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VECTOR-VALUED MAASS-POINCARÉ SERIES

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Abstract. Shortly before his death, Ramanujan wrote about his discovery of mock theta functions, functions with interesting analytic properties. Recently, Zweger showed that mock theta functions could be "completed" to satisfy the transformation properties of a weight 1/2 real analytic vector-valued modular form. Using Maass-Poincaré series, Bringmann and Ono proved the Andrews-Dragonette conjecture, establishing an exact formula for the coefficients of Ramanujan's mock theta function $f(q)$. In this paper we study vector-valued Maass-Poincaré series of all weights, and give their Fourier expansions.

1. Introduction

In his last letter to G. H. Hardy, Srinivasa Ramanujan [9] wrote about his discovery of "very interesting functions," such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

which he called mock theta functions. (Throughout we let $q := e^{2\pi i z}$.) Nearly seven decades later, the celebrated physicist Freeman Dyson [6] challenged those in attendance at the 1987 Ramanujan Centenary Conference to build a better understanding of the mock theta functions, remarking,"somehow it should be possible to build them (the mock theta functions) into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi."

Zwegers made a big step towards answering Dyson's challenge in his 2002 Ph.D. thesis [11]. He showed that many of the mock theta functions could be "completed" in such a way that they fit into the structure of real analytic weight 1/2 vector-valued modular forms. In [10], Zwegers gave the example of a vector-valued function related to the mock theta functions $f(q)$ and $\omega(q)$ where

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}. \tag{1.1}$$

Bringmann and Ono [3] built upon this work to place mock theta functions in the realm of weight 1/2 weak Maass forms using Dyson's own work in a surprising way, and they used this connection to prove new results about partition ranks.

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Moreover, in [4], Bringmann and Ono constructed a particular weight 1/2 Maass-Poincaré series and proved that this series is equal to the "Maass form completion" associated to $f(q)$. By writing the Fourier expansion of this Maass-Poincaré series, Bringmann and Ono proved the Andrews-Dragonette conjecture, giving the exact formula for the coefficients of $f(q)$, the first such exact formula for a mock theta function.

In [7] the author proved an exact formula for the coefficients of the mock theta function $\omega(q)$ by studying the transformation properties of the Maass-Poincaré series associated to $f(q)$. A crucial step was constructing a weight 1/2 vector-valued modular form from this Maass-Poincaré series and writing the corresponding Fourier expansions for each component. In this paper we explore the transformation properties of Maass-Poincaré series of all weights and give their Fourier expansions. In Section 2 we set notation. In Section 3 we construct weight k real analytic vector-valued modular forms from the weight k Maass-Poincaré series, thus realizing the transformation properties for this series over all of $\mathrm{SL}_2(\mathbb{Z})$. In Section 4 we give the Fourier expansions of the weight k Maass-Poincaré series and the corresponding series formed by its transformations. In Section 5 we end with a discussion of the Maass-Poincaré series of weights $k = 1/2$ and $k = 3/2$ for which these series need special consideration.

2. Notation

In this section we set notation and define the weight k Maass-Poincaré series. Throughout we will let $z := x + iy \in \mathbb{C}$, where $x, y \in \mathbb{R}$, and we let $\mathbb{H}$ denote the usual upper half plane $\mathbb{H} = \{z \in \mathbb{C} : y > 0\}$. We let $e(x) := e^{2\pi i x}$. We also let S and T be the usual generators of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Furthermore, as we will work with both half integral and integral weight modular forms and Maass forms, we define

$$\upsilon_k := \begin{cases} 1 & \text{if } k \in \mathbb{Z}, \\ -i & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{2.2}$$

We begin by recalling the definition of a vector-valued modular form.

