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THE TORSION OF p-RAMIFIED IWASAWA MODULES II

SATOSHI FUJII

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ABSTRACT. In this article we prove the existence of a non-trivial torsion of the 3-ramified Iwasawa mocule over the $\mathbb{Z}_3^2$-extension of an imaginary quadratic field.

1. INTRODUCTION

Let p be a prime number, $k/\mathbb{Q}$ a finite extension of the rational number field $\mathbb{Q}$ and $\widetilde{k}$ the compositum of *all* $\mathbb{Z}_p$-extensions of k, where $\mathbb{Z}_p$ is the additive group of p-adic integers. Then one has

$$\mathrm{Gal}(\widetilde{k}/k) \simeq \mathbb{Z}_p^{\oplus r_2(k)+1} \tag{1}$$

if $k/\mathbb{Q}$ is abelian, where $r_2(k)$ denotes the number of complex primes of k (see Corollary 5.32 and Theorem 13.4 of [17]). Let $M_{\widetilde{k}}/\widetilde{k}$ be the maximal pro-p abelian extension which is unramified outside all primes lying above p and ∞, and $X_{\widetilde{k}} = \mathrm{Gal}(M_{\widetilde{k}}/\widetilde{k})$ denotes its Galois group. The module $X_{\widetilde{k}}$ has natural $\Lambda_{\widetilde{k}/k} = \mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/k)]]$-module structure, and this subject had been deeply studied by Greenberg [5], Jannsen [10] and Nguyen Quang Do [15] including the case where k is a local field.

The author asked in a previous article [4] whether the torsion submodule $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}}$ of $X_{\widetilde{k}}$ over $\Lambda_{\widetilde{k}/k}$ is always trivial or not when k is not a totally real field and then obtained partial results for imaginary quadratic fields as follows:

Theorem A. (1) *For any prime number p, there exist infinitely many imaginary quadratic fields k such that* $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}} = 0$.

(2) *Let $p \neq 3$ be a prime number and let k be an imaginary quadratic field. If the class number h_k of k is not divided by p, then* $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}} = 0$.

These results were shown by observing the base field k and the cyclotomic $\mathbb{Z}_p$-extension k_∞ of k. When $p = 3$ and the completion of k at a prime lying above 3 contains the group of all 3-rd roots of unity μ_3, the fields k and k_∞ do not know whether $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}} = 0$ or not. In this article, we shall show that $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}} \neq 0$ in this exceptional case.

Theorem 1. *Let $p = 3$ and let $k = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field with a positive square-free integer m such that $3 \neq m \equiv 3 \bmod 9$. If $3 \nmid h_k$, then* $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}} X_{\widetilde{k}} \neq 0$.

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One can show that if $k=\mathbb{Q}(\mu_3)$ or if $3\nmid h_k$ and $m\not\equiv 3 \bmod 9$, then $X_{\widetilde{k}}\simeq\Lambda_{\widetilde{k}/k}$, in particular, $\mathrm{Tor}_{\widetilde{k}/k}X_{\widetilde{k}}=0$, for the prime 3. Indeed, these k's are 3-rational with $\mathrm{Gal}(\widetilde{k}/k)\simeq\mathbb{Z}_3^{\oplus 2}$ (see Example 3 of [13]). If k is 3-rational and $\mathrm{Gal}(\widetilde{k}/k)\simeq\mathbb{Z}_3^{\oplus 2}$, then $X_{\widetilde{k}}\simeq\Lambda_{\widetilde{k}/k}$ by Theorem 2 of [8]. See also the proof of Theorem 1.1 of [4]. Hence we have classified the non-triviality of $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}}X_{\widetilde{k}}$ under the assumption that $p\nmid h_k$. As a corollary to Theorem 1 and a result of Davenport and Heilbronn [2], [3] as extended by Horie and Nakagawa [7], we find that there exist infinitely many imaginary quadratic fields k with *positive density* such that $\mathrm{Tor}_{\Lambda_{\widetilde{k}/k}}X_{\widetilde{k}}\neq 0$ for the prime 3.

