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POSITIVE SOLUTIONS IN THE SENSE OF DISTRIBUTIONS OF SINGULAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. We obtain positive solutions in the sense of distributions of singular boundary value problems using perturbation and variational methods.

1. Introduction

In recent years fixed point theory and other methods have been used to obtain positive solutions of singular boundary value problems such as

$$\begin{cases} -u'' = u^{-q} + g(x,u), & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

with certain restrictions on $q > 0$ and various assumptions on the regular term g; see, e.g., Agarwal and O'Regan [1] for an extensive bibliography. In this paper we use perturbation and variational methods to obtain new existence and multiplicity results for a broad class of singular problems that includes (1.1) with no restrictions on q. Our techniques are applicable to other types of singular problems as well.

We consider the problem

$$\begin{cases} -u'' = f(x,u) + g_0(x,u) + \mu\, g_1(x,u), & 0 < x < 1, \\ u(x) > 0, & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{1.2}$$

where $f,\ g_0,\ g_1 \in C((0,1) \times (0,\infty), \mathbb{R})$ satisfy

(A_1) $\exists\, t_0 > 0$ and a nontrivial bounded function $f_0 \geq 0$ in $C(0,1)$ such that $f(x,t) \geq f_0(x),\ g_0(x,t), g_1(x,t) \geq 0$ for $t \leq t_0$,

(A_2) $\displaystyle\sup_{(x,s)\in(0,1)\times[t,\infty)} |f(x,s)|,\ \sup_{(x,s)\in(0,1)\times(0,t]} |g_1(x,s)| < \infty$ for each t,

(A_3) $|g_0(x,t)| \leq \lambda\, t + a_0$ for some constants $\lambda < \lambda_1$, the first Dirichlet eigenvalue of $-\dfrac{d^2}{dx^2}$ on $(0,1)$, and $a_0 \geq 0$

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and $\mu \geq 0$ is a small parameter.

Recall that a weak solution of (1.2) is a positive function u in the Sobolev space $H_0^1(0,1)$ satisfying

$$\int_0^1 \Big[u'(x)\,\varphi'(x) - \big(f(x,u) + g_0(x,u) + \mu\, g_1(x,u)\big)\,\varphi(x)\Big]\,dx = 0 \tag{1.3}$$

for all $\varphi \in H_0^1(0,1)$. We seek solutions in the sense of distributions: a solution of (1.2) is a function $u \in H^1_{\text{loc}}(0,1) \cap C_0[0,1]$, where $C_0[0,1]$ is the space of continuous functions on $[0,1]$ that vanish on the boundary, that is positive in $(0,1)$ and satisfies (1.3) for all test functions $\varphi \in C_0^\infty(0,1)$. Then u is in the Hölder space $C^{1,\alpha}_{\text{loc}}(0,1)$ for any $\alpha < 1$ by standard regularity arguments.

Theorem 1.1. *If* (A_1) - (A_3) *hold, then* $\exists\, \mu_0 > 0$ *such that* (1.2) *has a solution* u_1 *for each* $\mu \in [0,\mu_0)$.

Theorem 1.2. *If* $f(x,t)$ *is bounded on compact* t *intervals and nonincreasing in* t*, then* (1.2) *with* $g_0 = g_1 = 0$ *has at most one solution.*

Now we assume

- (A_4) $f(x,t)$ is nonincreasing and convex in t for $t \leq t_0$,
- (A_5) $|g_1(x,t)| \leq a_1\, t^{p-1} + a_2$ for some $p > 2$ and $a_1, a_2 \geq 0$,
- (A_6) $\exists\, t_1 > 0$ such that $0 < G_1(x,t) := \displaystyle\int_0^t g_1(x,s)\,ds \leq \frac{1}{p}\, t\, g_1(x,t)$ for $t \geq t_1$.

Theorem 1.3. *If* (A_1) - (A_6) *hold, then* $\exists\, \mu_0 > 0$ *such that* (1.2) *has two solutions* $u_1 \leq u_2$ *with* $u_2 - u_1 \in H_0^1(0,1)$ *for each* $\mu \in (0,\mu_0)$.