Definition 2.1. Suppose $k \in \frac{1}{2}\mathbb{Z}$. A meromorphic vector-valued function $\widetilde{P}(z) : \mathbb{H} \to \mathbb{C}^n$ is a *weight k meromorphic vector-valued modular form on* $\mathrm{SL}_2(\mathbb{Z})$ if there exist $n \times n$ complex-valued matrices M_T and M_S such that for all $z \in \mathbb{H}$, the function $\widetilde{P}(z)$ satisfies the transformation properties

$$\widetilde{P}(z+1) = M_T \widetilde{P}(z),$$

and

$$\widetilde{P}(-1/z) = (\upsilon_k z)^k M_S \widetilde{P}(-1/z).$$

We continue with the definition of a weight k weak Maass form. For d odd, define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Definition 2.2. Suppose that $k \in \frac{1}{2}\mathbb{Z}$, that N is a positive integer with $4|N$ if $k \in \frac{1}{2}+\mathbb{Z}$, and that ψ is a Dirichlet character with modulus N. A smooth function $f : \mathbb{H} \to \mathbb{C}$ is a *weight k harmonic weak Maass form on $\Gamma_0(N)$ with Nebentypus character ψ* if it satisfies the following conditions:

(1) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $z \in \mathbb{H}$, we have

$$f(Mz) = \begin{cases} \psi(d)(cz+d)^k f(z) & \text{if } k \in \mathbb{Z}, \\ \psi(d)\left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k}(cz+d)^k f(z) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

(2) We have that $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian defined by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

(3) The function $f(z)$ has at most linear exponential growth at all the cusps of $\Gamma_0(N)$.

Now we turn our attention to Maass-Poincaré series. In the case of $f(q)$, Bringmann and Ono [4] used a specific Maass-Poincaré series to give an exact formula for the coefficients of $f(q)$, confirming the Andrews-Dragonette conjecture. To define the Maass-Poincaré series first requires more notation. Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ denote the standard M-Whittaker function and W-Whittaker function which are solutions to the differential equation

$$\frac{\partial^2 u}{\partial z^2}+\left(-\frac{1}{4}+\frac{\nu}{z}+\frac{\frac{1}{4}-\mu^2}{z^2}\right)u = 0.$$

For $s \in \mathbb{C}, k \in \frac{1}{2}+\mathbb{Z}$, and $y \in \mathbb{R}\setminus\{0\}$, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.3}$$

and similarly let

$$\mathcal{W}_s(y) := |y|^{-\frac{k}{2}} W_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|). \tag{2.4}$$

Furthermore, for negative rational values m, define

$$\varphi_{s,k,m}(z) := \mathcal{M}_s\left(4m\pi y\right) e\left(mx\right),$$

which is an eigenfunction of Δ_k with eigenvalue $s(1-s)+\frac{k}{2}\left(1-\frac{k}{2}\right)$.

We are now ready to define our general Maass-Poincaré series. Let

$$P(N,\chi,m,k,s;z) := \sum_{M:=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k,m}(Mz), \tag{2.5}$$

where $\Gamma_\infty := \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$, $k \in \frac{1}{2}\mathbb{Z}$, $N \in \mathbb{N}$, $0 > m \in \mathbb{Q}$, $s \in \mathbb{C}$, and χ is a multiplier system for $\Gamma_0(N)$. Note that for this definition to make sense, the term arising from the summand T^n must be the same for all n; that is,

$$\chi(I)^{-1}\varphi_{s,k,m}(z) = \chi\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)^{-1}\varphi_{s,k,m}(z+n).$$

This gives us $\chi(T) = e(m)$.

Remark. Bringmann and Ono used the Fourier expansion of

$$P_f(z) := \frac{1}{\Gamma(3/2)} P(2, \chi_f, -1/24, 1/2, 3/4; z),$$

where χ_f is as defined in (3.1) of [4] in their study of $f(q)$.

