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ATOMIC CHARACTERIZATION OF THE HARDY SPACE $H^1_L(\mathbb{R})$ OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH NONNEGATIVE POTENTIALS

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ABSTRACT. Given a Schrödinger operator $L = \frac{d^2}{dx^2} - V(x)$ on $\mathbb{R}$ with nonnegative potential V, we present an atomic characterization of the associated Hardy space $H^1_L(\mathbb{R})$.

1. INTRODUCTION

Consider a Schrödinger operator $L = \frac{d^2}{dx^2} - V(x)$ on $\mathbb{R}$ with locally integrable, nonnegative potential $V \in L^1_{loc}(\mathbb{R})$, $V \geq 0$, $V \not\equiv 0$. It is well known that L generates a semigroup of operators $\{K_t : t \geq 0\}$ in $L^p(\mathbb{R})$, $1 \leq p < \infty$, with nonnegative integral kernels $\{k_t : t \geq 0\}$, see, e.g., [2]. We define the *Hardy space* $H^1_L(\mathbb{R})$ associated with the operator L as

$$H^1_L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \|f\|_{H^1_L} = \left\| \sup_{t \geq 0} |K_t f(\cdot)| \right\|_{L^1(\mathbb{R})} < \infty \right\}.$$

The aim of this note is to describe an *atomic decomposition* of $H^1_L(\mathbb{R})$. We start by introducing some definitions and notation.

Let $\{I_j : j \in \mathbb{N}\}$ be a cover of $\mathbb{R}$ by dyadic intervals (i.e., intervals of the form $[m2^k, (m+1)2^k]$, $k, m \in \mathbb{Z}$) with disjoint interiors. For any two closed dyadic intervals $I, J \subset \mathbb{R}$, we say that they are *neighbors* of each other if their intersection $I \cap J$ consists of a single point. Assume that the dyadic cover $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ has the following additional property: for each I_j there exist its neighbors $I_{j'}, I_{j''}$, and $I_{j'}, I_{j''}$ have lengths comparable to the length of I_j with uniform comparison constants $0 < C_1, C_2 < \infty$.

Throughout this paper we shall denote the length of an interval I by $|I|$. Moreover, given an interval $I \subset \mathbb{R}$, we denote by $I^* \subset \mathbb{R}$ the interval with the same center as I, but with length equal to $1 + \alpha$ times the length of I. The parameter $\alpha > 0$ will depend on constants in Lemma 2.1 and, thus, shall be chosen later. At this point we shall only mention that α needs to be sufficiently small, in particular, so that $\{I^{**}_j : j \in \mathbb{N}\}$ forms a locally finite cover of $\mathbb{R}$.

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We say that a is an $H^1_{\mathcal{I}}(\mathbb{R})$-*atom* if either a is a classical atom supported in I_j^* for some $j \in \mathbb{N}$ (that is, $\int a = 0$ and $|a| \le |I_j^*|^{-1}$, see, e.g., [3, Definition 1.2.3]) or if $a = \|I_j\| \mathbb{1}_{I_j}$ for some $j \in \mathbb{N}$. Next we define the Hardy space $H^1_{\mathcal{I}}(\mathbb{R})$ associated with $\mathcal{I}$ to be

$$H^1_{\mathcal{I}}(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : f(x) = \sum_{j\in\mathbb{N}} \lambda_j a_j(x),\ \sum_{j\in\mathbb{N}} |\lambda_j| < \infty \right\} \tag{1.1}$$

for any family $\{a_j : j \in \mathbb{N}\}$ of $H^1_{\mathcal{I}}(\mathbb{R})$-atoms. We let

$$\|f\|_{H^1_{\mathcal{I}}(\mathbb{R})} = \inf \left\{ \sum_{j\in\mathbb{N}} |\lambda_j| < \infty \right\},$$

where the infimum is taken over all representations of f in (1.1).

Our goal is to prove the following complete atomic characterization of Hardy spaces $H^L_1(\mathbb{R})$.

Theorem 1.1. *Let* $L = \frac{d^2}{dx^2} - V(x)$ *on* $\mathbb{R}$, $V \in L^1_{loc}(\mathbb{R})$, $V \ge 0$, $V \not\equiv 0$. *Then, there exists a family of dyadic intervals* $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ *such that*

$$H^1_L(\mathbb{R}) = H^1_{\mathcal{I}}(\mathbb{R}).$$

2. Auxiliary lemmas and Proof of Theorem 1.1

The proof of Theorem 1.1 is not self-contained and it uses a slightly modified argument from [1]. For the statements and proofs of auxiliary lemmas in this section we need the following definition: given an interval I, we denote by $I^{\#}$ an interval with the same center as I but with twice the length. We begin with the following lemma.