Example 1.4. The problem

$$\begin{cases} -u'' = e^{1/u} + \lambda\, u + \mu\, u^{p-1}, & 0 < x < 1,\\ u(x) > 0, & 0 < x < 1,\\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $p > 2$, has

- (i) a unique solution if $\lambda = \mu = 0$,
- (ii) at least one solution if $0 < \lambda < \lambda_1$ and $\mu = 0$,
- (iii) two ordered solutions if $0 \leq \lambda < \lambda_1$ and $\mu > 0$ is sufficiently small.

We will make use of the following variant of the mountain pass lemma due to Cerami [2] in getting the second solution in Theorem 1.3.

Proposition 1.5. *If* Φ *is a* C^1 *functional defined on a Banach space* H*, and* $\exists\, v_0, v_1 \in H$ *such that*

$$c := \inf_{\gamma \in \Gamma}\ \max_{v \in \gamma([0,1])}\ \Phi(v) > \Phi(v_0), \Phi(v_1) \tag{1.5}$$

where

$$\Gamma := \Big\{\gamma \in C([0,1],H) : \gamma(0) = v_0,\ \gamma(1) = v_1\Big\} \tag{1.6}$$

is the class of paths in H *joining* v_0 *and* v_1*, then there is a sequence* $(v_j) \subset H$ *such that*

$$|\Phi(v_j) - c| \to 0, \qquad (1 + \|v_j\|)\,\|\Phi'(v_j)\| \to 0. \tag{1.7}$$

2. Preliminaries

Consider

$$\begin{cases} -u'' = h(x,u), & 0 < x < 1, \\ u(x) > 0, & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{2.1}$$

where $h = f + g$ and $f,\, g \in C((0,1) \times (0,\infty), \mathbb{R})$ satisfy

$$\sup_{(x,s)\in(0,1)\times[t_1,t_2]} |f(x,s)|, \quad \sup_{(x,s)\in(0,1)\times(0,t_2]} |g(x,s)| < \infty \quad \forall\, 0 < t_1 \le t_2 < \infty. \tag{2.2}$$

We approximate (2.1) with the sequence of regular problems

$$\begin{cases} -u'' = f_j(x,u) + g(x,u), & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{2.3}$$

where $f_j(x,t) = f(x, \max\{t, \varepsilon_j\})$ and (ε_j) is a sequence of positive numbers decreasing to zero.

Proposition 2.1. *If* $(u_j) \subset H_0^1(0,1)$ *is a sequence of weak solutions of* (2.3) *such that*

$$\varepsilon_\delta := \inf_j \min_{[\delta,1-\delta]} u_j > 0 \quad \forall\, 0 < \delta \le 1/2, \tag{2.4}$$

$$M := \sup_j \max_{[0,1]} u_j < \infty, \tag{2.5}$$

then a subsequence converges pointwise to a solution u_1 *of* (2.1).

Proof. Take a sequence (δ_k) of positive numbers decreasing to zero. For all j so large that $\varepsilon_j < \varepsilon_{\delta_1}$, taking $(u_j - \varepsilon_{\delta_1})^+$ as the test function in (2.3) gives

$$\int_{\left\{u_j(x) > \varepsilon_{\delta_1}\right\}} u_j'(x)^2\, dx = \int_{\left\{u_j(x) > \varepsilon_{\delta_1}\right\}} h(x,u_j)\left(u_j(x) - \varepsilon_{\delta_1}\right) dx. \tag{2.6}$$

Since $u_j \ge \varepsilon_{\delta_1}$ on $[\delta_1, 1-\delta_1]$ by (2.4) and the right side is bounded by (2.2) and (2.5), (u_j) is bounded in $H^1(\delta_1, 1-\delta_1)$ and hence a subsequence $(u_{1,j})$ converges to some u^1 weakly in $H^1(\delta_1, 1-\delta_1)$ and strongly in $C[\delta_1, 1-\delta_1]$. Repeating with further and further subsequences, for each k we get a subsequence (u_{kj}) that converges to some u^k weakly in $H^1(\delta_k, 1-\delta_k)$ and strongly in $C[\delta_k, 1-\delta_k]$ with $(u_{k+1,j}) \subset (u_{kj})$. Then $u^{k+1}|_{[\delta_k, 1-\delta_k]} = u^k$, so

