

SHARP BOUNDS FOR THE VALENCE OF CERTAIN HARMONIC POLYNOMIALS

LUKAS GEYER

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ABSTRACT. In Khavinson and Świątek (2002) it was proved that harmonic polynomials $z - \overline{p(z)}$, where p is a holomorphic polynomial of degree $n > 1$, have at most $3n - 2$ complex zeros. We show that this bound is sharp for all n by proving a conjecture of Sarason and Crofoot about the existence of certain extremal polynomials p . We also count the number of equivalence classes of these polynomials.

1. INTRODUCTION AND RESULTS

In [KS] it was proved that harmonic polynomials $z - \overline{p(z)}$, $\deg p = n > 1$, have at most $3n - 2$ complex zeros. An example was provided that this bound is sharp for $n = 3$, and it was conjectured that it is sharp for every n . The main result of this note is the following theorem, which was conjectured by Crofoot and Sarason [KS, Conj. 1].

Theorem 1. *For every $n > 1$ there exists a complex analytic polynomial p of degree n and mutually distinct points z_1, z_2, \dots, z_{n-1} with $p'(z_j) = 0$ and $\overline{p(z_j)} = z_j$.*

In particular, this implies the sharpness of the bound $3n - 2$ for every $n > 1$.

Corollary 2. *For every $n > 1$ there exists a complex analytic polynomial p of degree n such that $\overline{p(z)} - z$ has $3n - 2$ zeros.*

It was proved in [KS] that Theorem 1 implies Corollary 2. However, for the convenience of the reader we will provide an alternative argument to derive this Corollary after the proof of the Theorem. The main ingredient in the proof of Theorem 1 is an explicit construction of a “topological polynomial” and a result of Levy that certain topological polynomials are Thurston-equivalent to polynomials.

In addition we are able to calculate the number of real polynomials p satisfying the conditions in Theorem 1, up to equivalence. Two polynomials p and q are *conjugate* if there is an affine linear map T with $p = T^{-1} \circ q \circ T$. They are *equivalent* if there are affine linear maps S and T with $p = S \circ q \circ T$.

Theorem 3. *The number E_n of equivalence classes of real polynomials p of degree n having $n - 1$ distinct critical points c_1, \dots, c_{n-1} and satisfying $p(c_j) = \overline{c_j}$ for*

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$j = 1, \dots, n - 1$ is $E_n = C_{\lfloor (n-1)/2 \rfloor}$, where $C_m = \frac{1}{m+1} \binom{2m}{m}$ is the m -th Catalan number. The number of conjugacy classes is $Q_n = E_n + C_k$ if $n = 4k + 3$, and $Q_n = E_n$ if $n \not\equiv 3 \pmod{4}$.

Remark. There are non-real polynomials satisfying the same equations. In particular, for any real solution p , and ω on the unit circle, the function $q(z) = \omega p(\omega z)$ is also a solution. We conjecture that all non-real solutions are equivalent to real solutions.

The proof of Theorem 3 relies on Poirier's results about Hubbard trees for post-critically finite polynomials from [Po1] and [Po2].

If one allows rational functions p instead of polynomials, the number of zeros of $\overline{p(z)} - z$ is at most $5n - 5$ (see [KN], which also outlines the connection of this problem to gravitational lensing), and the proof that this bound is sharp is much easier and very explicit [Rh].

2. PROOF OF THEOREM 1

We will construct a polynomial with real coefficients. There is a minor difference between the cases where n is even and the one where n is odd. In the case of even n , one of the points will be real; in the odd case, all z_j will be non-real. Note also that the polynomial p itself will map all z_j to their complex conjugates, so the second iterate fixes all z_j . These polynomials are special cases of post-critically finite rational maps. The main ingredient in the proof is Thurston's topological characterization of rational maps [DH], and a result of Levy on topological polynomials. For background see also [BFH] and [Pi].

