d$ (see [7, Sec. 8] for further details).

For any vector $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$, write $\widehat{\mathbf{v}} \in \mathsf{S}_{d-1}$ for the unique vector of unit length with the property that $\mathbf{v} = \lambda\widehat{\mathbf{v}}$ for some scalar $\lambda > 0$. From (2.3) and (2.4) it follows that

$$f(\mathbf{n}) = g(\mathbf{n}) + h(\widehat{\mathbf{n}}) \tag{2.6}$$

where

$$g(\mathbf{n}) = \frac{1}{\|\mathbf{n}\|} \sum_{v \in \mathcal{S}(\mathfrak{p})} \log |1 - \phi_v(\mathbf{n})|_v$$

and

$$\phi_v(\mathbf{n}) = \begin{cases} \xi_{\mathfrak{p}}^{-\mathbf{n}} & \text{if } |\xi_{\mathfrak{p}}^{\mathbf{n}}|_v > 1; \\ \xi_{\mathfrak{p}}^{\mathbf{n}} & \text{if } |\xi_{\mathfrak{p}}^{\mathbf{n}}|_v \leqslant 1. \end{cases}$$

Notice that $\phi_v(\mathbf{n}) \neq 1$ if $\mathbf{n} \neq 0$ by the assumption that the action is mixing.

To establish the lower bound $C_2 > 0$ in Theorem 1.1, first note that equality in (2.2) gives the expression

$$\frac{1}{\|\mathbf{n}\|} \log |\operatorname{F}(\alpha_M^{\mathbf{n}})|$$

as a finite sum of terms of the form (2.5) and, crucially, the assumption $\dim(X) > 0$ means at least one of these terms arises from a prime $\mathfrak{p}$ such that $\mathbb{K}(\mathfrak{p})$ is an algebraic number field (rather than a function field of positive characteristic). For the lower bound, it is therefore enough by (2.2) to consider only the asymptotic behavior of f with the assumption that it arises from such a prime.

We need to know that g does not make an asymptotic contribution. It is clear that g cannot be too large, since

$$\prod_{v \in \mathcal{S}(\mathfrak{p})} |1 - \phi_v(\mathbf{n})|_v \leq 2^{|\mathcal{S}(\mathfrak{p})|}.$$

On the other hand, it cannot be large and negative for the following reason. If v is an infinite (Archimedean) place, then Baker's theorem [1] can be used to find constants $A, B \geq 0$ such that

$$|1 - \phi_v(\mathbf{n})|_v \geq \frac{A}{\max\{n_i\}^B}$$

(see [4] or [9] for similar arguments; roughly speaking, the issue is to bound the proximity of $\xi_{\mathfrak{p}}^{\mathbf{n}}$ and 1 in terms of $\mathbf{n}$). If v is a finite place, then Yu's p-adic bounds for linear forms in logarithms [22] give a similar lower bound (note that $d \geq 2$ by assumption). Since $\mathcal{S}(\mathfrak{p})$ is finite, this shows that

$$g(\mathbf{n}) \to 0 \text{ as } \mathbf{n} \to \infty. \tag{2.7}$$

Assume, for a contradiction, that $C_2 = 0$. Then there is a sequence $\mathbf{n}_j \to \infty$ as $j \to \infty$ with the property that

$$\lim_{j \to \infty} f(\mathbf{n}_j) = 0. \tag{2.8}$$

It follows from (2.6) and (2.7) that

$$h(\widehat{\mathbf{n}_j}) \to 0 \text{ as } j \to \infty. \tag{2.9}$$

Now the entropy function h restricted to the unit sphere S_{d-1} is a continuous function on a compact set, so (2.9) implies that there is some $\mathbf{v} \in \mathsf{S}_{d-1}$ for which

$$h(\mathbf{v}) = 0. \tag{2.10}$$

If the ray through $\mathbf{v}$ happens to meet a point $\mathbf{m} \in \mathbb{Z}^d$, then we have an immediate contradiction: the automorphism $\alpha_{R_d/\mathfrak{p}}^{\mathbf{m}}$ would have zero entropy, contradicting (2.3). In order to show that any point $\mathbf{v}$ with (2.10) gives a contradiction we need to make a quantitative version of that argument.