$$u_1 := \begin{cases} u^1 & \text{on } [\delta_1, 1-\delta_1], \\ u^{k+1} & \text{on } [\delta_{k+1}, 1-\delta_{k+1}] \setminus [\delta_k, 1-\delta_k] \text{ for each } k \end{cases} \tag{2.7}$$

is a well-defined positive function in $H^1_{\text{loc}}(0,1) \cap C(0,1)$, to which the diagonal subsequence (u_{kk}) converges pointwise.

To see that $u_1 \in C_0[0,1]$, let $0 < \varepsilon < M$, $M_\varepsilon = \sup h((0,1) \times [\varepsilon, M]) > 0$, and $\varphi_\varepsilon > 0$ be the solution of

$$\begin{cases} -\varphi_\varepsilon'' = M_\varepsilon, & 0 < x < 1, \\ \varphi_\varepsilon(0) = \varphi_\varepsilon(1) = 0. \end{cases} \tag{2.8}$$

For all k so large that $\varepsilon_{kk} < \varepsilon$, taking $\varphi = (u_{kk} - \varepsilon - \varphi_\varepsilon)^+$ as the test function in

$$-u''_{kk} = f_{kk}(x, u_{kk}) + g(x, u_{kk}) \tag{2.9}$$

gives

$$\begin{aligned} \int_0^1 u'_{kk}(x)\, \varphi'(x)\, dx &= \int_0^1 h(x, u_{kk})\, \varphi(x)\, dx \\ &\le \int_0^1 M_\varepsilon\, \varphi(x)\, dx = \int_0^1 \varphi'_\varepsilon(x)\, \varphi'(x)\, dx \end{aligned} \tag{2.10}$$

by (2.8), so

$$\int_{\{u_{kk}(x) > \varepsilon + \varphi_\varepsilon(x)\}} \big(u'_{kk}(x) - \varphi'_\varepsilon(x)\big)^2\, dx \le 0 \tag{2.11}$$

and hence $u_{kk} \le \varepsilon + \varphi_\varepsilon$. Thus $0 < u_1 \le \varepsilon + \varphi_\varepsilon$, which implies that $u_1(x) \to 0$ as $x \to 0, 1$ since $\varphi_\varepsilon(x) \to 0$ as $x \to 0, 1$ and ε is arbitrary.

For any $\varphi \in C_0^\infty(0, 1)$,

$$\int_{\delta_k}^{1-\delta_k} \Big[u'_{kj}(x)\, \varphi'(x) - h(x, u_{kj})\, \varphi(x)\Big]\, dx = 0 \tag{2.12}$$

for a fixed k so large that $[\delta_k, 1 - \delta_k] \supset \operatorname{supp} \varphi$ and all j so large that $\varepsilon_{kj} < \varepsilon_{\delta_k}$, and passing to the limit gives

$$\int_{\delta_k}^{1-\delta_k} \Big[(u^k)'(x)\, \varphi'(x) - h(x, u^k)\, \varphi(x)\Big]\, dx = 0, \tag{2.13}$$

or

$$\int_0^1 \Big[u'_1(x)\, \varphi'(x) - h(x, u_1)\, \varphi(x)\Big]\, dx = 0 \tag{2.14}$$

since $u^k = u_1|_{[\delta_k, 1-\delta_k]}$ and $\varphi = 0$ outside $[\delta_k, 1 - \delta_k]$. □

Proposition 2.2. *If* (2.3) *has a sequence of weak sub- and supersolution pairs* $\underline{u}_j \le \overline{u}_j$ *in* $H_0^1(0, 1)$ *such that*

$$\inf_j \min_{[\delta, 1-\delta]} \underline{u}_j > 0 \quad \forall\, 0 < \delta \le 1/2, \qquad \sup_j \max_{[0,1]} \overline{u}_j < \infty, \tag{2.15}$$

then (2.1) *has a solution* u_1.

