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COMPLEX MONGE-AMPÈRE OF A MAXIMUM

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ABSTRACT. We derive a formula for $(dd^c u)^n$ where $u = \max_j u_j$ is a finite maximum. As an application, we compute the complex equilibrium measures of some generalized polyhedra.

Plurisubharmonic (psh) functions play a primary role in pluripotential theory. They are closely related to the operator $dd^c = 2i\partial\bar{\partial}$ (with notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$), which serves as a generalization of the Laplacian from $\mathbb{C}$ to $\mathbb{C}^{\dim}$ for $\dim > 1$. If u is smooth of class C^2, then for $1 \leq n \leq \dim$, the coefficients of the exterior power $(dd^c u)^n$ are given by the $n \times n$ subdeterminants of the matrix $(\partial^2 u/\partial z_i \partial \bar{z}_j)$. The top exterior power corresponds to $n = \dim$, and in this case we have the determinant of the full matrix, which gives the complex Monge-Ampère operator. The extension of the (nonlinear) operator $(dd^c)^n$ to nonsmooth functions has been studied by several authors (see, for instance, [3]). Here it will suffice to define $(dd^c)^n$ on psh functions which are continuous.

For an open set $E \subset \mathbb{C}^{\dim}$, the set of psh functions on E forms a cone which is closed under the operations of addition and of taking finite maxima. The relation between $(dd^c)^n$ and the additive structure is given by the formula

$$(dd^c(u_1 + u_2))^n = \sum_{n_1+n_2=n} \frac{n!}{n_1! n_2!} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2},$$

which holds in both the classical and generalized sense. Here we make the connection between $(dd^c)^n$ and the operation of taking a finite maximum. A few examples of this are known already: if $u = \max(0, \log(|z_1|^2 + \cdots + |z_k|^2))$, then $(dd^c u)^k$ is a multiple of surface measure on $\{|z_1|^2 + \cdots + |z_k|^2 = 1\}$. And if $u = \max(0, x_1, \ldots, x_k)$, then $(dd^c u)^k$ is a multiple of surface measure on $\{x_1 = \cdots = x_k = 0\}$. The case $(dd^c \max(u_1, u_2, u_3))^2$, u_j's pluriharmonic, is given in [4]. A different sort of formula for $(dd^c \max(u_1, u_2))^k$ is given in [2], which, as pointed out by the referee, follows easily from the formula of this paper.

More generally, let the functions $u_1, \ldots, u_m$ be smooth, and let

$$u := \max(u_1, \ldots, u_m)$$

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be their maximum. It follows that $dd^c u_j$ and $dd^c u$ are locally bounded below, and for the purpose of defining $(dd^c)^n$, u_j and u may be treated as psh functions. Taking the maximum stratifies E as follows: for each $J \subset \{1, \dots, m\}$ there is

$$E_J = E_J(u_1, \dots, u_m) := \{z \in E : u(z) = u_j(z)\ \forall j \in J,\ u(z) > u_i(z)\ \forall i \notin J\}.$$

The sets E_J form a partition of E, and as J increases, the subsequent E_J's lie inside the "boundaries" of the previous ones. Since $(dd^c u)^n$ is representable by integration, we may decompose it into a sum over the elements of the partition

$$(dd^c u)^n = \sum_J (dd^c u)^n|_{E_J}. \tag{1}$$

The first terms in (1) are easy to identify: E_j is open and thus $(dd^c u)^n|_{E_j} = (dd^c u_j)^n|_{E_j}$. For the rest of the terms, we write $J = \{j_1, \dots, j_\ell\}$, $j_1 < \cdots < j_\ell$, so $|J| = \ell$ is the number of elements, and we define the forms

$$\delta_J^c = \delta_J^c(u_1, \dots, u_m) := d^c(u_{j_1} - u_{j_2}) \wedge \cdots \wedge d^c(u_{j_{\ell-1}} - u_{j_\ell}), \tag{2}$$

$$\sigma_J^n = \sigma_J^n(dd^c u_1, \dots, dd^c u_m) := \sum_{\beta_1 + \cdots + \beta_\ell = n} (dd^c u_{j_1})^{\beta_1} \wedge \cdots \wedge (dd^c u_{j_\ell})^{\beta_\ell}. \tag{3}$$

