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A COHOMOLOGICAL CHARACTERIZATION OF LEIBNIZ CENTRAL EXTENSIONS OF LIE ALGEBRAS

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ABSTRACT. Mainly motivated by Pirashvili's spectral sequences on a Leibniz algebra, a cohomological characterization of Leibniz central extensions of Lie algebras is given. In particular, as applications, we obtain the cohomological version of Gao's main theorem for Kac-Moody algebras and answer a question in an earlier paper by Liu and Hu (2004).

1. INTRODUCTION

The Leibniz algebras introduced by Loday ([17]) are a nonantisymmetric generalization of Lie algebras. There is a (co)homology theory for these algebraic objects whose properties are similar to those of the classical Chevalley-Eilenberg cohomology theory for Lie algebras. Since a Lie algebra is a Leibniz algebra, it is interesting to study Leibniz (co)homology of Lie algebras since it may provide new invariants for Lie algebras. Lodder ([19]) obtained the Godbillon-Vey invariants for foliations by computing Leibniz cohomology of certain Lie algebras and mentioned how a Leibniz algebra arises naturally from vertex (operator) algebras. Recently, some interrelations with manifolds were investigated, which could lead to possible applications of Leibniz (co)homology in geometry (see [11, 19], etc.).

Central extensions play a central role in the theory of Lie algebras (see [1, 2, 3, 6, 7, 8, 13, 14, 15, 20, 22], etc.). Viewing a Lie algebra as a Leibniz algebra, it is natural to determine its Leibniz central extensions and compare the differences between Leibniz and Lie central extensions. Loday and Pirashvili ([18]) have shown that the Virasoro algebra is a universal central extension of the Witt algebra in the category of Leibniz algebras as well. It is well known that any Kac-Moody Lie algebra $\mathfrak{g}(A)$ is centrally closed ([8]). However, its universal central extension in the category of Leibniz algebras is not centrally closed for affine type as witnessed by Gao in [9]. Other cases of infinite-dimensional Lie algebras were discussed in [16, 25]. Their approach involves technical and lengthy computations.

The purpose of this note is to give a concise determination of the universal central extension of a class of Lie algebras in the category of Leibniz algebras.

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Our motivation comes from the long exact sequence used by Pirashvili ([19, 21]). Note that the invariant symmetric bilinear forms and Leibniz central extensions are connected by the Cartan-Koszul homomorphism (see [14]). In practice, it is easier to handle the invariant symmetric bilinear forms on a Lie algebra than to directly determine its Leibniz central extensions. On the other hand, we notice that determining Leibniz central extensions of a Lie algebra is equivalent to treating dual space derivations of this Lie algebra (cf. [7]). We combine these observations to study the (universal) Leibniz central extensions of Lie algebras. The current method avoids complicated computations when applied to some classes of infinite-dimensional Lie algebras we are interested in.

The organization of this note is as follows. Section 2 introduces some basic notions on Leibniz algebras. In Section 3, we present a short exact sequence as a variation of Pirashvili's long exact sequence. We then go on to describe the Leibniz central extensions of Lie algebras, in terms of the invariant symmetric bilinear forms and the Cartan-Koszul homomorphism (see Corollary 3.3), and to obtain the second Leibniz cohomology group with trivial coefficients by dual space derivations (Theorem 3.5). Section 4 provides some applications based on Corollary 3.3. As a consequence, a cohomological version of Gao's result ([9]) in the Kac-Moody algebras case (Theorem 4.18) is given. In Section 5, we study the Leibniz central extensions of the quadratic Lie algebras and construct a counterexample to address a question posed in [16].

2. Prerequisites on Leibniz algebras

2.1. **Leibniz algebra.** Let $\mathbb{K}$ be an algebraically closed field with $\operatorname{char}\mathbb{K} = 0$.

Definition 2.1. A Leibniz algebra is a $\mathbb{K}$-module L with a bilinear map $[-,-] : L \times L \longrightarrow L$ satisfying the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, for $x, y, z \in L$.

The center of L is defined as $\{z \in L \mid [z,L] = [L,z] = 0\}$. L is called *perfect* if $[L,L] = L$. If, in addition, $[x,x] = 0$, $\forall\, x \in L$, the Leibniz identity is equivalent to the Jacobi identity. In particular, Lie algebras are examples of Leibniz algebras.

Definition 2.2. Let L be a Leibniz algebra over $\mathbb{K}$. M is called a representation of L if M is a $\mathbb{K}$-vector space equipped with two actions (left and right) of L, i.e., $[-,-] : L \times M \longrightarrow M$ and $[-,-] : M \times L \longrightarrow M$ satisfying

$$\begin{aligned}
&(MLL) \qquad [m,[x,y]] = [[m,x],y] - [[m,y],x],\\
&(LML) \qquad [x,[m,y]] = [[x,m],y] - [[x,y],m],\\
&(LLM) \qquad [x,[y,m]] = [[x,y],m] - [[x,m],y],
\end{aligned}$$

for any $m \in M$ and $x, y \in L$.