Proof. By a standard argument (2.3) has a weak solution $u_j \in H_0^1(0, 1)$ in the order interval $[\underline{u}_j, \overline{u}_j]$, and the conclusion follows from Proposition 2.1. □

Now we assume that $f(x, t)$ is nonincreasing in t. Given a solution u_1 of (2.1) we seek a second solution of the form $u_2 = u_1 + v$ where $v \ge 0$ is then a solution of

$$\begin{cases} -v'' = h(x, u_1 + v) - h(x, u_1), \quad 0 < x < 1, \\ v(0) = v(1) = 0. \end{cases} \tag{2.16}$$

Let (δ_j) be a sequence of positive numbers decreasing to zero and consider the approximating sequence of regular problems

$$\begin{cases} -v'' = h(x, u_1 + v^+) - h(x, u_1), \quad \delta_j < x < 1 - \delta_j, \\ v(\delta_j) = v(1 - \delta_j) = 0. \end{cases} \tag{2.17}$$

Solutions of (2.17) are nonnegative by the maximum principle and coincide with the critical points of the C^1 functional

$$\Phi_j(v) = \int_{\delta_j}^{1-\delta_j} \left[\frac{1}{2}\, v'(x)^2 - \int_0^{v(x)^+} \big(h(x, u_1 + s) - h(x, u_1)\big)\, ds \right] dx \tag{2.18}$$

defined on $H^1_0(\delta_j, 1-\delta_j)$, which we view as a subspace of $H^1_0(0,1)$ by setting each $v = 0$ outside $(\delta_j, 1-\delta_j)$.

Proposition 2.3. *If* (v_j), $v_j \in H^1_0(\delta_j, 1-\delta_j)$ *is a bounded sequence in* $H^1_0(0,1)$ *such that*

$$\inf_j \Phi_j(v_j) > 0, \qquad \|\Phi_j'(v_j)\| \to 0 \tag{2.19}$$

($\|\cdot\|$ *is the norm in* $H^{-1}(\delta_j, 1-\delta_j)$), *then a subsequence converges pointwise to a nontrivial function* $v \geq 0$ *in* $H^1_0(0,1)$ *such that* $u_2 = u_1 + v$ *is a solution of* (2.1).

Proof. Since (v_j) is bounded in $H^1_0(0,1)$, a subsequence converges to some v weakly in $H^1_0(0,1)$ and strongly in $C_0[0,1]$. Since

$$\|v_j^-\|^2 = \Phi_j'(v_j)\, v_j^- \to 0 \tag{2.20}$$

by (2.19), $v \geq 0$ and hence $u_2 \geq u_1 > 0$ in $(0,1)$.

For any $\varphi \in C_0^\infty(0,1)$ and all j so large that $[\delta_j, 1-\delta_j] \supset \operatorname{supp}\varphi$,

$$\Phi_j'(v_j)\, \varphi = \int_0^1 \Big[v_j'(x)\, \varphi'(x) - \big(h(x, u_1 + v_j^+) - h(x, u_1)\big)\, \varphi(x) \Big]\, dx, \tag{2.21}$$

and passing to the limit and adding

$$\int_0^1 \Big[u_1'(x)\, \varphi'(x) - h(x, u_1)\, \varphi(x) \Big]\, dx = 0 \tag{2.22}$$

gives

$$\int_0^1 \Big[u_2'(x)\, \varphi'(x) - h(x, u_2)\, \varphi(x) \Big]\, dx = 0. \tag{2.23}$$

Since $f(x,t)$ is nonincreasing in t,

$$\begin{aligned} &\int_0^1 \left[\int_0^{v_j(x)^+} \big(f(x, u_1 + s) - f(x, u_1)\big)\, ds \right] dx \\ &\geq \int_0^1 \big(f(x, u_1 + v_j^+) - f(x, u_1)\big)\, v_j(x)^+\, dx \\ &= \|v_j^+\|^2 - \int_0^1 \big(g(x, u_1 + v_j^+) - g(x, u_1)\big)\, v_j(x)^+\, dx - \Phi_j'(v_j)\, v_j^+ \end{aligned} \tag{2.24}$$

and hence

$$\begin{aligned} \Phi_j(v_j) \leq \frac{1}{2}\, \|v_j^-\|^2 - \int_0^1 \bigg[&\int_0^{v_j(x)^+} \big(g(x, u_1 + s) - g(x, u_1)\big)\, ds \\ &- \big(g(x, u_1 + v_j^+) - g(x, u_1)\big)\, v_j(x)^+ \bigg]\, dx + \Phi_j'(v_j)\, v_j^+. \end{aligned} \tag{2.25}$$