If E_J is smooth, we let $[E_J]$ denote the current of integration over E_J, where we orient E_J so that the current $\delta_J^c \wedge [E_J]$ is positive (see Lemma 1). In this paper we identify the terms of the summation (1) as integrations on the strata E_J:

Theorem 1. *Let $u_j \in C^3$, $1 \le j \le m$, be given, and set $u = \max(u_1, \dots, u_m)$. If all of the sets E_J are smooth, then*

$$(dd^c u)^n = \sum_J \sigma_J^{n-|J|+1} \wedge \delta_J^c \wedge [E_J], \tag{$\star$}$$

where the sum is taken over all $J \subset \{1, \dots, m\}$ with $1 \le |J| \le n+1$.

A function that arises frequently is $v = \log(|f_1|^2 + \cdots + |f_N|^2)$, where the f_i's are holomorphic. Set $f = (f_1, \dots, f_N) : \mathbb{C}^{\dim} \to \mathbb{C}^N$ and let $\pi : \mathbb{C}^N - 0 \to \mathbb{P}^{N-1}$ be the projection. The powers $(dd^c v)^n$ may be determined on $\{v > -\infty\}$ using the fact that $dd^c \log(|z_1|^2 + \cdots + |z_N|^2) = \pi^* \omega_{FS}$, where ω_{FS} denotes the Fubini-Study Kähler form on $\mathbb{P}^{N-1}$. That is, $(dd^c v)^n$ is the pullback of ω_{FS}^n under $(\pi \circ f)^*$. In particular, $(dd^c v)^n = 0$ if $n \ge N$.

Such functions and their maxima arise naturally with generalized polyhedra. For $1 \le \alpha \le A$, let $p_{\alpha,1}, \dots, p_{\alpha,N_\alpha}$ be polynomials. Let $\deg_\alpha = \max_{1 \le i \le N_\alpha}(\deg(p_{\alpha,i}))$ be the maximum of the degrees. Define

$$u_\alpha = \frac{1}{2 \deg_\alpha} \log \sum_{i=1}^{N_\alpha} |p_{\alpha,i}|^2$$

and

$$K = \{z \in \mathbb{C}^{\dim} : \max_{1 \le \alpha \le A} (|p_{\alpha,1}|^2 + \cdots + |p_{\alpha,N_\alpha}|^2) \le 1\}.$$

A useful fact is $(dd^c(\max(u_{\alpha_1}, \dots, u_{\alpha_\ell})))^N = 0$ *on the set where the maximum is finite, whenever* $N \ge N_{\alpha_1} + \cdots + N_{\alpha_\ell}$. This may be seen because $(\star)$ involves sums of terms of the form $(dd^c u_{a_1})^{\beta_1} \wedge \cdots \wedge (dd^c u_{a_k})^{\beta_k}$ with $\beta_1 + \cdots + \beta_k = N + 1 - k$, which means that for some i, $\beta_i \ge N_{\alpha_i}$ and thus $(dd^c u_{\alpha_i})^{\beta_i} = 0$.

Theorem 2. *Let us set* $u := \max(0, u_1, \dots, u_A)$. *Suppose that* $u \geq \log^+ |z| - C$ *and that for every* J *such that* $E_J - K \neq \emptyset$ *we have* $\sum_{j \in J} N_j \leq \dim$. *Then* u *is the psh Green function for* K. *If the sets* E_J *are smooth at points of* K, *then the equilibrium measure* $\mu_K := (\frac{1}{2\pi} dd^c u)^{\dim}$ *is given by the formula* $(\star)$.

Proof. We will show first that u is the psh Green function of K. It is evident that u is continuous and bounded above by $\log^+ |z| + C$. Since we also have the lower bound $\log^+ |z| - C$, it will suffice to show that $(dd^c u)^{\dim}$ vanishes on the complement of K. Suppose that $E_J - K$ contains a point z_0. Then $u = \max_{j \in J} u_j$ in a neighborhood of z_0. Replacing u by $u^\epsilon := \max(\epsilon_0, u_1 + \epsilon_1, \dots, u_A + \epsilon_A)$ for small ϵ_j's we may assume that $E_J^\epsilon = E_J(\epsilon_0, u_1 + \epsilon_1, \dots, u_A + \epsilon_A)$ is smooth near z_0. We evaluate $(dd^c u)^{\dim}$. By the useful fact above, near z_0 we have $(dd^c u^\epsilon)^N = 0$ on $E_{\tilde{J}}^\epsilon - K$ for all $\tilde{J} \subset J$ if $N \geq \sum_{j \in \tilde{J}} N_j$; thus $(dd^c u^\epsilon)^{\dim} = 0$. Letting $\epsilon \to 0$, we have $(dd^c u)^{\dim} = 0$ near z_0, and hence on $\mathbb{C}^{\dim} - K$. The formula for μ_K then follows from Theorem 1. $\square$