Corollary 1. *Let $\#S$ denote the cardinality of a set S. Let $p=3$ and let Δ_k be the discriminant of a number field k. Let $K^-(x)$ be the set of imaginary quadratic fields k such that $|\Delta_k|<x$ and put $K^-(x,6,9)=\{k\in K^-(x)\mid \Delta_k\equiv 6 \bmod 9\}$. Then we have*

$$\liminf_{x\to\infty}\frac{\#\{k\in K^-(x,6,9)\mid \mathrm{Tor}_{\Lambda_{\widetilde{k}/k}}X_{\widetilde{k}}\neq 0\}}{\#K^-(x,6,9)}\geq\frac{1}{2}$$

and

$$\liminf_{x\to\infty}\frac{\#\{k\in K^-(x,6,9)\mid \mathrm{Tor}_{\Lambda_{\widetilde{k}/k}}X_{\widetilde{k}}\neq 0\}}{\#K^-(x)}\geq\frac{1}{16}.$$

(*Note that if* $3\neq m\equiv 3 \bmod 9$*, then* $\Delta_{\mathbb{Q}(\sqrt{-m})}\equiv 6 \bmod 9$.)

Corollary 1 is shown by the inequality

$$\#\{k\in K^-(x,6,9)\mid \mathrm{Tor}_{\Lambda_{\widetilde{k}/k}}X_{\widetilde{k}}\neq 0\}\geq\#\{k\in K^-(x,6,9)\mid 3\nmid h_k\}$$

and doing estimation similar to the proof of a result of Taya (Theorem 2 of [16]). His description is very easy to understand.

Furthermore, we have an application on the unramified Iwasawa modules. Let $Y_F=\mathrm{Gal}(L_F/F)$ be the Galois group of the maximal unramified pro-p abelian extension L_F/F of a number field F. It is conjectured by Greenberg that $Y_{\widetilde{k}}$ is a pseudo-null $\Lambda_{\widetilde{k}/k}$-module, i.e., there are two relatively prime annihilators, for any finite extension $k/\mathbb{Q}$ (Conjecture 3.5 of [6]). Several works on this conjecture have been done for imaginary quadratic fields and imaginary biquadratic fields (see [1], [9] and [12]), and no counterexample has been found yet. As a corollary to Theorem 1, we obtain the failure of the pseudo-nullity for a $\mathbb{Z}_3^{\oplus 2}$-extension of certain biquadratic imaginary fields.

Corollary 2. *Let $p=3$ and let $k=\mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field with a positive square-free integer m such that $3\neq m\equiv 3 \bmod 9$. Then $Y_{\widetilde{k}(\mu_3)}$ is not a pseudo-null $\Lambda_{\widetilde{k}(\mu_3)/k(\mu_3)}$-module, if $3\nmid h_k$.*

Corollary 2 immediately follows from Theorem 1 of this article and Proposition 3.6 of [11]. Note that the failure of the pseudo-nullity of $Y_{\widetilde{k}(\mu_3)}$ does not contradict Greenberg's conjecture since $\widetilde{k}(\mu_3)\subsetneq\widetilde{k(\mu_3)}$ by the isomorphism (1), and note that $k(\mu_3)$ is not contained in Bandini's results [1]. The pseudo-nullity of $Y_{\widetilde{k(\mu_3)}}$ in this case seems quite mysterious.

Here, we shall set the notation of this article. Let $\widetilde{M}_k/k$ be the maximal pro-3 extension unramified outside all primes lying above 3 (resp. maximal pro-3 extension) of an algebraic extension k of $\mathbb{Q}$ (resp. $\mathbb{Q}_3$), and let $G_k=\mathrm{Gal}(\widetilde{M}_k/k)$ be its Galois group. Then M_k is also defined to be the maximal abelian subfield of $\widetilde{M}_k$ and we then have $X_k = G_k^{\mathrm{ab}}$, where the symbol ab means the maximal pro-3 abelian quotient. If K/k is a pro-3 extension, then the complete group ring $\Lambda_{K/k} = \mathbb{Z}_3[[\mathrm{Gal}(K/k)]]$ acts on X_K via the inner automorphism. Let $\langle J\rangle \simeq \mathbb{Z}/2$ and let M be a $\mathbb{Z}_3[\langle J\rangle]$-module. Then we have the eigen spaces $M^{\pm} = \{x \in M \mid Jx = \pm x\}$ of M and the decomposition $M = M^+ \oplus M^-$. For any topological group G, let G' be the topological commutator group of G.