The right side goes to zero by (2.2) and (2.20) if $v = 0$, contrary to (2.19). □

3. Proofs

Proof of Theorem 1.1. We apply Proposition 2.2 with $g = g_0 + \mu\, g_1$. Let t_0 and f_0 be as in (A_1) and let $0 < \varepsilon \le 1$ be so small that the solution $\underline{u} > 0$ in $H^1_0(0,1)$ of

$$\begin{cases} -u'' = \varepsilon\, f_0(x), & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

is $\le t_0$. Then

$$-\underline{u}'' \le f_0(x) \le f_j(x, \underline{u}) + g(x, \underline{u}) \tag{3.2}$$

for all j so large that $\varepsilon_j < t_0$ and $\mu \ge 0$.

By (A_2) and (A_3),

$$\begin{cases} -u'' = f_j(x, u) + g_0(x, u) + 1, & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{3.3}$$

has a solution $\overline{u}_j \in H^1_0(0,1)$. By (A_1) and the maximum principle, $\overline{u}_j \ge \underline{u}$. Taking $(\overline{u}_j - t_0)^+$ as the test function in (3.3) gives

$$\|(\overline{u}_j - t_0)^+\|^2 \le \int_{\{\overline{u}_j(x) > t_0\}} \big(\lambda\, \overline{u}_j(x) + a_3\big)\big(\overline{u}_j(x) - t_0\big)\, dx \tag{3.4}$$

for some $a_3 \ge 0$, so $((\overline{u}_j - t_0)^+)$ is bounded in $H^1_0(0,1)$ and hence $(\overline{u}_j)$ is bounded in $C_0[0,1]$. Then

$$-\overline{u}_j'' \ge f_j(x, \overline{u}_j) + g_0(x, \overline{u}_j) + \mu\, g_1(x, \overline{u}_j) \tag{3.5}$$

for all sufficiently small $\mu \ge 0$. □

Proof of Theorem 1.2. Suppose u_1 and u_2 are both solutions. For any $\varepsilon > 0$, taking $u = u_1, u_2$ with $\varphi = (u_1 - u_2 - \varepsilon)^+$ in (1.3) and subtracting gives

$$\begin{aligned} &\int_{\{u_1(x) > u_2(x) + \varepsilon\}} \big(u_1'(x) - u_2'(x)\big)^2\, dx \\ &= \int_{\{u_1(x) > u_2(x) + \varepsilon\}} \big(f(x, u_1) - f(x, u_2)\big)\big(u_1(x) - u_2(x) - \varepsilon\big)\, dx \le 0 \end{aligned} \tag{3.6}$$

since $f(x,t)$ is nonincreasing in t, so $u_1 \le u_2 + \varepsilon$. Since ε is arbitrary, $u_1 \le u_2$, and the reverse inequality follows similarly. □

Proof of Theorem 1.3. We apply Proposition 1.5 to Φ_j and use Proposition 2.3 to get u_2. By (A_2) and (A_4), we may assume that $f(x,t) \ge 0$ and nonincreasing and convex in t for all t by replacing $f(x,t)$ and $g_0(x,t)$ with $f(x,t_0)$ and $g_0(x,t) + f(x,t) - f(x,t_0)$ for $t > t_0$, respectively.