2.2. **Cohomology of Leibniz algebras.** Let L be a Leibniz algebra over $\mathbb{K}$, and let M be a representation of L. Denote $C^n(L,M) := \operatorname{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$, $n \geq 0$. The

Loday coboundary map $d^n : C^n(L,M) \to C^{n+1}(L,M)$ is defined by

$$\begin{aligned}(d^n f)(x_1,\dots,x_{n+1}) &= [x_1, f(x_2,\dots,x_{n+1})] \\ &+ \sum_{i=2}^{n+1}(-1)^i[f(x_1,\dots,\hat{x}_i,\dots,x_{n+1}),x_i] \\ &+ \sum_{1\le i<j\le n+1}(-1)^{j+1}f(x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\hat{x}_j,\dots,x_{n+1}).\end{aligned}$$

Clearly, $d^{n+1}d^n = 0,\ n \geq 0$. $(C^*(L,M),d)$ is a well-defined cochain complex, whose cohomology is called the cohomology of the Leibniz algebra L with coefficients in the representation M: $\mathrm{HL}^*(L,M) := \mathrm{H}^*((C^*(L,M),d))$. Similarly, we have a chain complex $(C_*(L,M),d)$, whose homology is called the homology of the Leibniz algebra L with coefficients in the representation M: $\mathrm{HL}_*(L,M) := \mathrm{H}_*((C_*(L,M),d))$.

2.3. **Cohomology of Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let M be a $\mathfrak{g}$-module. Denote the Chevalley-Eilenberg cochain complex by

$$(\Omega^*(\mathfrak{g},M),\ \delta) := (\mathrm{Hom}(\wedge^*\mathfrak{g},M),\ \delta),$$

where δ is the Chevalley-Eilenberg coboundary map defined by

$$\begin{aligned}(\delta^n f)(x_1,\dots,x_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1}x_i\cdot f(x_1,\dots,\hat{x}_i,\dots,x_{n+1}) \\ &+ \sum_{1\le i<j\le n+1}(-1)^{i+j}f([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}).\end{aligned}$$

Then $\mathrm{H}^*(\mathfrak{g},M) := \mathrm{H}^*((\Omega^*(\mathfrak{g},M),\delta))$ is called the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M.

3. Leibniz cohomology and Leibniz central extensions of Lie algebras

3.1. **Central extensions of Leibniz algebras.** A central extension of L is a pair $(\hat{L},\pi)$, where $\hat{L}$ is a Leibniz algebra and $\pi : \hat{L} \longrightarrow L$ is a surjective homomorphism whose kernel lies in the center of $\hat{L}$. The pair $(\hat{L},\pi)$ is a universal central extension of L if for every central extension $(\tilde{L},\tau)$ of L, there is a unique homomorphism $\psi : \hat{L} \longrightarrow \tilde{L}$ for which $\tau\circ\psi = \pi$. The following result is known.

Proposition 3.1 ([18])**.** *There exists a one-to-one correspondence between the set of equivalent classes of one-dimensional Leibniz central extensions of L by $\mathbb{K}$ and the second Leibniz cohomology group $\mathrm{HL}^2(L,\mathbb{K})$.*

3.2. **Invariant symmetric bilinear forms.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let M be a $\mathfrak{g}$-module. A symmetric bilinear form ϕ on $\mathfrak{g}$ is called $\mathfrak{g}$-invariant if ϕ satisfies $\phi([x,y],z) = \phi(x,[y,z]),\ \forall\ x,y,z \in \mathfrak{g}$. Let $\mathrm{B}(\mathfrak{g},\mathbb{K})$ stand for the set of all $\mathbb{K}$-valued symmetric $\mathfrak{g}$-invariant bilinear forms on the Lie algebra $\mathfrak{g}$.

The short exact sequence below is a variation of the first 5 terms of Pirashvili's long exact sequence (see [21]).

Proposition 3.2. *For any Lie algebra $\mathfrak{g}$, there is an exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\mathfrak{g},\mathbb{K}) \xrightarrow{f} \mathrm{HL}^2(\mathfrak{g},\mathbb{K}) \xrightarrow{g} \mathrm{B}(\mathfrak{g},\mathbb{K}) \xrightarrow{h} \mathrm{H}^3(\mathfrak{g},\mathbb{K}),$$

where f is the natural embedding map, g is defined by $g(\alpha)(x,y)=\alpha(x,y)+\alpha(y,x)$, $\forall\ \alpha\in \mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, $x,\ y\in\mathfrak{g}$, and h is the Cartan-Koszul map ([14]) *defined by $h(\alpha)(x,y,z)=\alpha([x,y],z)$, $\forall\,\alpha\in \mathrm{B}(\mathfrak{g},k)$, $x,\ y,\ z\in\mathfrak{g}$.*

Corollary 3.3. $\frac{\mathrm{HL}^2(\mathfrak{g},\mathbb{K})}{\mathrm{H}^2(\mathfrak{g},\mathbb{K})}=\ker(h)$. *In particular,* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})=\mathrm{H}^2(\mathfrak{g},\mathbb{K})$ *if and only if* $\ker(h)=0$.