2. Proof of Theorem 1

Put $K = \mathbb{Q}_3(\mu_3)$ and let M_K/K be the maximal abelian pro-3 extension. Then, by class field theory, we have an isomorphism

$$K^\times \otimes \mathbb{Z}_3 \simeq \mathrm{Gal}(M_K/K),\ x \otimes \alpha \mapsto (x, M_K/K)^\alpha, \tag{2}$$

where $(\ , M_K/K)$ is the reciprocity map.

Put $\langle J\rangle = \mathrm{Gal}(K/\mathbb{Q}_3)$. Then J acts on both groups in (2) and the reciprocity map commutes with the action of J. Put $U = \{x \in K^\times | x \equiv 1 \bmod 3\}$. Since $\sqrt{-3}$ is a prime element of K, one sees that

$$K^\times \otimes \mathbb{Z}_3 = \mu_3 \oplus 3^{\mathbb{Z}_3} \oplus U.$$

Then

$$(K^\times \otimes \mathbb{Z}_3)^+ = 3^{\mathbb{Z}_3} \oplus U^+ \simeq \mathbb{Z}_3^{\oplus 2}$$

and

$$(K^\times \otimes \mathbb{Z}_3)^- = \mu_3 \oplus U^- \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}_3$$

as $\mathbb{Z}_3$-modules. This shows that there is a unique $\mathbb{Z}_3$-extension K_∞^a of K such that $K_\infty^a/\mathbb{Q}_3$ is a Galois extension and such that J acts on $\mathrm{Gal}(K_\infty^a/K)$ as -1. Let K_∞/K be the cyclotomic $\mathbb{Z}_3$-extension, which is obtained by adjoining the group of all 3-power-th roots of unity μ_{3^∞}. Put $K^{(2)} = K_\infty K_\infty^a$. Then $K^{(2)}/K$ is a unique $\mathbb{Z}_3^{\oplus 2}$-extension such that $K^{(2)}/\mathbb{Q}_3$ is a Galois extension, $K^{(2)}$ contains K_∞ and such that $\mathrm{Gal}(K^{(2)}/K)^{\pm} \simeq \mathbb{Z}_3$ as $\mathbb{Z}_3$-modules. Note that $K^{(2)}$ is the fixed field of $(\mu_3 \oplus 3^{\mathbb{Z}_3}, M_K/K)$.

Let $\Gamma = \mathrm{Gal}(K^{(2)}/K)$. We then define a subgroup $H_{K^{(2)}}$ of G_K to be the kernel of the natural morphism $G_K \to \Gamma$. Let $\gamma_c, \gamma_a \in G_K$ be lifts of a system of topological generators of $\Gamma \simeq \mathbb{Z}_3^{\oplus 2}$. Here we note that $X_{K^{(2)}} = H_{K^{(2)}}^{\mathrm{ab}}$. In fact, since $K \subseteq K^{(2)} \subseteq \widetilde{M}_K$, we sees that $\widetilde{M}_{K^{(2)}} = \widetilde{M}_K$. In particular, $M_{K^{(2)}}$ is the maximal abelian subextension of $\widetilde{M}_K/K^{(2)}$. We want to know the structure of $X_{K^{(2)}}$ as a $\Lambda := \mathbb{Z}_3[[\Gamma]]$-module.

Lemma 1. *There exist $\mathfrak{f}, \zeta \in H_{K^{(2)}}$ such that*

$$(3, M_K/K) = \mathfrak{f} \bmod G'_K$$

and

$$\left(\frac{-1+\sqrt{-3}}{2}, M_K/K\right) = \zeta \bmod G'_K,$$

and such that

$$X_{K^{(2)}} = \Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} + \Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}},$$

where we let $[\gamma_c, \gamma_a] = \gamma_c\gamma_a\gamma_c^{-1}\gamma_a^{-1}$. Furthermore, we have

$$X_{K^{(2)}} = \Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} \oplus (\Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}}),\ \Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} \simeq \Lambda$$

as Λ-modules.