A *topological polynomial* is an orientation-preserving branched covering f of the sphere to itself with $f^{-1}(\infty) = \{\infty\}$. Critical points are points where f is not locally injective, and the *post-critical set* $P(f)$ is the closure of $\{f^n(c) : n > 1, c \text{ critical}\}$. An orientation-preserving branched covering of the sphere is *post-critically finite* if the post-critical set contains finitely many points, or equivalently if all critical points are eventually periodic under iteration. Two post-critically finite maps f and g are *Thurston equivalent* if there exist orientation-preserving homeomorphisms ϕ and ψ , homotopic rel $P(f)$, with $\phi \circ f = g \circ \psi$.

Thurston gave a criterion for post-critically finite branched coverings to be equivalent to rational maps, and Levy proved the following for topological polynomials (see [Le], [BFH, Cor. 5.13]).

Theorem (Levy). *If f is a topological polynomial such that every critical point eventually lands in a periodic cycle containing a critical point, then f is Thurston equivalent to a polynomial. The polynomial is unique up to affine conjugation.*

We will first construct a topological polynomial f satisfying $f(\bar{z}) = \overline{f(z)}$ for all z , and $f(c_j) = \bar{c}_j$ for all critical points c_j . The cases of even and odd degrees are slightly different, so we will start with the case of odd degree $n = 2m + 1$ for some $m \geq 1$. Let g be a polynomial of degree $2m + 1$ with real coefficients and m distinct critical points c_1, \dots, c_m in the upper half-plane such that the corresponding critical values $v_j = g(c_j)$ are mutually distinct points in the lower half-plane. E.g., one could choose a small constant $\epsilon > 0$ and define $c_{2k-1} = ik$, $c_{2k} = i(k + \epsilon)$, and

$$g(z) = - \int_0^z \prod_{k=1}^m (\zeta^2 - c_k^2) d\zeta.$$

It is easy to check that g satisfies all the conditions if ϵ is sufficiently small. Now choose a homeomorphism h of the closed lower half-plane with $h(v_j) = \overline{c_j}$. Extend h by reflection to a homeomorphism of the plane and define $f = h \circ g$. Then f is a topological polynomial of degree $2m + 1$, commuting with reflection in the real line, mapping all its critical points to their complex conjugates. The post-critical set of f consists of all the critical points $c_1, \dots, c_m, \overline{c_1}, \dots, \overline{c_m}$ and ∞ . By Levy's result f is Thurston-equivalent to a polynomial p . The uniqueness part of Levy's theorem implies that p is conjugate to $\overline{p(\overline{z})}$, so p is conjugate to a real polynomial, and we may as well assume that p itself is real. It also follows that both ϕ and ψ are symmetric with respect to the real line. By the definition of Thurston-equivalence we know that $z_j = \phi(c_j) = \psi(c_j)$, so p has critical points $z_1, \dots, z_m, \overline{z_1}, \dots, \overline{z_m}$, and $p(z_j) = \overline{z_j}$ for all j .

If the degree n is even, we have to modify the initial step of the construction. Write $n = 2m + 2$ with some $m \geq 0$, and find a real polynomial g of degree n with distinct simple critical points $c_0 \in \mathbb{R}$, c_1, \dots, c_m in the upper half-plane, such that $v_j = g(c_j)$ are all distinct and in the lower half-plane for $1 \leq j \leq m$. Here one can take

$$g(z) = \int_0^z (\zeta - 1) \prod_{k=1}^m (\zeta^2 - c_k^2) d\zeta$$

with c_k chosen as above for $k \geq 1$. Now choose a homeomorphism h of the closed lower half-plane onto itself such that $h(g(c_0)) = c_0$ and $h(g(c_k)) = \overline{c_k}$ for $k \geq 1$. The rest of the proof is identical to the case of odd degree. \square