Let $\mathcal{D}^k$ denote the space of test forms of degree k. The currents of dimension k are defined as the dual of $\mathcal{D}^k$. Since we may decompose the k-forms into terms of bidegree (p, q), $\mathcal{D}^k = \bigoplus_{p+q=k} \mathcal{D}^{p,q}$, each current may be written as a sum of currents of bidimension (p, q). We have operators $\partial : \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}$ and $\bar{\partial} : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q+1}$; and their adjoints, which we denote again by ∂ and $\bar{\partial}$, act on the spaces of currents by duality. If u is psh, then $u[E]$ is a current which has the same dimension as E, and $dd^c(u[E])$ is a positive, closed current. If u is psh and continuous, we may define $(dd^c u)^n$ by induction on n (cf. [1]). Specifically, since $(dd^c u)^n$ is positive, then it is represented by integration. It follows that $u(dd^c u)^n$ is a well-defined current, and we set $(dd^c u)^{n+1} := dd^c(u\,(dd^c u)^n)$; in other words, its action on a test form φ is given by

$$\langle (dd^c u)^{n+1}, \varphi \rangle := \int dd^c \varphi \wedge u \wedge (dd^c u)^n.$$

This definition gives a continuous extension of $(dd^c)^n$ to the continuous, psh functions.

Let $M \subset \mathbb{C}^{\dim}$ be a smooth submanifold of locally finite volume. If M has codimension k, then we may orient M by choosing a simple k-form ν of unit length which annihilates the tangent space to M. We may define the current of integration $[M]$, which acts on a test form φ according to the formula

$$\langle \varphi, [M] \rangle := \int *(\varphi \wedge \nu)\, ||\nu||^{-1}\, dS_M,$$

where $*$ is the Hodge $*$-operator taking volume form to a scalar function, and dS_M the euclidean surface measure on M. Given a k-tuple $(\rho_1, \dots, \rho_k)$ of defining functions, we define an orientation as follows. $\mathbb{C}^{\dim}$ has a canonical orientation induced by its complex structure. We orient $M_1 = \{\rho_1 = 0\}$ as the boundary of $\{\rho_1 < 0\} \subset \mathbb{C}^{\dim}$. Thus $\nu_1 = d\rho_1$. We orient $M_2 = \{\rho_1 = \rho_2 = 0\}$ as the boundary of $\{\rho_2 < 0\} \cap M_2$ inside M_1. Thus $\nu_2 = d\rho_1 \wedge d\rho_2$. Continuing this way, we orient M using $\nu = d\rho_1 \wedge \dots \wedge d\rho_k$.

A (p, p) current T is said to be positive if $\langle T, i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p \rangle \geq 0$ for all smooth test forms α_j of type $(1,\ 0)$. Here we choose to orient E_J so that $\delta_J^c \wedge [E_J]$ is positive, a choice which is justified by the following.

Lemma 1. *We may orient E_J so that $\delta_J^c \wedge [E_J]$ is a positive current. If we let E'_J denote E_J with the orientation given (as above) by taking successive boundaries in terms of the defining functions $(\rho_1 = u_{j_1} - u_{j_2}, \dots, \rho_{\ell-1} = u_{j_{\ell-1}} - u_{j_\ell})$, then we have $[E_J] = (-1)^{\ell(\ell-1)/2}[E'_J]$.*

