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NOTE ON A REMARKABLE SUPERPOSITION FOR A NONLINEAR EQUATION

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Abstract. We give a simple proof of—and extend—a superposition principle for the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq 0$, discovered by Crandall and Zhang. An integral representation comes as a byproduct. It follows that a class of Riesz potentials is p-superharmonic.

1. Introduction

The Newtonian potentials

$$V(x) = c_n \int \frac{\rho(y)dy}{|x-y|^{n-2}}, \quad \rho \geq 0,$$

are important examples of superharmonic functions in the n-dimensional Euclidean space, $n \geq 3$. They are obtained through a superposition of fundamental solutions

$$\frac{A_j}{|x-y_j|^{n-2}}, \quad A_j \geq 0,$$

of the Laplace equation. The equation $\Delta V(x) = -\rho(x)$ holds.

For the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

it was recently discovered by M. Crandall and J. Zhang that a similar superposition of fundamental solutions is possible. Indeed, they proved in [CZ] that sums like

$$\sum A_j |x-a_j|^{\frac{p-n}{p-1}} \quad (2 < p < n)$$

are p-superharmonic functions, where $A_j \geq 0$. They also included exponents other than the natural $(p-n)/(p-1)$ and allowed p to vary between 1 and ∞. The Riesz potentials

$$\int \frac{\rho(y)\, dy}{|x-y|^{(n-p)/(p-1)}}$$

appear as the limit of such sums.

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The purpose of our note is to give an alternative proof of the following theorem for the Riesz potentials

$$V_\alpha(x) = \int |x-y|^\alpha \rho(y)dy.$$

Theorem. *Let $\rho \in C_0(\mathbb{R}^n)$, $n \geq 2$, be a nonnegative function. We have three cases depending on p:*

(i) $\mathbf{2 < p < n}$. *The function V_α is p-superharmonic, if*

$$\frac{p-n}{p-1} \leq \alpha < 0.$$

(ii) $\mathbf{p > n}$. *The function V_α is p-subharmonic, if*

$$\alpha \geq \frac{p-n}{p-1}.$$

If $p = \infty$, we may take $\alpha \geq 1$.

(iii) $\mathbf{p = n}$. *The function*

$$V_0(x) = \int \log(|x-y|)\rho(y)dy$$

is n-subharmonic.

Before proceeding, we make a comment about the case $1 < p < 2$, which exhibits a puzzling behaviour. While the fundamental solution

$$|x-a|^{\frac{p-n}{p-1}}$$

is p-superharmonic in the whole $\mathbb{R}^n$, the sum

$$|x-a|^{\frac{p-n}{p-1}} + |x-b|^{\frac{p-n}{p-1}}$$

is not, assuming of course that $a \neq b$. The sum is p-subharmonic when $x \neq a$ and $x \neq b$, but it is not p-subharmonic in the whole $\mathbb{R}^n$. A p-subharmonic function cannot take the value $+\infty$ in its domain of definition because of the comparison principle. This was about $p < 2$.

We recall from [L] that p-superharmonic functions are defined as lower semicontinuous functions $v : \mathbb{R}^n \longrightarrow (0, \infty]$ that obey the comparison principle with respect to the p-harmonic functions. A more direct characterization is available for smooth functions. When $p \geq 2$ the function $v \in C^2(\mathbb{R}^n)$ is p-superharmonic if and only if

$$-\operatorname{div}(|\nabla v(x)|^{p-2}\nabla v(x)) \geq 0$$

at each point x. From the identity

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = |\nabla v|^{p-4}\left\{|\nabla v|^2 \Delta v + (p-2)\Delta_\infty v\right\},$$

where

$$\Delta_\infty v = \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}$$

is the ∞-Laplacian operator, we can read off that the pointwise inequality

$$|\nabla v|^2 \Delta v + (p-2)\Delta_\infty v \leq 0$$

is an equivalent characterization of p-superharmonic functions v in $C^2(\mathbb{R}^n)$. (Incidentally, this is also valid in the case $1 < p < 2$. See [JLM].)

Thus we have a practical definition for functions of class C^2. The polar set $\Xi = \{x \colon v(x) = +\infty\}$ can be exempted if v is lower semicontinuous in $\mathbb{R}^n$ and

$v \in C^2(\mathbb{R}^n \setminus \Xi)$. An example is the fundamental solution $|x-a|^{(p-n)/(p-1)}, 1 < p < n$, where the point $x = a$ is exempted.

Finally, we mention that in Section 3 we need the concept of *weak p-supersolution.* We say that $v \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ is a weak p-supersolution if

$$\int_{\mathbb{R}^n} \left\langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \right\rangle \, dx \geq 0$$

holds for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^n)$. In [L] it was established that *locally bounded* p-superharmonic functions are weak p-supersolutions. On the other hand, lower semicontinuous weak p-supersolutions are p-superharmonic functions.