Remark 3.4. Note that the natural embedding $\Omega^*(\mathfrak{g},M)\hookrightarrow C^*(\mathfrak{g},M)$ induces a short exact sequence in the category of cochain complexes of the Lie algebra $\mathfrak{g}$:

$$0\longrightarrow\Omega^*(\mathfrak{g},M)\longrightarrow C^*(\mathfrak{g},M)\longrightarrow C^*_{rel}(\mathfrak{g},M)[2]\longrightarrow 0,$$

where $C^*_{rel}(\mathfrak{g},M)[2]:=\frac{C^*(\mathfrak{g},M)}{\Omega^*(\mathfrak{g},M)}$ is the quotient cochain complex. Pirashvili's long exact sequence below (see [19], [21])

$$0\to \mathrm{H}^2(\mathfrak{g},M)\to \mathrm{HL}^2(\mathfrak{g},M)\to \mathrm{H}^0_{rel}(\mathfrak{g},M)\to \mathrm{H}^3(\mathfrak{g},M)\to \mathrm{HL}^3(\mathfrak{g},M)\to\cdots$$

and isomorphisms $\mathrm{HL}^i(\mathfrak{g},M)=\mathrm{H}^i(\mathfrak{g},M)$, $i=0,\,1$, are the main tools to compare the higher Lie and Leibniz cohomology groups of a Lie algebra.

Let $M=\mathbb{K}$. We have

$$0\to \mathrm{H}^2(\mathfrak{g},\mathbb{K})\to \mathrm{HL}^2(\mathfrak{g},\mathbb{K})\to \mathrm{H}^0_{rel}(\mathfrak{g},\mathbb{K})\to \mathrm{H}^3(\mathfrak{g},\mathbb{K})\to \mathrm{HL}^3(\mathfrak{g},\mathbb{K})\to\cdots.$$

As a consequence of the spectral sequences, Pirashvili claimed (with no proof, see [21]) : $\mathrm{H}^0_{rel}(\mathfrak{g},\mathbb{K})\cong \mathrm{B}(\mathfrak{g},\mathbb{K})$, while a direct elementary proof of Proposition 3.2 can be found in arXiv.org: math.QA/0605399 by the authors.

3.3. **Dual space derivations.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let M be a $\mathfrak{g}$-module. A linear map $d:\mathfrak{g}\longrightarrow M$ is a derivation if $d([x,y])=x\cdot d(y)-y\cdot d(x)$, for $x,\ y\in\mathfrak{g}$. The derivations of the form $x\mapsto x\cdot m$ for some $m\in M$ are called inner derivations. $\mathrm{Der}(\mathfrak{g},M)$ and $\mathrm{Inn}(\mathfrak{g},M)$ denote the spaces of derivations and inner derivations, respectively. Clearly, $\mathrm{H}^1(\mathfrak{g},M)=\mathrm{Der}(\mathfrak{g},M)/\mathrm{Inn}(\mathfrak{g},M)$ is the first cohomology group of $\mathfrak{g}$ with coefficients in M.

Theorem 3.5. $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})=\mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)=\mathrm{Der}(\mathfrak{g},\mathfrak{g}^*)\,/\,\mathrm{Inn}(\mathfrak{g},\mathfrak{g}^*)$, *for any Lie algebra* $\mathfrak{g}$, *where* $\mathfrak{g}^*$ *is the dual* $\mathfrak{g}$*-module.*

Proof. Define a map $\theta:\mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)\longrightarrow \mathrm{HL}^2(\mathfrak{g},\mathbb{K})$ as $\theta(\alpha)(x,y):=\alpha(y)(x)$ for any $\alpha\in\mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$ and $x,\ y\in\mathfrak{g}$. Indeed, since

$$\begin{aligned}\theta(\alpha)(x,[y,z])&=\alpha([y,z])(x)=(y\cdot\alpha(z))(x)-(z\cdot\alpha(y))(x)\\&=\alpha(z)([x,y])-\alpha(y)([x,z])=\theta(\alpha)([x,y],z)+\theta(\alpha)([z,x],y),\end{aligned}$$

$\theta(\alpha)\in\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, θ is well defined. If $\theta(\alpha)=\bar{0}$ in $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, there exists an element β in $C^1(\mathfrak{g},\mathbb{K})$ such that, for any $x,y\in\mathfrak{g}$, $d^1(\beta)(x,y)=\theta(\alpha)(x,y)$. That is, $-\beta([x,y])=\alpha(y)(x)$. So α is a 1-coboundary in $\Omega^1(\mathfrak{g},\mathfrak{g}^*)$, i.e., $\alpha=\bar{0}\in\mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$. Hence, θ is injective. On the other hand, for any $\beta\in\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, we define a map $\alpha\in\mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{g}^*)$ by $\alpha(x)(y):=\beta(y,x)$, for any $x,y\in\mathfrak{g}$. Since

$$\begin{aligned}\alpha([x,y])(z)&=\beta(z,[x,y])=\beta([z,x],y)-\beta([z,y],x)=\alpha(y)([z,x])-\alpha(x)([z,y]),\\&=(x\cdot\alpha(x))(z)-(y\cdot\alpha(x))(z),\qquad\text{for any}\ \ z\in\mathfrak{g},\end{aligned}$$

we get $\alpha([x,y])=x\cdot\alpha(x)-y\cdot\alpha(x)$ for any $x,y\in\mathfrak{g}$, which means $\alpha\in\mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$. Hence, θ is surjective. □

Denote $\mathrm{SDer}(\mathfrak{g},\mathfrak{g}^*):=\{\phi\in\mathrm{Der}(\mathfrak{g},\mathfrak{g}^*)\mid\phi(x)(y)+\phi(y)(x)=0,\ \forall\ x,\ y\in\mathfrak{g}\}$. By Theorem 3.5 above and Proposition 1.3 (2) in [6], we have

Corollary 3.6. $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})/\,\mathrm{H}^2(\mathfrak{g},\mathbb{K}) = \mathrm{Der}(\mathfrak{g},\mathfrak{g}^*)\,/\,\mathrm{SDer}(\mathfrak{g},\mathfrak{g}^*)\big(\subset H^0_{rel}(\mathfrak{g},\mathbb{K})\big)$.