Since f is nonincreasing in t,

$$\int_0^{v(x)^+} \big(f(x, u_1 + s) - f(x, u_1)\big)\, ds \le 0, \tag{3.7}$$

and hence

$$\Phi_j(v) \ge \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|v\|^2 - a_4\left(1 + \mu\, \|v\|^{p-1}\right)\|v\| \quad \forall v \tag{3.8}$$

for some $a_4 \geq 0$ by (A_3) and (A_5). So $\exists\, c_0, R, \mu_0 > 0$ such that

$$\inf_{v \in H_0^1(\delta_j, 1-\delta_j),\, \|v\|=R} \Phi_j(v) \geq c_0 \quad \forall \mu \in (0, \mu_0). \tag{3.9}$$

Since $f(x,t) \geq 0$,

$$\int_0^1 \left[\int_0^{v(x)^+} \big(f(x, u_1 + s) - f(x, u_1)\big)\, ds \right] dx \geq - \int_0^1 f(x, u_1)\, v(x)^+\, dx, \tag{3.10}$$

and integrating (A_6) gives

$$g_1(x,t) \geq a_5\, t^{p-1}, \quad t \geq t_1, \tag{3.11}$$

for some $a_5 > 0$, so for each $\mu > 0$, $\exists\, v_1 > 0$ in $H_0^1(\delta_1, 1-\delta_1)$, $\|v_1\| > R$ such that

$$\Phi_j(v_1) = \Phi_1(v_1) \leq 0 \quad \forall j. \tag{3.12}$$

Noting that $\Phi_j(0) = 0$ and setting

$$c_j := \inf_{\gamma \in \Gamma_j} \max_{v \in \gamma([0,1])} \Phi_j(v) \geq c_0 \tag{3.13}$$

where

$$\Gamma_j := \Big\{ \gamma \in C([0,1], H_0^1(\delta_j, 1-\delta_j)) : \gamma(0) = 0,\ \gamma(1) = v_1 \Big\}, \tag{3.14}$$

Proposition 1.5 now gives a $v_j \in H_0^1(\delta_j, 1-\delta_j)$ such that

$$|\Phi_j(v_j) - c_j| \to 0, \qquad (1 + \|v_j\|)\, \|\Phi_j'(v_j)\| \to 0. \tag{3.15}$$

Since $H_0^1(\delta_j, 1-\delta_j) \subset H_0^1(\delta_{j+1}, 1-\delta_{j+1})$, $\Gamma_j \subset \Gamma_{j+1}$ and hence $c_j \geq c_{j+1}$, so

$$\frac{c_0}{2} \leq \Phi_j(v_j) \leq 2\, c_1 \tag{3.16}$$

for all sufficiently large j.

We have

$$\begin{aligned} \Phi_j(v_j) - \frac{1}{2}\, \Phi_j'(v_j)\, v_j^+ = \frac{1}{2}\, \|v_j^-\|^2 + \int_0^1 \bigg[& \frac{1}{2} \big(h(x, u_1 + v_j^+) + h(x, u_1)\big)\, v_j(x)^+ \\ & - \int_{u_1(x)}^{u_1(x) + v_j(x)^+} h(x,s)\, ds \bigg] dx \end{aligned} \tag{3.17}$$

where $h = f + g$. Since $f(x,t)$ is convex in t,

$$\frac{1}{2} \big(f(x, u_1 + v_j^+) + f(x, u_1)\big)\, v_j^+ - \int_{u_1}^{u_1 + v_j^+} f(x,s)\, ds \geq 0, \tag{3.18}$$

and (A_3), (A_5), (A_6), and (3.11) imply that the integrals of the corresponding expressions for g_0 and g_1 are bounded from below by $-a_6 \big(\|v_j^+\|_{L^2(0,1)}^2 + 1\big)$ and $a_7\, \|v_j^+\|_{L^p(0,1)}^p - a_8$ for some $a_6, a_8 \geq 0$, $a_7 > 0$, respectively. So $\big(v_j^+\big)$ is bounded in $L^p(0,1)$, and hence it follows from

$$\frac{1}{2}\, \|v_j\|^2 \leq \Phi_j(v_j) + \int_0^1 \left[\int_0^{v_j(x)^+} \big(g(x, u_1 + s) - g(x, u_1)\big)\, ds \right] dx \tag{3.19}$$

that (v_j) is bounded in $H_0^1(0,1)$. The conclusion follows since (2.19) holds by (3.15) and (3.16). ∎

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