The explicit formula (2.3) for h means there is a list of vectors

$$\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{R}^d$$

with the property that for any $\mathbf{u} \in \mathbb{R}^d$,

$$h(\mathbf{u}) = \mathbf{u} \cdot \mathbf{a}_k,$$

for some k, $1 \leqslant k \leqslant r$. It follows that for any $\mathbf{n} \in \mathbb{Z}^d$ and $\mathbf{u} \in \mathbb{R}^d$,

$$|h(\mathbf{n}) - h(\mathbf{u})| \leqslant \max_{1 \leqslant k \leqslant r} \max_{1 \leqslant i \leqslant d} \{|a_{k,i}|\} \|\mathbf{n} - \mathbf{u}\|,$$

where $\mathbf{a}_k = (a_{k,1}, \dots, a_{k,d})$. In particular, since $h(\lambda \mathbf{v}) = 0$ for all $\lambda > 0$ we may find a sequence $(\mathbf{m}_j)$ of vectors in $\mathbb{Z}^d$ with $\|\mathbf{m}_j - \lambda_j \mathbf{v}\| \to 0$ as $j \to \infty$ (for some sequence (λ_j) of scalars) and hence have

$$h(\mathbf{m}_j) \to 0 \text{ as } j \to \infty. \tag{2.11}$$

On the other hand, the dual group of $R_d/\mathfrak{p}$ is connected and finite-dimensional. Hence, by Yuzvinskiĭ's formula, for any $\mathbf{n} \neq 0$,

$$h(\mathbf{n}) \geqslant \mathsf{m}(P) > 0$$

where $\mathsf{m}(P) = \int_0^1 \log |P(e^{2\pi \mathrm{i} s})| \mathrm{d}s$ denotes the logarithmic Mahler measure of some polynomial P of degree no greater than $\dim_{\mathbb{Q}}(\mathbb{K}(\mathfrak{p}))$. It follows that there is a constant $C > 0$ (depending only on $\mathfrak{p}$) for which

$$h(\mathbf{n}) > C > 0 \text{ for any } \mathbf{n} \neq 0$$

(the existence of a lower bound for the non-zero logarithmic Mahler measure of polynomials of bounded degree is well known; see [2] or [8] for the background). This lower bound contradicts (2.11), so (2.8) is impossible. Thus $C_2 > 0$.

The upper bound is clear: the inequality (2.2) together with the explicit formula (2.4) gives a uniform constant C_1 with the property that

$$|\,\mathrm{F}(\alpha^{\mathbf{n}})| \leq C_1^{\max_{1 \leqslant i \leqslant d}\{n_i\}}. \qquad \square$$

Remark 2.2. Baker's theorem provides the key estimates in many dynamical problems; see [4] and [9] for examples. As pointed out by Lind [14], in order to establish the logarithmic growth rate of periodic points for a quasihyperbolic toral automorphism (a typical application), sometimes all that is needed is a weaker and earlier result due to Gel′fond [10]. Here we need something closer to the full weight of the theorems of Baker and Yu, because we are in a higher rank.

Remark 2.3. Lehmer's problem [13] asks if there is a uniform lower bound for all positive Mahler measures. As shown by Lind (see [15]) this is equivalent to a uniform lower bound for the topological entropy of any mixing compact group automorphism. If 'Lehmer's conjecture' (that there is such a bound and that it is attained by the expected polynomial) holds, then the topological entropy of any compact group automorphism with positive entropy is at least

$$0.162\cdots = \mathsf{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1).$$

This does not imply a uniform bound for C_2 in Theorem 1.1, because C_2 is influenced by the geometry of the acting group as well as the collection of maps in its image. For example, the $\mathbb{Z}^2$-action α corresponding to the module $R_2/\langle u_1 - 2, u_1^k u_2 - 3\rangle$ has the property that $h(\alpha^{(k,1)}) = \log 3$, so the corresponding constant C_2 cannot exceed $\log 3/\sqrt{1+k^2}$.

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