Proof of Corollary 2. We will show that $z - \overline{p(z)}$ has at least $3n - 2$ zeros. Together with the result from [KS], this shows that it has exactly $3n - 2$ zeros. First we observe that p has no degenerate fixed points, i.e., all fixed points of $z \mapsto \overline{p(z)}$ satisfy $|p'(z)| \neq 1$. (For holomorphic maps, a fixed point is degenerate if the derivative is exactly 1; for anti-holomorphic maps it is degenerate whenever the derivative has modulus 1.) If this were the case, then $\overline{p(\overline{p(z)})}$ would have a parabolic fixed point, which would have a critical point in its attracting basin. However, all critical points of this map are mapped to the critical points of p and those are fixed. Having only non-degenerate fixed points, the sum of the Lefschetz indices of the fixed points in the Riemann sphere has to be equal to $-n + 1$, as the induced maps on 0- and 2-dimensional homology are the identity and multiplication by $-n$, respectively. It already has super-attracting fixed points at ∞ and z_1, \dots, z_{n-1} . The Lefschetz index of a super-attracting fixed point is 1; thus the sum of the indices of all those fixed points is n . We conclude that p must have at least $2n - 1$ fixed points with index -1 , i.e., repelling fixed points for \overline{p} . This proves sharpness. \square

3. COUNTING CONJUGACY AND EQUIVALENCE CLASSES

We will use results about Hubbard trees by Poirier (see [Po1] and [Po2]) in order to prove Theorem 3. Hubbard trees were first introduced in [BFH] in order to classify post-critically finite polynomials without periodic critical points. In that case the Julia set is a dendrite, and the Hubbard tree is the embedded subtree of the Julia set spanned by the post-critical set. It has a finite number of vertices and edges. The polynomial induces a map on the tree, and conversely an embedded finite tree with a map satisfying certain conditions gives rise to a post-critically finite polynomial, unique up to affine conjugation. Poirier generalized this to include the

case of post-critically finite polynomials with critical periodic points. In this case all Fatou components are simply connected with a distinguished base point, the unique preperiodic point in the component. The Hubbard tree in this case intersects both the Julia and the Fatou set, and the part in a Fatou set consists of preimages of radial lines under the Riemann map mapping the component to the unit disk and the basepoint to 0. This construction determines an angle between any two edges meeting at a point in the Fatou set. The important information needed to reconstruct the map from the (marked) tree is the local degree at every vertex, the angles between any two edges meeting at a Fatou vertex, and the mapping on vertices. There is no well-defined angle between edges meeting at points in the Julia set, but there is a well-defined cyclic order, and we define the “angles” between the edges to be all the same. In that way we end up with an angle function on the tree defined for any pair of incident edges.

An abstract Hubbard tree is a finite tree embedded in the plane, together with a mapping $F : V \rightarrow V$ on the vertices, a local degree function $\delta : V \rightarrow \{1, 2, \dots\}$ on the vertices, and an angle function α defined for any pair of incident edges l_k, l_j . It has to satisfy $\alpha(l_k, l_j) \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha(l_j, l_k) = -\alpha(l_k, l_j)$, and $\alpha(l_j, l_k) + \alpha(l_k, l_m) = \alpha(l_j, l_m)$ whenever l_k, l_j, l_m meet at a vertex. The map F satisfies $F(v) \neq F(w)$ whenever v and w are adjacent vertices. In this way F induces a mapping on the edges, mapping the edge $[v, w]$ homeomorphically to the unique path between $F(v)$ and $F(w)$, i.e., mapping an edge to a union of edges. If two edges l_j and l_k meet at v , then $F(l_j)$ and $F(l_k)$ meet at $F(v)$, and we require that $\alpha(F(l_j), F(l_k)) = \delta(v)\alpha(l_j, l_k)$, which is just the fact that holomorphic maps of local degree δ multiply angles by δ . A vertex is a Fatou vertex if it eventually lands in a periodic vertex v with $\delta(v) > 1$. All other vertices are Julia vertices. The last and not quite as obvious requirement for the Hubbard tree is that F is expanding; i.e., whenever v and w are adjacent Julia vertices, there exists an n such that $F^n(v)$ and $F^n(w)$ are not adjacent.

The *degree* of the Hubbard tree is $d = 1 + \sum_{v \in V} (\delta(v) - 1)$. Poirier’s main result in [Po2] is that any such abstract Hubbard tree corresponds to a polynomial of degree d , unique up to affine conjugation. The uniqueness part easily implies that any symmetries of the Hubbard tree correspond to symmetries of the polynomial.