3. Vector-valued modular forms

In this section we study the transformation properties of $P(N, \chi, m, k, s; z)$, but first we begin by looking at analytic properties. Since $\varphi_{s,k,m}(z) = O\left(y^{\mathrm{Re}(s)-\frac{k}{2}}\right)$ as $y \to 0$, we have

$$\chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k,m}(z) = O\left(|cz+d|^{-k}\left(\frac{y}{|cz+d|^2}\right)^{\mathrm{Re}(s)-\frac{k}{2}}\right),$$

and so $P(N, \chi, m, k, s; z)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Now suppose $V = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$. If $\mathrm{Re}(s) > 1$ let $M' = MV \in \Gamma_0(N)$. By reordering, we obtain

$$\begin{aligned} &(3.1)\quad P(N, \chi, m, k, s; Vz) \\ &\qquad = \sum_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = M' \in \Gamma_\infty \backslash \Gamma_0(N)} \chi(M'V^{-1})^{-1}(c'z+d')^{-k}(\gamma z+\delta)^k \varphi_{s,k,m}(M'z). \end{aligned}$$

Hence, for all $V \in \Gamma_0(N)$, we have

$$P(N, \chi, m, k, s; Vz) = \chi(V)(\gamma z+\delta)^k P(N, \chi, m, k, s; z). \tag{3.2}$$

Remark. Given the discussion above, if $k < 0$ and $s = 1 - k/2$ or $k > 2$ and $s = k/2$, then $P(N, \chi, m, k, s; z)$ is a weight k harmonic weak Maass form on $\Gamma_0(N)$ with Nebentypus if χ is compatible with the first condition in Definition 2.2.

Remark. In the case of the mock theta function $f(q)$, the corresponding function $P_f(z)$ requires the change of variables $z \mapsto 24z$ for χ_f to satisfy the correct properties. In this case, $P_f(24z)$ is a weight 1/2 harmonic weak Maass form on $\Gamma_0(144)$ with Nebentypus $\left(\frac{12}{d}\right)$. Further, $\mathrm{Re}(s) = 3/4 < 1$, and so special care is required in this case, as discussed in the final section of this paper.

Given the nice transformation properties of $P(N, \chi, m, k, s; z)$ on $\Gamma_0(N)$, it is natural to ask about the transformation properties on all of $\mathrm{SL}_2(\mathbb{Z})$. Indeed, it is the transformation $P_f(-1/z)$ that leads to an exact formula for the coefficients of the mock theta function $\omega(q)$.

Looking at (3.1) for general $V \in \mathrm{SL}_2(\mathbb{Z})$ allows us to extend the Maass-Poincaré series to sums over the various cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Moreover, we create a vector-valued function from these series,

$$\widetilde{P}(z) := \left(P_I(z), P_{V_2}(z), \ldots, P_{V_n}(z)\right), \tag{3.3}$$

where $\{I, V_2, \ldots, V_n\}$ form a set of coset representatives for $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$. Here we have that

$$n = [\Gamma_0(N) : \mathrm{SL}_2(\mathbb{Z})] = N \prod_{p|N}(1 + 1/p).$$

Each $P_{V_i}(z)$ has the form

$$P_{V_i} = \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)=M=M'V_i \\ M'\in\Gamma_\infty\backslash\Gamma_0(N)}} \chi_{V_i}(M)^{-1}(cz+d)^{-k}\varphi_{s,k,m}(Mz), \tag{3.4}$$

where χ_{V_i} is a multiplier system for matrices in the same coset as V_i. In the case where N is prime, we can define this vector-valued function so that it is a weight k real analytic vector-valued modular form over $\mathrm{SL}_2(\mathbb{Z})$.

The following theorem is straightforward to prove.

Theorem 3.1. *Fix a prime p and a weight $k \in \frac{1}{2}\mathbb{Z}$. Let υ_k be defined as above. Given a Maass-Poincaré series $P(p,\chi,m,k,s;z)$, choose roots of unity s_2, $t_3, \dots,$ t_{p+1} and define*

$$\chi_S(M) := (\upsilon_k)^k s_2 \chi(MS^{-1}), \tag{3.5}$$

$$\chi_{ST^j}(M) := (\upsilon_k)^k s_2 t_3 \cdots t_{j+2}\chi(M(ST^j)^{-1}). \tag{3.6}$$