Corollary 3.7. *If* $\mathfrak{g}$ *is a finite-dimensional Lie algebra over* $\mathbb{K}$ *with a nondegenerate invariant symmetric bilinear form* ψ*, then* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{Der}(\mathfrak{g},\mathfrak{g})\,/\,\mathrm{Inn}(\mathfrak{g},\mathfrak{g})$. *If, in addition,* $\mathfrak{g}$ *is simple, then* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = 0$.

4. Applications: Leibniz central extensions of some Lie algebras

4.1. **Lie algebras of Virasoro type.** Let $\mathbb{K}[t,t^{-1}]$ be the Laurent polynomial algebra over $\mathbb{K}$ and $\frac{d}{dt}$ be the differential operator on $\mathbb{K}[t,t^{-1}]$. Set $L_n = -t^{n+1}\frac{d}{dt}$ and $I_n = t^n$ for $n \in \mathbb{Z}$.

Definition 4.1. The Witt algebra $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}L_n$ is defined by $[L_m, L_n] = (m-n)L_{m+n},\ m,\, n \in \mathbb{Z}$.

It is well known that $\mathrm{H}^2(\mathcal{W},\mathbb{K}) = \mathrm{H}_2(\mathcal{W},\mathbb{K}) = \mathbb{K}\alpha$, $\alpha(L_m, L_n) = \delta_{m+n,0}\frac{m^3-m}{12}$.

Definition 4.2. ([1]) The Lie algebra $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{C}I_m$ is defined by $[L_m, L_n] = (m-n)L_{m+n}$, $[I_m, I_n] = 0$, $[L_m, I_n] = -nI_{m+n}$, $m,\, n \in \mathbb{Z}$.

In [1], the authors proved that $\dim \mathrm{H}^2(\mathcal{H},\mathbb{K}) = \dim \mathrm{H}_2(\mathcal{H},\mathbb{K}) = 3$. In fact, as the universal central extension of $\mathcal{H}$, the twisted Heisenberg-Virasoro algebra H_{Vir} has a basis $\{I_m, L_m, C_I, C_L, C_{LI} \mid m \in \mathbb{Z}\}$ satisfying the following relations:

$$\begin{aligned}
[I_m, I_n] &= n\delta_{m+n,0}C_I,\\
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(m^3-m)C_L,\\
[L_m, I_n] &= -nI_{m+n} + \delta_{m+n,0}(m^2-m)C_{LI},\\
[H_{\mathrm{Vir}}, C_L] &= [H_{\mathrm{Vir}}, C_I] = [H_{\mathrm{Vir}}, C_{LI}] = 0.
\end{aligned}$$

Definition 4.3. The Lie algebra $\mathcal{D} = \bigoplus_{m\in\mathbb{Z}, n\in\mathbb{Z}_+} \mathbb{K}t^mD^n$ of differential operators is defined by $[\,t^mD^r, t^nD^s\,] = t^{m+n}((D+n)^r - (D+m)^s)$, $\quad m,\, n \in \mathbb{Z},\ r,\, s \in \mathbb{Z}_+$, where $D = t\frac{d}{dt}$.

By [15], $\dim \mathrm{H}^2(\mathcal{D},\mathbb{K}) = \dim \mathrm{H}_2(\mathcal{D},\mathbb{K}) = 1$, and the universal central extension of $\mathcal{D}$ is a Lie algebra $\mathcal{W}_{1+\infty} = \bigoplus_{m\in\mathbb{Z}, n\in\mathbb{Z}_+} \mathbb{K}t^mD^n \oplus \mathbb{K}C$ with definition $[t^mD^r, t^nD^s] = t^{m+n}((D+n)^r - (D+m)^s) + \psi(t^mD^r, t^nD^s)C$, $\psi(t^{m+r}D^r, t^{n+s}D^s) = \delta_{m+n,0}(-1)^r r!s!\binom{m+r}{r+s+1}$, for $m,\, n \in \mathbb{Z},\ r,\, s \in \mathbb{Z}_+$.

Remark 4.4. $\mathcal{W}$ and $\mathcal{H}$ are Lie subalgebras of $\mathcal{D}$.

Proposition 4.5. *There is no nontrivial invariant symmetric bilinear form on a Lie algebra* $\mathfrak{g}$*, where* $\mathfrak{g} = \mathcal{W}, \mathcal{H}$*, or* $\mathcal{D}$*.*

Proof. Assume that f is an invariant symmetric bilinear form on $\mathfrak{g}$. Note that

(1) $\mathfrak{g} = \mathcal{W}$ is generated as a Lie algebra by L_{-2}, L_3 with $f(L_i, L_j) = 0$ for $i,\, j \in \{-2, 3\}$. So $f \equiv 0$.

(2) $\mathfrak{g} = \mathcal{H}$ is generated as a Lie algebra by L_{-2}, L_3 and I_1 with $f(x, y) = 0$ for $x, y \in \{L_{-2}, L_3, I_1\}$. So $f \equiv 0$.