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2. Proof of the Theorem

We assume $n \geq 2$. The following calculations are formal, but are easy to justify because $\alpha > 2 - n$. Notice that we have

$$\alpha \geq \frac{p-n}{p-1} > 2 - n \quad \text{when} \quad p > 2.$$

Differentiating

$$V(x) = \int |x-y|^\alpha \rho(y) dy$$

under the integral sign we obtain

$$\frac{\partial V}{\partial x_i} = \alpha \int |x-y|^{\alpha-2} (x_i - y_i) \rho(y) dy$$

and

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \alpha(\alpha-2) \int |x-y|^{\alpha-4} (x_i-y_i)(x_j-y_j) \rho(y) dy$$
$$+ \alpha \delta_{ij} \int |x-y|^{\alpha-2} \rho(y) dy.$$

Aiming at $\Delta_\infty V$, we write the product of the integrals in

$$\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j}$$

as a triple integral in disjoint variables.* This yields the formula

$$\Delta_\infty V(x) = \alpha^3 \int |x-c|^{\alpha-2} \rho(c) \, dc \left| \int |x-a|^{\alpha-2} (x-a) \rho(a) \, da \right|^2$$
$$+ \alpha^3 (\alpha - 2) \int |x-c|^{\alpha-2} \rho(c) \left\langle \frac{x-c}{|x-c|}, \int |x-a|^{\alpha-2} (x-a) \rho(a) \, da \right\rangle^2 dc$$

in vector notation. Keeping α within the prescribed range we have only harmless singularities.

*The principle is clear from the example

$$\left(\int e^x \, dx \right)^2 \int e^{2x} \, dx = \iiint e^{a+b+2c} \, da \, db \, dc.$$

By the Cauchy-Schwarz inequality we have

$$\left|\left\langle \frac{x-c}{|x-c|}, \int |x-a|^{\alpha-2}(x-a)\rho(a)\,da \right\rangle\right| \leq \left|\int |x-a|^{\alpha-2}(x-a)\rho(a)\,da\right|.$$

In easily understandable notation we can therefore write the above formula as

$$\Delta_\infty V(x) = \alpha^3 A(x) + \alpha^3(\alpha-2)B(x)$$

where

$$0 \leq B(x) \leq A(x).$$

From this we can read off that $\Delta_\infty V(x) \geq 0$ when $\alpha \geq 1$. This settles the case $p = \infty$. There is a more succinct representation. Lagrange's identity

$$|X \wedge Y|^2 = \frac{1}{2}\sum (X_i Y_j - X_j Y_i)^2 = |X|^2|Y|^2 - \langle X, Y\rangle^2$$

for vectors enables us to write

$$\begin{aligned} C(x) &= A(x) - B(x) \\ &= \int |x-c|^{\alpha-2}\rho(c)\left|\frac{x-c}{|x-c|} \wedge \int |x-a|^{\alpha-2}(x-a)\rho(a)\,da\right|^2 dc. \end{aligned}$$

Notice that $C(x) \geq 0$. Thus we have arrived at the representation formula

$$\boldsymbol{\Delta_\infty \mathbf{V}(\mathbf{x}) = \alpha^3 \mathbf{C}(\mathbf{x}) + \alpha^3(\alpha - 1)\mathbf{B}(\mathbf{x})},$$

which is particularly appealing for $\alpha = 1$ and

$$V(x) = \int |x-y|\rho(y)dy.$$

Continuing the calculations, we find that

$$\Delta V(x) = \alpha(\alpha - 2 + n)\int |x-c|^{\alpha-2}\rho(c)\,dc,$$

and hence, after some simple manipulations

$$|\nabla V|^2 \Delta V = \alpha^3(\alpha - 2 + n)A.$$

It follows that

$$\begin{aligned} |\nabla V|^2\Delta V \quad &+ \quad (p-2)\Delta_\infty V \\ &= \quad \alpha^3(n+\alpha+p-4)A(x) + \alpha^3(\alpha-2)(p-2)B(x) \\ &= \quad \alpha^3\left[(2-\alpha)(p-2)C(x) + (n-p+\alpha(p-1))B(x)\right]. \end{aligned}$$

In this formula we have command over the sign of

$$|\nabla V|^2\Delta V + (p-2)\Delta_\infty V,$$

at least in the cases needed for the theorem. We may add that the logarithmic integral in the borderline case $p = n$ requires a separate, but similar, calculation leading to

$$\begin{gathered} |\nabla V_0|^2\Delta V_0 + (n-2)\Delta_\infty V_0 \\ = 2(n-2)C(x) \end{gathered}$$

where $\alpha = 0$ in the expression for $C(x)$. This concludes our proof of the Theorem.