Proof. The k-form defining the orientation of E_J will be given by $\pm d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1}$, with the sign $\pm$ to be determined. In the notation above, we have

$$\delta_J^c \wedge [E'_J] = d^c\rho_1 \wedge \cdots \wedge d^c\rho_{\ell-1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1} \, ||d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1}||^{-1} \, dS,$$

where dS is the euclidean surface measure on E_J. Now since $d\rho = \partial\rho + \bar\partial\rho$ and $d^c\rho = i(\bar\partial\rho - \partial\rho)$ we see that

$$\begin{aligned}(-1)^{\frac{(\ell-2)(\ell-1)}{2}} 2^{1-\ell} d^c\rho_1 \wedge \cdots \wedge d^c\rho_{\ell-1} \wedge d\rho_1 \wedge \cdots \wedge d\rho_{\ell-1} \\ = (i\partial\rho_1 \wedge \bar\partial\rho_1) \wedge \cdots \wedge (i\partial\rho_{\ell-1} \wedge \bar\partial\rho_{\ell-1}).\end{aligned}$$

It follows that we may orient E_J to make $\delta_J^c \wedge [E_J]$ positive, and the relation between the orientations on E_J and E'_J is as claimed. □

We will use the notation

$$d_J^c = d_J^c(u_1, \dots, u_m) := d^c u_{j_1} \wedge \cdots \wedge d^c u_{j_\ell}. \tag{4}$$

For $1 \le t \le \ell$, we write $J(\hat t)$ to denote the set J with the t-th element removed, or if $s \in J$, $J(\hat s)$ denotes the set J with s removed. The meaning will be clear from the context.

Lemma 2. *We have the following identities:*

(1) $d(d_J^c(u)) = \sum_{t=1}^{\ell} (-1)^{t-1} d^c_{J(\hat t)}(u) \wedge dd^c u_{j_t}$.
(2) $d^c u_{j_t} \wedge \delta_J^c(u) = (-1)^{\ell-1} d_J^c(u)$ *for any* $1 \le t \le \ell$.
(3) $\sum_{t=1}^{\ell} (-1)^{\ell-t} d^c_{J(\hat t)} = \delta_J^c$.
(4) $dd^c u_{j_t} \wedge \sigma_J^n + \sigma^{n+1}_{J(\hat t)} = \sigma_J^{n+1}$.

Proof. These identities follow from the product rule and anticommutation of 1-forms. □

Lemma 3. *If α is a smooth form of type (a,a) and if β is a smooth form of type (b,b), then the forms $d\alpha \wedge d^c\beta$ and $d\beta \wedge d^c\alpha$ have the same parts of type $(a+b+1, a+b+1)$.*

Proof. Expand $d\alpha \wedge d^c\beta$ into terms of the form $\partial\alpha \wedge \bar\partial\beta$, etc., and compare bidegrees. □

Lemma 4. *$d[E_J] = [\partial E_J] = \sum_{\tilde J} \epsilon^J_{\tilde J}[E_{\tilde J}]$, where the sum is taken over all $\tilde J$ such that $J \subset \tilde J \subset \{1, \dots, m\}$ and $|\tilde J| = |J| + 1$. For each such $\tilde J$, there is an s such that $j_1 < \cdots < j_k < s < j_{k+1} < \cdots < j_\ell$ and $J = \tilde J(\hat s)$, and we have $\epsilon^J_{\tilde J} = (-1)^k$.*

Proof. By Stokes' Theorem, we have $d[E_J] = [\partial E_J] = \sum [E'_{\tilde J}]$, where $E'_{\tilde J}$ denotes the manifold $E_{\tilde J}$ with the induced boundary orientation on ∂E_J. Thus we need to compare the orientations of $E'_{\tilde J}$ and $E_{\tilde J}$. As in the discussion before Lemma 1, the orientations of E_J and $E'_{\tilde J}$ are given by the defining functions $(u_{j_2} - u_{j_1}, \dots, u_{j_\ell} - u_{j_{\ell-1}})$ and $(u_{j_2} - u_{j_1}, \dots, u_{j_\ell} - u_{j_{\ell-1}}, \rho)$, respectively, where we may take ρ to be either $u_s - u_{j_{k+1}}$ or $u_s - u_{j_k}$. By Lemma 1, the orientation of E_J is given by

$$\nu_{\tilde J} = (-1)^{\ell(\ell-1)/2} A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B,$$

where $A = d(u_{j_2} - u_{j_1}) \wedge \cdots \wedge d(u_{j_k} - u_{j_{k-1}})$ and $B = d(u_{j_{k+2}} - u_{j_{k+1}}) \wedge \cdots \wedge d(u_{j_\ell} - u_{j_{\ell-1}})$. Thus the orientation of $E'_{\tilde{J}}$ is given by the form

$$\nu'_{\tilde{J}} = (-1)^{\ell(\ell-1)/2} A \wedge d(u_{j_{k+1}} - u_{j_k}) \wedge B \wedge d(u_s - u_{j_k}).$$