(3) $\mathfrak{g} = \mathcal{D}$ is generated as a Lie algebra by t, t^{-1} and D^2 (see [23]). By a direct computation, we have

$$\begin{aligned}
t^2(\frac{d}{dt})^2 &= D^2 - D,\\
t^m(\frac{d}{dt})^n &= \frac{1}{n+1}\Big[t^m(\frac{d}{dt})^{n+1}, t\Big], \quad m \in \mathbb{Z},\, n \in \mathbb{Z}_+.
\end{aligned}$$

By (1) and (2), $f(t^{\pm 1}, t^{\pm 1}) = f(t^{\pm 1}, D) = f(D, D) = 0$. Then

$$\begin{aligned} f\Big(t(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) &= f\Big(t(\frac{d}{dt})^2, -\frac{1}{3}[t, t^2(\frac{d}{dt})^3]\Big) = -\frac{1}{3} f\Big([t(\frac{d}{dt})^2, t], t^2(\frac{d}{dt})^3\Big) \\ &= -\frac{2}{3} f\Big(t\frac{d}{dt}, t^2(\frac{d}{dt})^3\Big) = -\frac{2}{3} f\Big(t\frac{d}{dt}, -\frac{1}{4}[t, t^2(\frac{d}{dt})^4]\Big) \\ &= \frac{1}{6} f\Big([t\frac{d}{dt}, t], t^2(\frac{d}{dt})^4\Big) = \frac{1}{6} f\Big(t, t^2(\frac{d}{dt})^4\Big) \\ &= \frac{1}{6} f\Big(t, -\frac{1}{5}[t, t^2(\frac{d}{dt})^5]\Big) = -\frac{1}{30} f\Big([t, t], t^2(\frac{d}{dt})^5\Big) \\ &= 0. \end{aligned}$$

Similarly, we have $f(D, t^2(\frac{d}{dt})^2) = 0$. Then,

$$\begin{aligned} f(D^2, D^2) &= f\Big(D + t^2(\frac{d}{dt})^2, D + t^2(\frac{d}{dt})^2\Big) \\ &= f(D, D) + 2f\Big(D, t^2(\frac{d}{dt})^2\Big) + f\Big(t^2(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) \\ &= 2f\Big(D, t^2(\frac{d}{dt})^2\Big) + f\Big(t^2(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) \\ &= 0. \end{aligned}$$

In summary, $B(\mathfrak{g}, \mathbb{K}) = 0$ for $\mathfrak{g} = \mathcal{W}$, $\mathcal{H}$, or $\mathcal{D}$. □

Corollary 4.6. $\mathrm{HL}^2(\mathfrak{g}, \mathbb{K}) = \mathrm{H}^2(\mathfrak{g}, \mathbb{K})$*, for* $\mathfrak{g} = \mathcal{W}$, $\mathcal{H}$*, or* $\mathcal{D}$.

Remark 4.7. Corollary 4.6 for $\mathfrak{g} = \mathcal{W}$ was obtained by [18], and for $\mathfrak{g} = \mathcal{D}$ by [16].

4.2. Lie algebras of Block type.

Definition 4.8 ([5])**.** Let A be a torsion-free abelian group. Then $\phi : A \times A \longrightarrow \mathbb{K}$ is a nondegenerate, skew-symmetric, $\mathbb{Z}$-bilinear function. Then the degenerate Block algebra $\mathcal{L}(A, \phi) = \bigoplus_{x \in A - \{0\}} \mathbb{K}e_x$ is defined by $[e_x, e_y] = \phi(x, y)e_{x+y}$.

Definition 4.9 ([13])**.** Let $A = \mathbb{Z} \times \mathbb{Z}$, $\phi((m, n), (m_1, n_1)) = nm_1 - mn_1$ be a skew-symmetric bi-additive function. The Virasoro-like algebra

$$\mathcal{V} = \bigoplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \mathbb{K}e_{m,n}$$

is defined by $[e_{m,n}, e_{m_1,n_1}] = (nm_1 - mn_1)e_{m+n,m_1+n_1}$.

Definition 4.10 ([13])**.** Let $A = \mathbb{Z} \times \mathbb{Z}$, $\phi((m, n), (m_1, n_1)) = q^{nm_1} - q^{mn_1}$ with a fixed $q \in \mathbb{K}^*$ (q not a root of unity), where ϕ is a skew-symmetric bi-additive function. Then the q-analogue Virasoro-like algebra $\mathcal{V}_q = \bigoplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \mathbb{K}e_{m,n}$ is defined by $[e_{m,n}, e_{m_1,n_1}] = (q^{nm_1} - q^{mn_1})e_{m+n,m_1+n_1}$.

Lemma 4.11. ([25]) $\mathrm{B}(\mathcal{L}(A, \phi), \mathbb{K}) = \mathbb{K}\alpha$*, where* $\alpha(e_x, e_y) = \delta_{x+y,0}$.

Proposition 4.12. $\mathrm{HL}^2(\mathcal{L}(A, \phi), \mathbb{K}) = \mathrm{H}^2(\mathcal{L}(A, \phi), \mathbb{K})$.