Lemma 2.1. *Let* $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ *be a family of maximal dyadic intervals defined by the stopping time condition*

$$|I| \int_{I^{\#\#\#\#}} V \le 1.$$

Then, $\mathcal{I}^{**} = \{I_j^{\#\#} : j \in \mathbb{N}\}$ *forms a locally finite cover of* $\mathbb{R}$ *and, for each* $I_j \in \mathcal{I}$, *its neighbors are well defined and have lengths comparable to* $|I_j|$.

Lemma 2.1 seems to be known. Since we do not know the exact reference, we provide here a complete proof.

Proof. Denote by I^d the smallest dyadic interval containing I and bigger than I. For $x \in \mathbb{R}$ denote by I_x the maximal dyadic interval I such that

$$x \in I \qquad \text{and} \qquad |I| \int_{I^{\#\#\#\#}} V \le 1.$$

Since $\lim_{I\to 0} |I| \int_{I^{\#\#\#\#}} V = 0$ and $\lim_{I\to\infty} |I| \int_{I^{\#\#\#\#}} V = \infty$, the interval I_x is well defined. Moreover, $\mathbb{R} = \bigcup_{x\in\mathbb{R}} I_x$. Using standard properties of dyadic intervals we can choose a countable subfamily $\mathcal{I}$ of $\{I_x : x \in \mathbb{R}\}$ such that $\mathbb{R} = \bigcup_{j=1}^{\infty} I_j$ and the intervals $\{I_j : j \in \mathbb{N}\}$ have disjoint interiors.

Fix $I \in \mathcal{I}$. Assume that for each $\epsilon > 0$ there exists some dyadic interval $J \in \mathcal{I}$ such that the following conditions hold:

$$dist\,\{I, J\} \le |I| \qquad \text{and} \qquad |J| \le \varepsilon |I|.$$

Choosing ε sufficiently small, one can verify that $(J^d)^{\#\#\#\#} \subset I^{\#\#\#\#}$. By the definition of J we obtain

$$1 \geq |I| \int_{I^{\#\#\#\#}} V \geq |I| \int_{(J^d)^{\#\#\#\#}} V = \frac{|I|}{|J^d|} |J^d| \int_{(J^d)^{\#\#\#\#}} V \geq \frac{|I|}{|J^d|} \geq \frac{1}{2\varepsilon}.$$

Since ε is arbitrary, the above estimate yields a contradiction. Thus, there exist universal constants $0 < C_1, C_2 < \infty$, such that for each $I \in \mathcal{I}$ its neighbors are well defined and their lengths are comparable to $|I|$, with universal comparison constants C_1 and C_2.

We close the proof of Lemma 2.1 by choosing α sufficiently small so that $\mathcal{I}^{**} = \{I_j^{**} : j \in \mathbb{N}\}$ is a locally finite cover of $\mathbb{R}$. □

In Lemmas 2.2 and 2.3 below we indicate the necessary modifications of [1] required to prove Theorem 1.1. Let $\mathcal{I}$ be as described in Lemma 2.1. We start by recalling condition (D) from [1]:

$$\exists C, \epsilon > 0, \text{ such that } \forall\, y \in I^*,\ \forall\, j \geq 0, \qquad \int k_{2^j |I|^2}(x,y) dx \leq C j^{-1-\epsilon}.$$

In order to verify that condition (D) is satisfied, we shall use the well-known method of *superharmonic weights.*

Lemma 2.2. *Let $\mathcal{I}$ be the cover of $\mathbb{R}$ defined in Lemma 2.1 and let $I \in \mathcal{I}$. There exists a constant $C > 0$ such that*

$$\forall\, y \in I^{\#\#\#\#},\ \forall\, j \geq 0, \qquad \int k_{2^j|I|^2}(x,y) dx \leq C 2^{-\frac{j}{2}}.$$

Proof. By the definition of $I \in \mathcal{I}$, we may choose an interval J with the properties

$$I^{\#\#\#\#} \subset J \subset (I^d)^{\#\#\#\#}$$

and

$$|J| \int_J V = 1.$$

Define the function

$$\phi_I(x) = \int_J V(y) \left(C|J| + \frac{1}{2}|x-y| \right) dy,$$

where $C > 0$ is some positive constant. With appropriate choice of the constant C, function ϕ_I satisfies $\phi_I(x) \geq 1 + \frac{|x|}{4|J|}$ and $\phi_I(y) \leq C'$ for some other constant $C' > 0$. We shall show first that $L(\phi_I)(x) \leq 0$. Indeed, we have

$$\frac{d^2}{dx^2}|x-y| = 2\delta_0(x-y),$$

where δ_x denotes the Dirac delta at $x \in \mathbb{R}$ (point mass). Consequently,

$$L(\phi_I)(x) = \frac{d^2}{dx^2}\phi_I(x) - V(x)\phi_I(x) = V(x)(1_J(x) - \phi_I(x)) \leq 0.$$