By Lemma 1 again, the orientation of $E_{\tilde{J}}$ is given by the form

$$\nu_{\tilde{J}} = (-1)^{(\ell+1)\ell/2} A \wedge d(u_s - u_{j_k}) \wedge d(u_{j_{k+1}} - u_s) \wedge B.$$

Since the degree of B is $\ell - (k+1)$, we find that $\nu'_{\tilde{J}} = (-1)^k \nu_{\tilde{J}}$, which completes the proof. □

Proposition 1. *Let M be a smooth submanifold with boundary, and let χ be a smooth form on E so that $\chi \wedge [M]$ is a current of bidimension (p,p). If v is a smooth function and ϕ is a smooth form of bidegree $(p-1, p-1)$, then*

$$\int_M v \wedge \chi \wedge dd^c\phi = \int_M d(d^c v \wedge \chi) \wedge \phi - \int_{\partial M} d^c v \wedge \chi \wedge \phi + \int_M \chi \wedge d(v \wedge d^c\phi).$$

Proof. By the product rule, we have

$$\int_M v \wedge \chi \wedge dd^c\phi = \int_M \chi \wedge \left(d(v \wedge d^c\phi) - dv \wedge d^c\phi\right).$$

Note that v is a $(0,\ 0)$-form, ϕ is of bidegree $(p-1, p-1)$, and $\chi \wedge [M]$ is a current of bidimension (p,p). Thus the only nonzero terms integrated against this current can come from (p,p)-forms. Thus by Lemma 3 we may replace $dv \wedge d^c\phi$ by $d\phi \wedge d^c v$ in the right-hand integral. Now we integrate by parts in the right-hand integral to obtain the desired formula. □

Proposition 2. *We have*

$$dd^c(u \wedge \delta^c_J \wedge [E_J]) = (-1)^{\ell+1}\left(d(d^c_J(u)) \wedge [E_J] - d^c_J(u) \wedge [\partial E_J]\right) - d^c(u \wedge d(\delta^c_J \wedge [E_J])).$$

Proof. Let us note first that $d(\delta^c_J \wedge [E_J]) = d(\delta^c_J) \wedge [E_J] + (-1)^{\ell-1}\delta^c_J \wedge [\partial E_J]$, so this current is represented by integration. Thus, since $u = u_j$ for all j on the sets E_J and ∂E_J, the current $u_j \wedge d(\delta^c_J \wedge [E_J]) = u \wedge d(\delta^c_J \wedge [E_J])$ is the same for all $j \in J$. Similarly, we may substitute u_j for u in the left-hand term of the equation.

By Lemma 1, $\delta^c_J \wedge [E_J]$ is a current of bidegree $(|J|, |J|)$. Thus we evaluate the left-hand term by testing it against a form of type $(\dim - |J| - 1, \dim - |J| - 1)$:

$$\langle \phi, dd^c(u \wedge \delta^c_J \wedge [E_J]) \rangle = \langle dd^c\phi, u \wedge \delta^c_J \wedge [E_J] \rangle.$$

Now we apply Proposition 1 with $v = u_j$, $\chi = \delta^c_J$ and $M = E_J$ to obtain

$$\int_{E_J} u_j \wedge \delta^c_J \wedge dd^c\phi = \int_{E_J} d(d^c u_j \wedge \delta^c_J) \wedge \phi - \int_{\partial E_J} d^c u_j \wedge \delta^c_J \wedge \phi + \int_{E_J} \delta^c_J \wedge d(u_j \wedge d^c\phi).$$

Our formula now follows by applying Lemma 2(2) to the first and second integrals, rewriting the terms as currents, then substituting u for u_j. □

Proof of Theorem 1. In the case $n = 1$, we have $J = \{j\}$ and $\delta^c_J = 1$, so the statement of Proposition 2 becomes

$$dd^c(u \wedge [E_j]) = dd^c(u_j \wedge [E_j]) = dd^c u_j \wedge [E_j] - d^c u_j \wedge [\partial E_j] - d^c(u \wedge d[E_j]).$$

We identify $dd^c u$ with $dd^c u \wedge [E] = dd^c(\sum [E_j])$, which gives

$$dd^c u \wedge [E] = \sum_j dd^c u_j \wedge [E_j] + \sum_{j_1 < j_2} \delta^c_{j_1,j_2} \wedge [E_{j_1,j_2}],$$

where the second sum is a consequence of Lemmas 4 and 2(3) applied to $\sum d^c u_j \wedge [\partial E_j]$, and the other terms vanish since $d(\sum [E_j]) = d[E] = 0$.