Proof. By Corollary 3.3 and Lemma 4.11, it suffices to prove that the image of α under the Cartan-Koszul homomorphism h is nonzero. If $h(\alpha) = \bar{0} \in \mathrm{H}^3(\mathfrak{g}, \mathbb{K})$, there is a $\psi \in \Lambda^2\mathfrak{g}$ such that

$$\begin{aligned} h(\alpha)(e_x, e_y, e_z) &= \alpha([e_x, e_y], e_z) = d(\psi)(e_x, e_y, e_z) \\ &= \psi(e_x, [e_y, e_z]) + \psi(e_y, [e_z, e_x]) + \psi(e_z, [e_x, e_y]), \end{aligned}$$

for $e_x, e_y, e_z \in \mathcal{L}(A,\phi)$. Let $x+y+z=0$ and $\phi(x,y)\neq 0$. Then

$$\phi(x,y)\psi(e_x,e_{-x})+\phi(x,y)\psi(e_y,e_{-y})+\phi(x,y)\psi(e_z,e_{-z})=\phi(x,y);$$

that is, $\psi(e_x,e_{-x})+\psi(e_y,e_{-y})+\psi(e_z,e_{-z})=1$. On the other hand, since $-x-y-z=0$, the above identity holds for $-x,-y,-z$. As ψ is skew-symmetric, we have $\psi(e_x,e_{-x})+\psi(e_y,e_{-y})+\psi(e_z,e_{-z})=-1$. This is impossible. So $h(\alpha)\neq\bar{0}$. □

It follows from [5] that

Corollary 4.13.

$$\mathrm{HL}^2(\mathcal{L}(A,\phi),\mathbb{K})=\{\,[\alpha_\mu]\mid \alpha_\mu(x,y)=\delta_{x+y,0}\mu(x),\forall\ x,\ y\in A,\ \ \mu\in \mathrm{Hom}_{\mathbb{K}}(A,\mathbb{K})\}.$$

Remark 4.14. With a different method, Corollary 4.13 was given in [25]. For the (q-analogue) Virasoro-like algebras, the same result was given in [16].

4.3. **Kac-Moody algebras.**

Lemma 4.15. *For a Lie algebra* $\mathfrak{g}$ *with* $\dim \mathrm{B}(\mathfrak{g},\mathbb{K})\leq 1$*, in both cases below*:

- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K})=0$; *or*
- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K})=\mathbb{K}\phi$ $(\phi\neq 0)$*, and there exists a subalgebra* $\mathfrak{a}\cong\mathfrak{sl}(2,\mathbb{K})$ *such that* $\phi|_{\mathfrak{a}}\neq 0$*,*

then $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})=\mathrm{H}^2(\mathfrak{g},\mathbb{K})$.

Proof. The first case is clear by Corollary 3.3. Now for nonzero $\phi\in\mathrm{B}(\mathfrak{g},\mathbb{K})$, if there is a subalgebra $\mathfrak{a}$ as an $\mathfrak{sl}(2,\mathbb{K})$-copy such that $\phi|_{\mathfrak{a}}\neq 0$, then $\phi|_{\mathfrak{a}}$ is a nonzero scalar multiple of the Killing form on $\mathfrak{a}$. Let h be the Cartan-Koszul homomorphism: $h(\phi)(a,b,c)=\phi([a,b],c)$ for any $a,b,c\in\mathfrak{g}$. If $h(\phi)=\bar{0}$, then there is a $\theta\in\Lambda^2\mathfrak{g}$ such that $\delta^2(\theta)=h(\phi)$, i.e., $\theta(a,[b,c])+\theta(b,[c,a])+\theta(c,[a,b])=\phi([a,b],c)$, for $a,\ b,\ c\in\mathfrak{g}$. Take $x,\ y,\ h\in\mathfrak{a}$ satisfying $[x,y]=h,\ [h,y]=-2y,\ [h,x]=2x$. Then

$$\theta(x,[y,h])+\theta(y,[h,x])+\theta(h,[x,y])=\phi([x,y],h),$$
$$2\theta(x,y)+2\theta(y,x)+\theta(h,h)=\phi(h,h).$$

Therefore, $\phi(h,h)=0$. This contradicts the property of the Killing form. □

Lemma 4.16. $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})=\mathrm{Hom}(\mathrm{HL}_2(\mathfrak{g},\mathbb{K}),\mathbb{K})$*, for any perfect Leibniz algebra* $\mathfrak{g}$.

Proof. Following the universal coefficient theorem for Leibniz algebras in [4], one has the following short exact sequence:

$$0\longrightarrow \mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K})\longrightarrow \mathrm{HL}^2(\mathfrak{g},\mathbb{K})\longrightarrow \mathrm{Hom}(\mathrm{HL}_2(\mathfrak{g},\mathbb{K}),\mathbb{K})\longrightarrow 0,$$

where $\mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K})$ denotes the abelian extension of $\mathrm{HL}_1(\mathfrak{g},\mathbb{K})$ by $\mathbb{K}$. It is clear that $\mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K})=0$ since $\mathrm{HL}_1(\mathfrak{g},\mathbb{K})=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]=0$. □

Let $\mathfrak{g}$ be a perfect Lie algebra ($[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$) over $\mathbb{K}$. Then, by [10], there exist a universal central extension $\pi:\hat{\mathfrak{g}}\twoheadrightarrow\mathfrak{g}$ in the category of Leibniz algebras and a universal central extension $\tilde{\pi}:\tilde{\mathfrak{g}}\twoheadrightarrow\mathfrak{g}$ in the category of Lie algebras.

Lemma 4.17. $\mathrm{HL}_2(\tilde{\mathfrak{g}},\mathbb{K})=\ker\{\mathrm{HL}_2(\mathfrak{g},\mathbb{K})\twoheadrightarrow\mathrm{H}_2(\mathfrak{g},\mathbb{K})\}$.