Proof of Theorem 3. We will first prove the result about the number of conjugacy classes. By Poirier’s theorem we only have to count the number of corresponding Hubbard trees up to symmetry. We will first establish some general properties of the corresponding Hubbard trees, after which we will treat odd and even degrees separately. The following claim gives a complete characterization of the Hubbard trees in question; the rest of the argument is counting. \square

Claim 1. The induced map on the Hubbard tree is $z \mapsto \bar{z}$. The only vertices in the Hubbard tree are the critical points. All angles are $1/3$ or $2/3$. Conversely, if T is a finite tree that is symmetric with respect to complex conjugation and has local degree 2 at every vertex, then T with the map $f : T \rightarrow T$, $f(z) = \bar{z}$ arises as a Hubbard tree of a polynomial satisfying the requirements in Theorem 3.

Proof. Let c_1 and c_2 be critical points such that the path γ between them does not contain any other critical point. Then γ will be mapped homeomorphically to the path connecting $p(c_1) = \bar{c}_1$ and $p(c_2) = \bar{c}_2$, which is the complex conjugate of γ . Since the whole post-critical set consists of critical points, this argument

covers the whole Hubbard tree. Any non-critical vertices would have to be branch points, and the induced map $z \mapsto \bar{z}$ reverses the cyclic order of edges at branch points, contradicting the fact that p is locally conformal at all non-critical vertices. Finally, let l_1 and l_2 be two edges meeting at a critical vertex c . Then $\delta(c) = 2$, and $2\alpha(l_1, l_2) = \alpha(p(l_1), p(l_2)) = \alpha(\bar{l}_1, \bar{l}_2) = -\alpha(l_1, l_2)$ in \mathbb{R}/\mathbb{Z} ; thus $3\alpha(l_1, l_2) \in \mathbb{Z}$.

Conversely, if all local degrees are 2, and all angles are $1/3$ or $2/3$, complex conjugation $f(z) = \bar{z}$ satisfies $\delta(v)\alpha(l_1, l_2) = \alpha(f(l_1), f(l_2))$ for any two edges meeting at a vertex v . The property that f on the tree T be expanding is trivially satisfied as there are no Julia vertices at all. \mathbb{R} -symmetric Hubbard trees give rise to \mathbb{R} -symmetric polynomials, so this shows the last part of the claim. \square

Claim 2. The number of vertices on the real line is at most 2. There is a bijective correspondence between Hubbard trees with 0 real vertices and those with 2 real vertices.

Remark. The bijective correspondence replaces the vertical edge that intersects the real line by a horizontal edge that is a real interval. See Figure 1 for examples of Hubbard trees up to degree 8.

Proof. Let c_1 and c_2 be real critical points. Since the Hubbard tree is a tree symmetric with respect to the real line, the subtree spanned by c_1 and c_2 has to be the real interval between them. Since no angle in the Hubbard tree is $1/2$, there can be no other vertex in that interval; i.e., there cannot be other critical points between c_1 and c_2 . This shows the first claim. In order to construct the bijective correspondence, we only need to give the modification of the tree in the upper half-plane. The part in the lower half-plane is determined by symmetry and the map on the tree is always complex conjugation. If there are two real critical points $c_1 < c_2$, let T_1 and T_2 be the subtrees in the upper half-plane attached at c_1 and c_2 , respectively. The correspondence is given by replacing the edge $l = [c_1, c_2]$ by $\tilde{l} = [-i, i]$, and attaching T_1 and T_2 to the left and right of i , respectively, so that $\alpha(T_2, T_1) = \alpha(T_1, \tilde{l}) = \alpha(\tilde{l}, T_2) = 1/3$. The inverse of this operation is given by replacing the edge that crosses the real line by a real interval and attaching the left and right subtrees at the endpoints of this interval, again with angles $1/3$. Observe also that any symmetry with respect to the imaginary axis will be preserved by this correspondence and its inverse.