It is remarkable that the factor $n - p + \alpha(p-1)$ in front of $B(x)$ reveals the natural exponent $\alpha = (p-n)/(p-1)$; the term vanishes for this α. Thus

$$|\nabla \mathbf{V}|^{\mathbf{2}} \mathbf{\Delta V} + (\mathbf{p}-\mathbf{2})\mathbf{\Delta}_\infty \mathbf{V} = \alpha^{\mathbf{3}}(\mathbf{2}-\alpha)(\mathbf{p}-\mathbf{2})\mathbf{C}(\mathbf{x})$$

when $\alpha = (p-n)/(p-1)$, and $p > 2$.

To this one may add that the method is rather flexible. For example, in the case of a variable exponent it works for

$$\int |x-y|^{\alpha(y)} \rho(y) dy.$$

One can also consider $V(x) + \langle a, x\rangle$ with an extra linear term.

3. Riesz potentials

So far, we have assumed that the nonnegative density ρ in the Riesz potential

$$V(x) = \int |x-y|^\alpha \rho(y) dy$$

is smooth. The restriction can easily be relieved because of the following theorem: the pointwise limit of an increasing sequence of p-superharmonic functions is either a p-superharmonic function or identically $+\infty$. Thus we immediately reach the case with lower semicontinuous ρ's. We point out that the discrete case

$$\sum A_j |x - a_j|^\alpha$$

follows if one regards the integrals as sums in disguise and takes into account a special reasoning concerning the poles a_j.

We can do more than that. Indeed, we can allow rather general measures.

Proposition. *Let μ be a Radon measure on $\mathbb{R}^n$ satisfying the growth condition*

$$\int_{|y|\geq 1} |y|^\alpha \, d\mu(y) < \infty.$$

The theorem holds for the Riesz potentials

$$V(x) = \int |x-y|^\alpha d\mu(y).$$

In other words, we have replaced $\rho(y)dy$ with $d\mu(y)$. The growth condition guarantees that $V(x) < \infty$ almost everywhere. In fact $V(x) \equiv \infty$ if $\int |y|^\alpha d\mu(y) = +\infty$. See [P, Theorem 3.4, p. 78] about this.

Proof. Because of the increasing limit

$$V(x) = \lim_{R\to\infty} \int_{|y|<R} |x-y|^\alpha \, d\mu(y)$$

we may, in the proof, assume that the measure μ has compact support. To simplify the exposition, we confine ourselves to the case

$$\alpha = \frac{p-n}{p-1}, \quad 2 < p < n.$$

The passage from integrals of the type $\int |x-y|^\alpha \rho(y)dy$ to the more general kind with the Radon measure is accomplished through a regularization, for example

$$\rho_t(y) = \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{|y-\xi|^2}{4t}} d\mu(\xi),$$

where the heat kernel is present. We have

$$\int \rho_t(y)dy = \int d\mu(\xi) = \mu(\mathbb{R}^n) = M.$$

Let us denote

$$V_k(x) = \int |x-y|^\alpha \rho_{t_k}(y)dy \qquad (k = 1, 2, 3, \cdots)$$

where $t_k \longrightarrow 0+$ as $k \longrightarrow \infty$. According to the theorem each V_k is p-superharmonic. It is not difficult to see that $V_k \longrightarrow V$ a.e., at least for a subsequence. The Proposition follows from the general theorem about a.e. convergence in [KM, Theorem 1.17], which assures that the limit fuction is p-superharmonic.

In the present situation a more direct proof is possible. It is based on a compactness argument in $W^{1,p-1}_{\text{loc}}(\mathbb{R}^n)$.

Alternative proof. A direct calculation yields

$$\int_{B_R} |\nabla V_k|^{p-1} dx \le 2\left(M\frac{n-p}{n-1}\right)^{p-1} \omega_{n-1} R, \quad k = 1, 2, 3, \cdots.$$

To obtain such a bound, free of k, we proceed as follows:

$$\begin{aligned}
|\nabla V_k(x)| &\le |\alpha| \int |x-y|^{\alpha-1} \rho_k(y)dy, \\
|\nabla V_k(x)|^{p-1} &\le |\alpha|^{p-1} \int |x-y|^{(\alpha-1)(p-1)} \rho_k(y)dy \left(\int \rho_k(y)dy\right)^{p-2} \\
&= |\alpha|^{p-1} M^{p-2} \int |x-y|^{1-n} \rho_k(y)dy, \\
\int_{B_R} |\nabla V_k(x)|^{p-1} dx &\le |\alpha|^{p-1} M^{p-2} \int \rho_k(y) \int_{B_R} |x-y|^{1-n} dx\, dy.
\end{aligned}$$

The inner integral can be estimated as

$$\int_{B_R} |x-y|^{1-n} dx \le 2R\omega_{n-1}$$

since $y \in B_R$. This yields the desired bound.