Using the notation of (3.4)*, the function*

$$\widetilde{P}(z) := (P(p,\chi,m,k,s;z), P_S(z), P_{ST}(z), \dots, P_{ST^{p-1}}(z))^T$$

is a vector-valued real analytic modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$. It satisfies the transformation properties

$$\widetilde{P}(z+1) = M_T\widetilde{P}(z); \qquad \widetilde{P}(-1/z) = (\upsilon_k z)^k M_S \widetilde{P}(z),$$

where

$$M_{Ti,j} := \begin{cases} e(m) & (i,j) = (1,1), \\ t_{i+1} & (i,j) = (i,i+1) \quad \text{for } 2 \le i \le p, \\ t_3^{-1}\cdots t_{p+1}^{-1}\chi(ST^pS^{-1}) & (i,j) = (p+1,2), \\ 0 & \text{otherwise}, \end{cases}$$

and

$$M_{Si,j} := \begin{cases} s_2^{-1} & (i,j) = (2,1), \\ s_2 & (i,j) = (1,2), \\ \upsilon_k^{-k}\chi(ST^{i-2}ST^{-j+2}S^{-1})t_{i+1}\cdots t_j & \text{for } 3 \le i \le j \le p+1 \\ & \text{with } (i-2)(j-2) \equiv -1 (\operatorname{mod} p), \\ \upsilon_k^{-k}\chi(ST^{i-2}ST^{-j+2}S^{-1})t_{j+1}^{-1}\cdots t_i^{-1} & \text{for } 3 \le j < i \le p+1 \\ & \text{with } (i-2)(j-2) \equiv -1 (\operatorname{mod} p), \\ 0 & \text{otherwise}, \end{cases}$$

where the product $t_{i+2}\cdots t_{j+1}$ is taken as 1 if $i = j$.

We can see that the transformation matrices satisfy the correct form by considering how the cosets of $\Gamma_0(p)$ permute when multiplied on the right by T and S. This is where the complexity arises for writing down such matrices for the general $\Gamma_0(N)$.

Example. Consider the first non-trivial case, $p = 2$. If we choose t_3 and s_2, then we have $\chi_S(M) = \upsilon_k^k s_2 \chi(MS^{-1})$, $\chi_{ST}(M) = \upsilon_k^k s_2 t_3 \chi(MT^{-1}S^{-1})$,

$$\widetilde{P}(z+1) = \begin{pmatrix} e(m) & 0 & 0 \\ 0 & 0 & t_3 \\ 0 & \chi(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix})t_3^{-1} & 0 \end{pmatrix} \widetilde{P}(z),$$

and

$$\widetilde{P}(-1/z) = (\upsilon_k z)^k \begin{pmatrix} 0 & s_2 & 0 \\ s_2^{-1} & 0 & 0 \\ 0 & 0 & \upsilon_k^{-k}\chi(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}) \end{pmatrix} \widetilde{P}(z).$$

In the case of $P_f(z)$, the choices $t_3 = e(1/3)$ and $s_2 = 1$ yield integer coefficients for the holomorphic part of the Fourier expansion of $P_f(z)$, that is, for $f(q)$ and $\omega(q)$.

4. Fourier expansion

In this section we determine the Fourier expansion for each $P_V(z)$. In the previous section we showed that in the case of N prime, these multiplier systems arise from the values $P(N, s, k, m; Vz)$. In other words, $\chi_V(M) = c_V \chi(MV^{-1})$, where c_V is a fixed root of unity. It is easy to see that in the case where N is not prime, we must have a similar occurrence for our function to be a vector-valued modular form, and as we are ultimately interested in $P(N, s, k, m; Vz)$, throughout this section we assume $\chi_V(M) = c_V \chi(MV^{-1})$ for some constant c_V.