Proof. See 4.6 in [18] or Corollary 2.7 in [10]. □

Let R be a unital, commutative and associative algebra over $\mathbb{K}$. The R-module of Kähler differentials $\Omega^1_{R|\mathbb{K}}$ is generated by $\mathbb{K}$-linear symbols $\mathrm{d}\,a$ for $a \in R$ with the relation $\mathrm{d}(ab) = a\,\mathrm{d}\,b + b\,\mathrm{d}\,a$, for any a, $b \in R$. In particular, if $R = \mathbb{K}[t, t^{-1}]$, then $\Omega^1_{R|\mathbb{K}} = \bigoplus_{m\in\mathbb{Z}} \mathbb{K}t^m\,\mathrm{d}\,t$. For any positive integer r, let

$$\Omega^1_{R|\mathbb{K}}(r) = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}t^{ir-1}\,\mathrm{d}\,t, \quad \Omega'^1_{R|\mathbb{K}}(r) = \bigoplus_{i\in\mathbb{Z}-\{0\}} \mathbb{K}t^{ir-1}\,\mathrm{d}\,t$$

be two $\mathbb{K}$-subspaces of $\Omega^1_{R|\mathbb{K}}$. Then $\Omega'^1_{R|\mathbb{K}}(r)$ is a subspace of $\Omega^1_{R|\mathbb{K}}(r)$ of codimension 1.

Theorem 4.18. *Let $A = (a_{ij})_{1\le i,j\le n}$ denote an $n \times n$-matrix of rank ℓ with entries in $\mathbb{K}$. Using the notation in* [12]*, denote by $\mathfrak{g}(A)$ the Kac-Moody Lie algebra associated to A. Let $\mathfrak{g}'(A)$ be the derived algebra of $\mathfrak{g}(A)$, $\mathfrak{c}$ the center of $\mathfrak{g}'(A)$, and $\bar{\mathfrak{g}}(A) = \mathfrak{g}'(A)/\mathfrak{c}$. Thus,*

(1) *if $a_{ii} \neq 0$, $1 \le i \le n$, then* $\dim \mathrm{HL}^2(\mathfrak{g}(A), \mathbb{K}) = (n-\ell)^2$*;*

(2) *if A is an indecomposable generalized Cartan matrix of affine $X_n^{(r)}$ type, then*

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = (\Omega^1_{R|\mathbb{K}}(r))^*, \qquad \mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = (\Omega'^1_{R|\mathbb{K}}(r))^*;$$

(3) *if A is an indecomposable generalized Cartan matrix of nonaffine type, then*

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*, \qquad \mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = 0.$$

Proof. (1) Theorem 3.2 in [7] tells us that $\dim \mathrm{H}^1(\mathfrak{g}(A), (\mathfrak{g}(A))^*) = (n-\ell)^2$ under the assumption $a_{ii} \neq 0$ for $1 \le i \le n$. Then Theorem 3.5 above gives the result.

(2) Let A be an indecomposable generalized Cartan matrix of affine $X_n^{(r)}$ type. By Gabber-Kac's radical theorem (Theorem 9.11 and remarks in [12], p. 159), $\mathfrak{g}'(A)$ and $\bar{\mathfrak{g}}(A)$ can be presented in term of $3n$ generators f_i, h_i, e_i $(1 \le i \le n)$ and the Chevalley-Serre relations. By Theorem 3.17 in [9],[1] which says that $\mathrm{HL}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \Omega^1_{R|\mathbb{K}}(r)$, together with Lemma 4.16, we get $\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = (\Omega^1_{R|\mathbb{K}}(r))^*$.

Because $\mathfrak{g}'(A)$ is the universal covering of $\bar{\mathfrak{g}}(A)$ in the category of Lie algebras (see [22]), we have the following exact sequence:

$$0 \longrightarrow \mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) \longrightarrow \mathfrak{g}'(A) \longrightarrow \bar{\mathfrak{g}}(A) \longrightarrow 0,$$

where $\mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \Omega^1_{R|\mathbb{K}}(r)/\Omega'^1_{R|\mathbb{K}}(r) = t^{-1}\,\mathrm{d}\,t$. Lemma 4.17 then yields

$$\mathrm{HL}_2(\mathfrak{g}'(A), \mathbb{K}) = \ker\{\mathrm{HL}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) \twoheadrightarrow \mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K})\} = \Omega'^1_{R|\mathbb{K}}(r).$$

So Lemma 4.16 gives the second result.

(3) Let A be an indecomposable generalized Cartan matrix of nonaffine type. Recall Berman's Theorem 3.1 in [3], which says that $\bar{\mathfrak{g}}(A)$ possesses a nondegenerate invariant symmetric bilinear form ϕ if and only if A is symmetrizable. Moreover, such forms on $\bar{\mathfrak{g}}(A)$ are unique up to scalars.

If A is symmetrizable, then Berman's result above means $\dim \mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) \ge 1$. On the other hand, for any $0 \neq \psi \in \mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K})$, ψ is necessarily nondegenerate because $\bar{\mathfrak{g}}(A)$ is simple (by Theorem 4.3 [12], Exercise 1.4 [12]). So Berman's result above insures $\mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathbb{K}\phi$. The nondegeneracy of ϕ implies the nontrivial property of its restriction to an $\mathfrak{sl}(2, \mathbb{K})$-copy (see [2]). Lemma 4.15 shows that $\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathrm{H}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*$ (the 2nd "=" was proved by Theorem 2.3 [3]).