This, in turn, yields

$$\frac{d}{dt} \int K_t(\delta_y)(x)\phi_I(x) dx = \int K_t(\delta_y)(x) L(\phi_I)(x) dx \leq 0.$$

By the above,

$$\int K_t(\delta_y)(x)\phi_I(x) dx \leq \phi_I(y). \tag{2.1}$$

Without loss of generality, we may assume that $|I| = 1$. Then, (2.1) implies in particular that

$$\forall\, t \geq 0, \quad \int k_t(0,y)dy \leq C(1+t)^{-\frac{1}{2}}. \tag{2.2}$$

Indeed, in order to see (2.2), we shall consider the function

$$\rho(t) = \int k_t(0,y)dy.$$

Observe that $\|k_t\|_{L^\infty} \leq Ct^{-\frac{1}{2}}$ implies that we have

$$k_{2t}(0,z) = \int k_t(0,y)k_t(y,z)dy \leq Ct^{-\frac{1}{2}} \int k_t(0,y)dy = C\rho(t)t^{-\frac{1}{2}}.$$

Hence, using (2.1), we get for $R > 0$

$$\rho(2t) \leq \int_{|x|\leq R} k_{2t}(0,y)dy + R^{-1} \int_{|x|\geq R} k_{2t}(0,y)(1+|y|)dy \leq 2RC\rho(t)t^{-\frac{1}{2}} + CR^{-1}.$$

Taking a minimum with respect to R, we obtain

$$\rho(2t) \leq Ct^{-\frac{1}{4}}(\rho(t))^{\frac{1}{2}}.$$

Iterating the above inequality yields the desired estimate (2.2). Lemma 2.2 follows. □

Let $\{\phi_j : j \in \mathbb{N}\}$ be a smooth resolution of identity associated with $\{I_j : j \in \mathbb{N}\}$; that is, let it be a family of smooth functions with the properties

$$\begin{aligned} & \operatorname{supp} \phi_j \subset I_j^*, \\ & 0 \leq \phi_j \leq 1, \\ \forall x \in \mathbb{R}, \quad & \left| \frac{d^n}{dx^n} \phi_j(x) \right| \leq C_n |I_j|^{-n}, \\ \forall x \in \mathbb{R}, \quad & \sum_{j\in\mathbb{N}} \phi_j(x) = 1. \end{aligned}$$

Here, we choose the parameter α depending on the constants in Lemma 2.1. Moreover, we choose α to be sufficiently small, for the purpose of the next lemma, which is an analog of Lemma 3.11 of [1].

The kernel of the heat semigroup generated by Δ is denoted by p_t:

$$p_t(x) = (4\pi t)^{-1/2} \exp(-x^2/4t),$$

and P_t denotes the associated convolution operator. For the purpose of comparison, it may be noted here that P_t arises from the special case of a Schrödinger operator with zero potential.

Lemma 2.3. *Let $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ be the cover of $\mathbb{R}$ defined in Lemma 2.1 and let $\{\phi_j : j \in \mathbb{N}\}$ be a smooth resolution of identity associated with $\mathcal{I}$ and satisfying the above properties. The following estimate holds:*

$$\forall\, j \in \mathbb{N}, \quad \left\| \sup_{0<t\leq|I_j|^2} (K_t - P_t)(\phi_j f) \right\|_{L^1(\mathbb{R})} \leq C\|\phi_j f\|_{L^1(\mathbb{R})}.$$

Proof. We write $V(x) = V'(x) + V''(x)$, where $V'(x) = \mathbb{1}_{I_j^{**}}(x)V(x)$ and $V''(x) = V(x) - V'(x)$. Using the *perturbation formula*, i.e., $P_t = K_t + \int_0^t P_{t-s}VK_s ds$ (see, e.g., [1]) we obtain

$$(K_t - P_t)(\phi_j f)(x) = \int_0^t P_{t-s}V'K_s(\phi_j f)(x)ds + \int_0^t P_{t-s}V''K_s(\phi_j f)(x)ds. \tag{2.3}$$