Proof. First, we shall introduce the five-term exact sequence

$$0 \to \mathrm{H}^1(G/H, A^H) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(H, A)^{G/H} \to \mathrm{H}^2(G/H, A^H) \to \mathrm{H}^2(G, A).$$

Here G, H and A stand for a profinite group, a closed normal subgroup of G and a discrete G-module, respectively. For reference, see some books about the Galois cohomology, for example, chapter 1, §6, of [14]. By applying the dual of the five-term sequence with coefficients in $\mathbb{Q}_3/\mathbb{Z}_3$ to the exact sequence $1 \to H_{K^{(2)}} \to G_K \to \Gamma \to 1$, the sequence

$$\mathrm{H}_2(\Gamma, \mathbb{Z}_3) \longrightarrow (X_{K^{(2)}})_\Gamma \longrightarrow G_K^{\mathrm{ab}} \longrightarrow \Gamma \longrightarrow 0 \tag{3}$$

is exact. From the fact that $\mathrm{Ker}(G_K^{\mathrm{ab}} \to \Gamma) = (\mu_3 \oplus 3^{\mathbb{Z}_3}, M_K/K)$ and the above exact sequence, there exist $\mathfrak{f}, \zeta \in H_{K^{(2)}}$ such that $(3, M_K/K) = \mathfrak{f} \bmod G'_K$ and $(\frac{-1+\sqrt{-3}}{2}, M_K/K) = \zeta \bmod G'_K$. Let $\mathcal{G} = \langle \widetilde{\gamma}_a, \widetilde{\gamma}_c \rangle$ be a free pro-3 group of rank 2 and $1 \to \mathcal{H} \to \mathcal{G} \to \Gamma \to 1$ a minimal presentation of Γ by $\mathcal{G}$ with $\widetilde{\gamma}_a \mapsto \gamma_a|_{K^{(2)}}, \widetilde{\gamma}_c \mapsto \gamma_c|_{K^{(2)}}$. Then $\mathcal{H}$ is generated by the commutator $[\widetilde{\gamma}_a, \widetilde{\gamma}_c]$ as a closed normal subgroup of $\mathcal{G}$. Let $\mathcal{G} \to G_K$ be a morphism defined by $\widetilde{\gamma}_a \mapsto \gamma_a, \widetilde{\gamma}_c \mapsto \gamma_c$. Then we obtain the exact commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}_2(\Gamma, \mathbb{Z}_3) & \longrightarrow & (X_{K^{(2)}})_\Gamma & \longrightarrow & G_K^{\mathrm{ab}} & \longrightarrow & \Gamma & \longrightarrow & 0 \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
\mathrm{H}_2(\Gamma, \mathbb{Z}_3) & \xrightarrow{\sim} & \mathcal{H}/[\mathcal{H}, \mathcal{G}] & \longrightarrow & \mathcal{G}^{\mathrm{ab}} & \xrightarrow{\sim} & \Gamma & \longrightarrow & 0,
\end{array}$$

since $\mathcal{G}^{\mathrm{ab}} \simeq \Gamma$ and $\mathrm{H}_2(\mathcal{G}, \mathbb{Z}_3) = 0$ because $\mathcal{G}$ is a free pro-3 group of rank 2. It follows that

$$\begin{aligned}
\mathrm{Image}(\mathrm{H}_2(\Gamma, \mathbb{Z}_3) \to (X_{K^{(2)}})_\Gamma) &= \mathrm{Image}(\mathcal{H}/[\mathcal{H}, \mathcal{G}] \to (X_{K^{(2)}})_\Gamma) \\
&= [\gamma_c, \gamma_a]^{\mathbb{Z}_3}[H_{K^{(2)}}, G_K]/[H_{K^{(2)}}, G_K].
\end{aligned}$$

Combining the above, by Nakayama's lemma, we have

$$X_{K^{(2)}} = \Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} + \Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}}.$$

Thus the first assertion follows.