We are ready to calculate the Fourier expansion of $P_V(z)$. Here we assume that $\mathrm{Re}(s)$ is sufficiently large. If $V \in \Gamma_0(N)$, then our sum contains

$$\varphi_{s,k,m}(z) = \mathcal{M}_s\left(4\pi m y\right) e\left(-mx\right). \tag{4.1}$$

Now consider the other terms in the series. By the definition of $\chi_V(M)$,

$$\chi_V^{-1}(MT^N) = \chi_V^{-1}(M)\chi^{-1}(VT^N V^{-1})$$

as $VT^N V^{-1} \in \Gamma_0(N)$. Define $0 \le \kappa_V < 1$ by

$$e(\kappa_V) = \chi^{-1}(VT^N V^{-1}),$$

and note that this depends only on the coset of V. In the case of $\Gamma_0(p)$, it is easy to see from the previous section that $e(\kappa_{ST^i}) = \chi^{-1}(ST^p S^{-1})$ for each $1 \le i \le p-1$.

As V and VT^N are in the same coset of $\Gamma_0(N)$, we can rewrite our sum as

$$\begin{aligned} &\sum_c{}' c^{-k} \sum_{d (\mathrm{mod}\, Nc)}{}'' \chi_V^{-1}(M) e\left(\frac{ma}{c}\right) \\ &\times \sum_{n \in \mathbb{Z}} e(\kappa_V n)(z + d/c + Nn)^{-k} \\ &\times \mathcal{M}_s\left(\frac{4\pi m y}{c^2 |z + d/c + N|^2}\right) e\left(\frac{-m}{c^2}\mathrm{Re}\left(\frac{1}{z + d/c + N}\right)\right). \end{aligned}$$

Here $\sum'$ and $\sum''$ indicate that we are summing only over the specific c and d values that fit the congruence conditions modulo N for M to be in the same coset as V. For example, for the coset $\Gamma_0(3)ST$, the first sum is over $c > 0$ such that $(c, 3) = 1$ and the second sum is over $d \pmod{3c}$ such that $d \equiv c \pmod 3$.

Let $K_c^V(u, v, N, \psi)$ be the Kloosterman-type sum given by

$$K_c^V(u, v, N, \psi) := \sum_{d (\mathrm{mod}\, Nc)}{}'' \psi^{-1}\begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix} e\left(\frac{Nu\bar{d} + vd}{Nc}\right), \tag{4.2}$$

where $d\bar{d} \equiv 1 \pmod{c}$, $c\bar{c} \equiv 1 (\operatorname{mod} d)$, and the sum $\sum''$ indicates that this sum is over d such that $\left(\begin{smallmatrix} \bar{d} & -\bar{c} \\ c & d \end{smallmatrix}\right)$ is in the same coset of $\Gamma_0(N)$ as V. If we let $I_k(\bullet)$ and $J_k(\bullet)$ be the usual I-Bessel function and J-Bessel function of order k, then we have the following coefficients for this series.

Theorem 4.1. *Assume the notation above and define the Fourier coefficients* $c(n, y, s)$ *by*

$$P(N, V, s, k, m; z) = c(V) + \sum_{n \in \mathbb{Z}} c(n, y, s) e\left(\frac{n - \kappa_V}{N} x\right).$$

The following are true:

(1) *We have that*

$$c(V) = \begin{cases} \mathcal{M}_s(4\pi m y) e(mx) & \text{if } V \in \Gamma_0(N), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If* $n \in \mathbb{Z}$ *with* $n - \kappa_V > 0$, *then*

$$c(n, y, s) = \frac{i^{-k} 2\pi N^{k/2 - 3/2} |\frac{n - \kappa_V}{m}|^{k/2 - 1/2} \Gamma(2s)}{\Gamma(s + k/2)}$$
$$\times \sum_{c > 0}{}' \frac{K_c^V(m, n - \kappa_V, N, \chi_V)}{c} I_{2s-1}\left(\frac{4\pi}{Nc}\sqrt{|Nm(n - \kappa_V)|}\right) \mathcal{W}_s\left(\frac{4\pi}{N}(n - \kappa_V) y\right).$$