[1]Note that the author in [9] used different notation.

If A is nonsymmetrizable, there exists no nondegenerate invariant symmetric bilinear form on $\bar{\mathfrak{g}}(A)$. Since $\bar{\mathfrak{g}}(A)$ is simple (by Theorem 4.3 [12], Exercise 1.4 [12]), there is no invariant symmetric bilinear form on $\bar{\mathfrak{g}}(A)$, i.e., $\mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) = 0$. Then Corollary 3.3 or Lemma 4.15 gives $\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathrm{H}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*$ (the 2nd "=" was proved by Theorem 2.3 [3]).

Furthermore, since $\mathfrak{g}'(A)$ is the universal Leibniz central extension of $\bar{\mathfrak{g}}(A) = \mathfrak{g}'(A)/\mathfrak{c}$, consequently, $\mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = 0$. ☐

Remark 4.19. Theorem 4.18 (2), (3) give a criterion to distinguish between affine and nonaffine Kac-Moody algebras by means of the vanishing property of the second Leibniz cohomology groups of $\mathfrak{g}'(A)$ with trivial coefficients, where the homological versions of Theorem 4.18 (2) and the second statement of (3) were due to Gao ([9]). However, strictly speaking, owing to Gabber-Kac's Theorem (see [12]), the definition of $\mathfrak{g}'(A)$ for nonsymmetrizable cases adopted by Gao is different from ours used here, since our $\mathfrak{g}'(A)$ in Kac's notation is the quotient of the former. So in this sense, we get the same result for the quotient object.

5. Quadratic Leibniz algebras and their central extensions

5.1. Quadratic Leibniz algebra.

Definition 5.1. $(\mathfrak{g}, \phi)$ is called a quadratic Leibniz algebra if ϕ is a symmetric invariant bilinear form on the Leibniz algebra $\mathfrak{g}$.

Lemma 5.2. *Let $\mathfrak{g}$ be a Lie algebra. If $(\mathfrak{g}, \phi)$ is a quadratic Leibniz algebra and d is a derivation of $\mathfrak{g}$, then $f(x, y) := \phi(x, dy)$ is a Leibniz 2-cocycle on $\mathfrak{g}$.*

Proof. Since $f(x, [y, z]) = \phi(x, d[y, z]) = f([x, y], z) - f([x, z], y)$ for $x,\ y,\ z \in \mathfrak{g}$, f is a Leibniz 2-cocycle on $\mathfrak{g}$. ☐

Corollary 5.3 ([12])**.** *If d is a skew-derivation,* i.e., $\phi(dx, y) + \phi(x, dy) = 0$*, then $f(x, y) := \phi(x, dy)$ is a Lie 2-cocycle on $\mathfrak{g}$.*

5.2. A negative answer to a question in [16].

A question in [16] is posed: Is it true that each Leibniz 2-cocycle on an infinite-dimensional Lie algebra is also a Lie 2-cocycle?

Proposition 5.4. *Assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{K}$. Construct the Lie algebra $\mathfrak{g} \otimes \mathbb{K}((t))$ with bracket: $[x \otimes r, y \otimes s]' = [x, y] \otimes rs,\ x,\ y \in \mathfrak{g},\ r,\ s \in \mathbb{K}((t))$. Then $\mathfrak{g} \otimes \mathbb{K}((t))$ is an infinite-dimensional simple Lie algebra over $\mathbb{K}$ and*

$$\mathrm{H}^2(\mathfrak{g} \otimes \mathbb{K}((t)), \mathbb{K}) \subsetneq \mathrm{HL}^2(\mathfrak{g} \otimes \mathbb{K}((t)), \mathbb{K}).$$

Proof. Define $\phi(x \otimes r, y \otimes s) = (x, y)\,\mathrm{Res}(rs)$, where Res is a linear function on $\mathbb{K}((t))$ and takes the coefficient of t^{-1} for every series, and $(\,,)$ is the Killing form on $\mathfrak{g}$. Then ϕ is an invariant symmetric bilinear form on $\mathfrak{g} \otimes \mathbb{K}((t))$ and $t^k \frac{d}{dt}, k \in \mathbb{Z} - \{0\}$ is a derivation of $\mathbb{K}((t))$. By Lemma 5.2, we get a nontrivial Leibniz 2-cocycle of the Lie algebra $\mathfrak{g} \otimes \mathbb{K}((t))$:

$$f(x \otimes \sum_{m \geq N} a_m t^m,\ y \otimes \sum_{n \geq N} b_n t^n) = (x, y) \sum_{m,n \geq N} n a_m b_n \delta_{m+n+k,0}, \tag{5.1}$$

where f is well defined since the summation in (5.1) is finite, and f is not skew-symmetric, that is, $\mathrm{H}^2(\mathfrak{g} \otimes \mathbb{K}((t)), \mathbb{K}) \subsetneq \mathrm{HL}^2(\mathfrak{g} \otimes \mathbb{K}((t)), \mathbb{K})$. ☐

Remark 5.5. $\mathfrak{g}\otimes\mathbb{K}((t))$ is a simple Lie algebra (see [24]), and $\dim \mathrm{H}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K}) = 1$, but $\dim \mathrm{HL}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K}) = \infty$, which then leads to a negative answer to the above question [16].

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