In order to estimate the first summand in (2.3), we start by noting that

$$\int_0^t p_s(x)ds \le t^{\frac{1}{2}} \exp(-x^2/4t).$$

By the Feynman-Kac formula, which states that $k_t(x, y) \le p_t(x - y)$, we obtain that there exists a constant $\gamma > 0$, depending on the constants in Lemma 2.1, such that for $0 < t \le |I_j|^2$,

$$\begin{aligned}\int_0^{\frac{t}{2}} P_{t-s}V'K_s(\delta_y)(x)ds &\le \left(\sup_{\frac{t}{2}\le s\le t} p_s\right) * \left(V' \int_0^t P_s(\delta_y)ds\right)(x) \\ &\le Ct^{1/2}p_t * V'(x) \\ &\le C|I_j|p_{|I_j|^2} * V'(x) \\ &\le C|I_j|^{-1}\exp(-\gamma x^2/4|I_j|^2)|I_j| \int_{\mathbb{R}} V'(x)dx \\ &\le C|I_j|^{-1}\exp(-\gamma x^2/4|I_j|^2).\end{aligned} \tag{2.4}$$

In the above calulation C denotes an arbitrary positive constant. Similar to the estimation in (2.4) we may show that

$$\int_{\frac{t}{2}}^t P_{t-s}V'K_s(\delta_y)(x)ds \le t^{-1/2} \int_{\frac{t}{2}}^t P_{t-s}(V')(x)ds \le C|I_j|^{-1}\exp(-\gamma x^2/4|I_j|^2).$$

Thus, we have shown that $\int_0^t P_{t-s}V'K_s(\delta_y)(x)ds \le C|I_j|^{-1}\exp(-\gamma x^2/4|I_j|^2)$, independently of y and $t \le |I_j|^2$, which, in turn, implies that

$$\left\| \sup_{0<t\le|I_j|^2} \int_0^t P_{t-s}V'K_s(\phi_j f)(x)ds \right\|_{L^1(\mathbb{R})} \le C\|\phi_j f\|_{L^1(\mathbb{R})}.$$

The second summand in (2.3) has been estimated in [1, Lemma 3.11]. For the sake of readability, we recall that argument here. We observe that for $y \in (I_j^{**})^c$, $x \in I_j^*$, and for $0 < s \le t \le |I_j|^2$, we have

$$p_{t-s}(x - y) \le C|I_j|^{-1}\exp(-c|x - y|^2/|I_j|^2).$$

The above inequality, together with [1, Lemma 3.11], is used to estimate that

$$\left| \int_0^t P_{t-s}V''K_s(\phi_{I_j} f)(x)ds \right| \le C\frac{1}{|I_j|}\|\phi_j f\|_{L^1(\mathbb{R})}.$$

This, in turn, implies that

$$\left\| \sup_{0<t\le|I_j|^2} \left| \int_0^t P_{t-s}V''K_s(\phi_j f)(x)ds \right| \right\|_{L^1(I_j^{**})} \le C\|\phi_j f\|_{L^1(\mathbb{R})}. \qquad \square$$

Remark. We would like to note here that the estimate of V'' in [1] is obtained using an additional assumption, the so-called condition (K), which states that there exist constants $C, \epsilon > 0$ such that for all $x \in \mathbb{R}^d$, for any dyadic cube $Q \in \mathcal{Q}$, and for $t \leq diam\,(Q)^2$,

$$\int_0^{2t} (\mathbb{1}_{Q^{***}} V) * p_s(x) ds \leq C \left(\frac{t}{diam\,(Q)} \right)^{\epsilon}. \tag{2.5}$$

In the proof of Lemma 2.3 above, we see that (2.5) always holds for $V \in L^1_{loc}(\mathbb{R})$, $\epsilon = 1/2$. This may be seen, e.g., as a consequence of (2.4).

Let us now recall the main result of [1] (Theorem 2.2) in the form adapted to the presentation in this paper.

Theorem 2.4. *Let $\mathcal{I} = \{I_j : j \in \mathbb{N}\}$ be the cover of $\mathbb{R}$ derived in Lemma* 2.1 *and let $\{\phi_j : j \in \mathbb{N}\}$ be the associated resolution of identity. Assume that the following conditions are satisfied:*

(1) $\forall\, y \in I^{****}$, $\forall\, j \in \mathbb{N}$, $\int k_{2^j|I|^2}(x, y) dx \leq C 2^{-j/2}$;

(2) $\forall\, j \in \mathbb{N}$, $\left\| \sup_{0<t\leq|I_j|^2} (K_t - P_t)(\phi_j f) \right\|_{L^1(\mathbb{R})} \leq C \|\phi_j f\|_{L^1(\mathbb{R})}$.

Then, $H^1_L(\mathbb{R}) = H^1_{\mathcal{I}}(\mathbb{R})$.

Proof. Follows by direct inspection of the proof of the main theorem in [1]. □

Proof of Theorem 1.1. Combining Theorem 2.4 with Lemmas 2.1–2.3, we obtain the thesis of Theorem 1.1. □

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