(3) *If* $n \in \mathbb{Z}$ *with* $n - \kappa_V < 0$, *then*

$$c(n, y, s) = \frac{i^{-k} 2\pi N^{k/2 - 3/2} |\frac{n - \kappa_V}{m}|^{k/2 - 1/2} \Gamma(2s)}{\Gamma(s - k/2)}$$
$$\times \sum_{c > 0}{}' \frac{K_c^V(m, n - \kappa_V, N, \chi_V)}{c} J_{2s-1}\left(\frac{4\pi}{Nc}\sqrt{|Nm(n - \kappa_V)|}\right) \mathcal{W}_s\left(\frac{4\pi}{N}(n - \kappa_V) y\right).$$

(4) *If* $n \in \mathbb{Z}$ *with* $n - \kappa_V = 0$, *then*

$$c(n, y, s) = \frac{i^{-k} 4^{1-k/2} \pi^{1+s-k/2} |m|^{s-k/2} y^{1-s-k/2} \Gamma(2s - 1)}{\Gamma(s - k/2)\Gamma(s + k/2)} \sum_{c > 0}{}' \frac{K_c(Nm, 0, N, \chi_V)}{c}.$$

Proof of Theorem 4.1. Consider the function

$$g(z) := \sum_{n \in \mathbb{Z}} (z + n)^k \mathcal{M}_s\left(\frac{4\pi m y}{c^2 |z + n|^2}\right) e\left(\frac{-m}{c^2} \operatorname{Re}\left(\frac{1}{z + n}\right)\right),$$

which has a Fourier expansion in the form

$$g(z) = \sum_{n \in \mathbb{Z}} a_y(n) e^{2\pi i n x},$$

where

$$a_y(n) = \int_{\mathbb{R}} z^k \mathcal{M}_s\left(\frac{4\pi m y}{c^2 |z|^2}\right) e\left(\frac{-mx}{c^2 |z|^2} - nx\right) dx.$$

This integral is computed on page 357 of [8] (see also page 33 of [5]). The theorem follows by using Poisson summation and taking the value of this function at $z + d/c$. ☐

Remark. In some cases we can simplify the formulae above. For example, if $V \in \Gamma_0(N)$, then $e(\kappa_V) = e(mN)$. Fixing $\ell \in \mathbb{Z}$ with $\kappa_V = Mn - \ell$, we get

$$\sum_{\substack{d (\operatorname{mod} c)\\(d,c)=1}} \chi^{-1}(M) e\left(\frac{Nm\bar{d} + ((n-\ell) + Nm)d}{Nc}\right) \sum_{k=0}^{N-1} e\left(\frac{(n-\ell)(k)}{N}\right).$$

The inner sum is N if $N|(n-\ell)$, and is 0 otherwise.

When $s = 1 - k/2$ or $s = k/2$, that is, $\Delta_k P(N, V, s, k, m; z) = 0$, we can use the following identities for $y > 0$ to further simplify the Fourier expansion (see, for example, Chapter 4 of [1]):

$$\mathcal{M}_{k/2}(-y) = e^{y/2}, \tag{4.3}$$

$$\mathcal{M}_{1-k/2}(-y) = (k-1)(\Gamma(1-k, y) - \Gamma(1-k))e^{y/2}, \tag{4.4}$$

$$\mathcal{W}_{1-k/2}(y) = \mathcal{W}_{k/2}(y) = e^{-y/2}, \tag{4.5}$$

$$\mathcal{W}_{1-k/2}(-y) = \mathcal{W}_{k/2}(-y) = e^{y/2}\Gamma(1-k, y). \tag{4.6}$$

Corollary 4.1. *Assume the notation above, and suppose that $k < 0$ and $s = 1-k/2$. Moreover, define the Fourier coefficients $a(n, y, s)$ by*

$$P(N, V, s, k, m; z) = a(V)q^m + \sum_{n \in \mathbb{Z}} a(n, y, s) q^{\frac{n}{N} - \frac{\kappa_V}{N}}.$$

The following are true:

(1) *We have that*

$$a(V) := \begin{cases} (k-1)(\Gamma(1-k, y) - \Gamma(1-k)) & \text{if } V \in \Gamma_0(N), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $n \in \mathbb{Z}$ with $n - \kappa_V > 0$, then*

$$a(n, y, s) = i^{-k} 2\pi N^{k/2-3/2} \left|\frac{n - \kappa_V}{m}\right|^{k/2-1/2} \Gamma(2-k) \times {\sum_{c>0}}' \frac{K_c^V(m, n-\kappa_V, N, \chi_V)}{c} I_{k-1}\left(\frac{4\pi}{Nc}\sqrt{|Nm(n-\kappa_V)|}\right).$$

(3) *If $n \in \mathbb{Z}$ with $n - \kappa_V < 0$, then*

$$a(n, y, s) = i^{-k} 2\pi N^{k/2-3/2} (1-k) \left|\frac{n - \kappa_V}{m}\right|^{k/2-1/2} \Gamma\left(1-k, \frac{-4\pi(n-\kappa_V)y}{N}\right) \times {\sum_{c>0}}' \frac{K_c^V(m, n-\kappa_V, N, \chi_V)}{c} J_{2s-1}\left(\frac{4\pi}{Nc}\sqrt{|Nm(n-\kappa_V)|}\right).$$

(4) *If $n \in \mathbb{Z}$ with $n - \kappa_V = 0$, then*

$$a(n, y, s) = i^{-k} 4^{1-k/2} \pi^{2-k} |m|^{1-k} {\sum_{c>0}}' \frac{K_c^V(m, 0, N, \chi_V)}{c}.$$

If $s = k/2$, the sum simplifies further due to the Gamma factor in the denominator. In this case our function is holomorphic.

Corollary 4.2. *Assume the notation above and suppose that $k > 2$ and $s = k/2$. Define the Fourier coefficients $b(n,s)$ by*

$$P(N, V, s, k, m; z) = b(V)q^m + \sum_{n \in \mathbb{Z} > 0} b(n,s) q^{\frac{n}{N} - \frac{\kappa_V}{N}}.$$

The following are true:

(1) *We have that*

$$b(V) = \begin{cases} 1 & \textit{if } V \in \Gamma_0(N), \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *If $n \in \mathbb{Z}$ with $n > 0$, then*

$$b(n,s) = i^{-k} 2\pi N^{k/2-3/2} \left| \frac{n - \kappa_V}{m} \right|^{k/2-1/2}$$
$$\times {\sum_{c>0}}' \frac{K_c^V(Nm, n - \kappa_V, N, \chi_V)}{c} I_{k-1}\left(\frac{4\pi}{Nc} \sqrt{|Nm(n - \kappa_V)|} \right).$$

5. Maass-Poincaré series for weights 1/2 and 3/2

The mock theta functions are the holomorphic projections of weight 1/2 weak Maass forms. Recall that in this case, the Maass-Poincaré series $P(N, \chi, m, 1/2, 3/4; z)$ is not absolutely convergent, and thus is not well defined. We can, however, make these series well defined, as is seen with $P_f(z)$.

We can define the series $P(N, \chi, m, 1/2, 3/4; z)$ and similarly define the series $P(N, \chi, m, 3/2, 3/4; z)$ through a process of analytic continuation on the related series $P(N, \chi, m, k, 1 - k/2; z)$ and $P(N, \chi, m, k, k/2; z)$, respectively, by taking the value to be the limit of Fourier expansion as $k \to 1/2$ and $k \to 3/2$.

Bringmann and Ono construct a method for this type of argument for $P_f(z)$ (see Section 4 in [4]). In their proof that the Fourier expansion of this series is well defined Bringmann and Ono express the coefficients in terms of "Kloosterman-type" sums, where we take into account the multiplier system χ_f. For half-integral weights, these sums are essentially Salié sums, and can be expressed in terms of binary quadratic forms. For an example of this method, see [4] and [2], respectively.

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