Now we proceed by induction, assuming that Theorem 1 has been proved for n. Then we have

$$(dd^c u)^{n+1} = dd^c(u \wedge (dd^c u)^n) = dd^c \left(u \sum_J \sigma_J^{n-|J|+1} \wedge \delta_J^c \wedge [E_J] \right).$$

Since $\sigma_J^{n-|J|+1}$ is an even form which is both d- and d^c-closed, this expression is

$$= \sum_J \sigma_J^{n-|J|+1} \wedge dd^c(u \wedge \delta_J^c \wedge [E_J]).$$

We apply Proposition 2 to obtain

$$= \sum_J \sigma_J^{n-|J|+1} \wedge \Big((-1)^{|J|+1} \big\{ d(d_J^c(u)) \wedge [E_J] - d_J^c \wedge [\partial E_J] \big\} \\ - \big\{ d^c(u \wedge d(\delta_J^c \wedge [E_J])) \big\} \Big) = I + II.$$

Since $d^c \sigma_J^{n-|J|+1} = 0$, we have

$$II = -d^c \sum ud \left(\sigma^{n-|J|+1} \wedge \delta_J^c \wedge [E_J] \right).$$

Then, by our induction hypothesis,

$$II = -d^c(u \wedge d((dd^c u)^n)) = -d^c(u \wedge 0) = 0.$$

Now use Lemma 2(1) in the left-hand summation in I to obtain

$$(dd^c u)^{n+1} = \sum \sigma_J^{n-|J|+1} \wedge (-1)^{|J|+1} \sum_{t=1}^{|J|} (-1)^{t-1} dd^c u_{j_t} \wedge d^c_{J(\hat{t})} \wedge [E_J] \\ - \sum (-1)^{|J|+1} \sigma_J^{n-|J|+1} \wedge d_J^c \wedge [\partial E_J] = A + B.$$

In the notation of Lemma 4, we have

$$B = \sum_J \sigma_J^{n-|J|+1} \wedge (-1)^{|J|} d_J^c \wedge \sum_{\tilde{J}} \epsilon_{\tilde{J}}^J [E_{\tilde{J}}].$$

Now let us rewrite B, summing over $\tilde{J}$ on the outside, and summing over subsets $J = \tilde{J}(\hat{s})$ on the inside. By Lemma 4, $\epsilon_{\tilde{J}}^{\tilde{J}(\hat{s})} = (-1)^{s-1}$, $1 \le s \le |\tilde{J}|$. This gives

$$B = \sum_{\tilde{J}} \left(\sum_{s=1}^{|\tilde{J}|} (-1)^{|\tilde{J}|-s} \sigma_{\tilde{J}(\hat{s})}^{n-|\tilde{J}|+2} \wedge d^c_{\tilde{J}(\hat{s})} \right) \wedge [E_{\tilde{J}}].$$

Since we are summing over all subsets $\tilde{J}$, we can remove the tilde from $\tilde{J}$. Further, we can set $s = t$, which lets us rewrite $A + B$ as

$$\sum_J \sum_t \left(\sigma_J^{n-|J|+1} \wedge dd^c u_{j_t} + \sigma_{J(\hat{t})}^{n-|J|+2} \right) \wedge d^c_{J(\hat{t})} \wedge (-1)^{|J|-t} d_J^c(\hat{t}) \wedge [E_J].$$

Finally, by Lemma 2(3) and (4), we have

$$(dd^c u)^{n+1} = \sum_J \sigma_J^{n-|J|+2} \wedge \left(\sum_t (-1)^{|J|-t} d^c_{J(\hat{t})} \right) \wedge [E_J] = \sum_J \sigma_J^{n-|J|+2} \wedge \delta_J^c \wedge [E_J],$$

which completes the proof. □

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