According to the celebrated Banach-Saks theorem there exists a sequence of indices $k_1 < k_2 < \cdots$ such that for the arithmetic means

$$W_j = \frac{V_{k_1} + V_{k_2} + \cdots + V_{k_j}}{j}$$

we have that $\nabla W_j \longrightarrow \nabla V$ strongly in $L^{p-1}(B_R)$. Now we take advantage of the linear structure by concluding that each W_j is a p-superharmonic function, because it can be written as a Riesz potential with the density $(\rho_{k_1} + \cdots + \rho_{k_j})/j$. Hence

$$\int \langle |\nabla W_j|^{p-2} \nabla W_j,\ \nabla\varphi\rangle\, dx \ge 0$$

for each nonnegative test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Given φ, we take a ball B_R containing its support. The strong convergence of the sequence $\{\nabla W_j\}$ in $L^{p-1}(B_R)$ and the elementary vector inequality

$$\left| |b|^{p-2}b - |a|^{p-2}a \right| \le (p-1)|b-a|(|b|+|a|)^{p-2}, \ \ p \ge 2,$$

enable us to pass to the limit under the integral sign so that also

$$\int \langle |\nabla V|^{p-2}\nabla V, \ \nabla\varphi\rangle dx \ge 0.$$

We could conclude that V is a weak supersolution if we knew that V belongs to $W^{1,p}_{\text{loc}}(\mathbb{R})$. Unfortunately, this is not always the case. For example, the fundamental solution is not in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$. A simple correction is required. Also the cut-off functions*

$$W_j^L = \min\{W_j(x), L\}$$

are weak p-supersolutions. The ordinary Caccioppoli estimate

$$\int \zeta^p |\nabla W_j^L|^p dx \le p^p L^p \int |\nabla\zeta|^p dx$$

is available; cf. [L, Corollary 2.5]. By weak lower semicontinuity it also holds for

$$V^L = \min\{V(x), L\}.$$

Therefore, $\nabla V^L \in L^p_{\text{loc}}(\mathbb{R}^n)$, so that V^L is in the right Sobolev space.

As before, we can conclude that

$$\int \langle |\nabla V^L|^{p-2}\nabla V^L, \ \nabla\varphi\rangle dx \ge 0,$$

but this time it follows that V^L is a weak p-supersolution. Then the increasing limit $V = \lim_{L\to\infty} V^L$ is p-superharmonic.

Strictly speaking, the conclusion is that the function

$$\tilde{V}(x) = \operatorname{ess}\liminf_{y\to x} \ V(y)$$

is p-superharmonic, because it is the increasing limit of the p-superharmonic functions $\operatorname{ess}\liminf \ V^L(y)$ as $L \longrightarrow \infty$. See [KM, Proposition 1.7]. The Lemma below concludes our proof. $\square$

Lemma (Brelot)**.** *At each point x_0 we have*

$$V(x_0) = \tilde{V}(x_0) = ess\liminf_{x\to x_0} V(x),$$

when $2-n < \alpha < 0$.

For the sake of completeness we present the proof.

Proof. The function $|x-y|^\alpha$ is superharmonic and therefore we have

$$\fint_{B(x_0,r)} |x-y|^\alpha dx \le |x_0-y|^\alpha$$

*This proof does not work if one cuts the original functions V_j instead.

for the volume average over the ball $B(x_0, r)$. It follows that

$$\begin{aligned}
\fint_{B(x_0,r)} V(x)dx &= \fint_{B(x_0,r)} \int |x-y|^\alpha d\mu(y)dx \\
&= \int \left(\fint_{B(x_0,r)} |x-y|^\alpha dx \right) d\mu(y) \\
&\le \int |x_0 - y|^\alpha d\mu(y) = V(x_0).
\end{aligned}$$

This is merely a restatement of the fact that V is a superharmonic function (in the ordinary sense).

It follows from Fatou's lemma that V is lower semicontinuous. Hence

$$\begin{aligned}
V(x_0) &\le \liminf_{x \to x_0} V(x) \le \operatorname{ess} \liminf_{x \to x_0} V(x) \\
&\le \liminf_{r \to 0} \fint_{B(x_0,r)} V(x)dx \le V(x_0)
\end{aligned}$$

where the last inequality was proved above. Thus equality holds at each step. □

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