It remains to count the Hubbard trees. Let us start with the easier case of even degree $n = 2m + 2$ with $m \geq 0$. Here we have an odd number of critical points, so by Claim 2 there is exactly one real critical point c_1 . The number of edges meeting at this point must be even by symmetry. In the case $m = 0$, there is only one critical point, so there are no edges. In all other cases the number of edges meeting at c_0 must be 2, since it can be at most 3 by Claim 1. Also the angles between the real axis and the edges must be $1/6$ and $1/3$, respectively. There are exactly two choices, and they correspond to each other under the symmetry $z \mapsto -z$. Since we count Hubbard trees up to symmetry, we may fix an orientation. It also follows that all those Hubbard trees have no rotational symmetry. We can view the part of the tree in the upper half-plane as a finite part of an infinite rooted binary tree. The number of vertices, excluding the root, is m , and it is well known (and easy to prove from the recursive formula $C_m = \sum_{k=0}^{m-1} C_k C_{m-1-k}$) that there are $C_m = \frac{1}{m+1} \binom{2m}{m}$ of those.

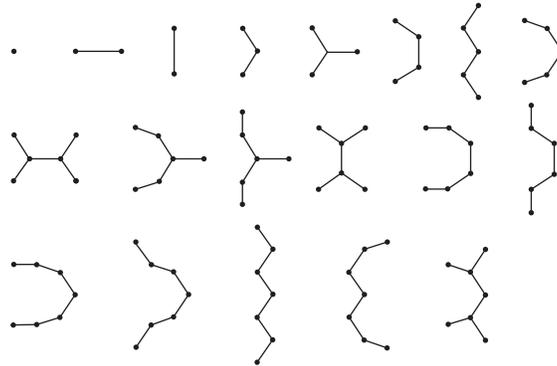


FIGURE 1. Hubbard trees for degrees $2 \leq n \leq 8$. The real line is the horizontal line of symmetry for each tree, and the induced mapping on the trees is complex conjugation. All angles are multiples of $2\pi/3$.

In the case of odd degree $n = 2m + 1$ there are $2m$ critical points. Let us count the number of trees that have no vertex on the real line. Then the part in the upper half-plane is again a finite part of a rooted binary tree with m vertices, so there are C_m of those. For every such tree the image under $z \mapsto -z$ has the same form. If $m = 2k$, this implies that there are $C_m/2$ up to symmetry. However, if $m = 2k + 1$, there are trees that are symmetric under $z \mapsto -z$. The number of those corresponds to the number of subtrees with k vertices attached to the right of the first vertex in the upper half-plane. This means that there are C_k symmetric trees, so we have $(C_m + C_k)/2$ inequivalent Hubbard trees in this case. By Claim 2, we have the same number of trees with 2 vertices on the real line, so the number of inequivalent Hubbard trees is C_{2k} for $n = 4k + 1$ and $C_{2k+1} + C_k$ for $n = 4k + 3$.

Let us now count the number of equivalence classes. The equivalence of two solutions p and q implies the existence of affine linear maps S and T with $SpS^{-1} = Sq$, i.e., the existence of an affine map $R = ST$ such that R maps the critical points of q to itself. In particular this implies that RqR^{-1} is again a solution. The only rotations R which map one of our symmetric Hubbard trees to a symmetric Hubbard tree are $R(z) = i^p z$ with $p \in \{1, 2, 3\}$. In the case $R(z) = -z$, the map $p = Rq$ has the same underlying Hubbard tree as q with the map $z \mapsto -\bar{z}$ on it. This corresponds to the Hubbard tree of q rotated by 90 degrees with complex conjugation as the map. This can only happen if $n = 4k + 3$, and the number of trees which admit this is C_k . The case $R(z) = \pm i$ can actually not occur, since the map $p = Rq$ would have the same Hubbard tree and the map on it would be reflection in the line $(1 \pm i)\mathbb{R}$. However, none of the Hubbard trees has one of those lines as a line of symmetry.

Combining all this, the number of equivalence classes is the same as the number of conjugacy classes except in the case $n = 4k + 3$, where it becomes C_{2k+1} . Thus the number of equivalence classes is $Q_n = C_{\lfloor (n-1)/2 \rfloor}$ for all n . \square

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DEPARTMENT OF MATHEMATICS, MONTANA STATE UNIVERSITY, P.O. BOX 172400, BOZEMAN, MONTANA 59717-2400

E-mail address: geyer@math.montana.edu