Since $h_{\mathbb{Q}(\mu_3)} = 1$, the extension $\widetilde{M}_{\mathbb{Q}(\mu_3)}/\mathbb{Q}$ is totally ramified at 3. Thus

$$\mathrm{Gal}(\widetilde{M}_{\mathbb{Q}(\mu_3)}/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}_3\widetilde{M}_{\mathbb{Q}(\mu_3)}/\mathbb{Q}_3).$$

Note that there is a unique $\mathbb{Z}_3^{\oplus 2}$-extension $\widetilde{\mathbb{Q}(\mu_3)}/\mathbb{Q}(\mu_3)$ by the isomorphism (1), which is the compositum of all $\mathbb{Z}_3$-extensions of $\mathbb{Q}(\mu_3)$. By the natural morphism, we may identify $\langle J \rangle = \mathrm{Gal}(K/\mathbb{Q}_3)$ with $\mathrm{Gal}(\mathbb{Q}(\mu_3)/\mathbb{Q})$. Then one sees that $\mathrm{Gal}(\widetilde{\mathbb{Q}(\mu_3)}/\mathbb{Q}(\mu_3))^\pm \simeq \mathbb{Z}_3$ as $\mathbb{Z}_3$-modules. This shows that $K^{(2)} = K\widetilde{\mathbb{Q}(\mu_3)}$, whence we can identify Γ with $\mathrm{Gal}(\widetilde{\mathbb{Q}(\mu_3)}/\mathbb{Q}(\mu_3))$. Since $G_{\mathbb{Q}(\mu_3)}$ is a free pro-3 group of rank 2 ($\mathbb{Q}(\mu_3)$ is 3-rational), we find that $G_{\mathbb{Q}(\mu_3)}$ is generated by the images $\gamma_c|_{\widetilde{M}_{\mathbb{Q}(\mu_3)}}$ and $\gamma_a|_{\widetilde{M}_{\mathbb{Q}(\mu_3)}}$ of γ_c and γ_a. Put $H_{\widetilde{\mathbb{Q}(\mu_3)}} = \mathrm{Gal}(\widetilde{M}_{\mathbb{Q}(\mu_3)}/\widetilde{\mathbb{Q}(\mu_3)})$. Since $X_{\widetilde{\mathbb{Q}(\mu_3)}} = H^{\mathrm{ab}}_{\widetilde{\mathbb{Q}(\mu_3)}}$ by the same reason mentioned before Lemma 1, we then have

$$H^{\mathrm{ab}}_{\widetilde{\mathbb{Q}(\mu_3)}} = X_{\widetilde{\mathbb{Q}(\mu_3)}} = \Lambda[\gamma_c|_{\widetilde{M}_{\mathbb{Q}(\mu_3)}}, \gamma_a|_{\widetilde{M}_{\mathbb{Q}(\mu_3)}}]H'_{\widetilde{\mathbb{Q}(\mu_3)}} \simeq \Lambda.$$

By observing the natural surjective morphism $X_{K^{(2)}} \twoheadrightarrow X_{\widetilde{\mathbb{Q}(\mu_3)}}$, we can conclude that $\Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} \simeq \Lambda$ and $X_{K^{(2)}} = \Lambda[\gamma_c, \gamma_a]H'_{K^{(2)}} \oplus (\Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}})$. □

Let $k = \mathbb{Q}(\sqrt{-m})$ with $3 \neq m \equiv 3 \bmod 9$. Then one can easily see that the completion of k at the prime above 3 is K. Similarly to $\mathbb{Q}(\mu_3)$, we find that $K^{(2)} = K\widetilde{k}$, whence we identify Γ and $\Lambda_{\widetilde{k}/k}$ with $\mathrm{Gal}(\widetilde{k}/k)$ and Λ, respectively.

Suppose that $3 \nmid h_k$. Then the natural morphism $G_K \to G_k$ is surjective. It follows from the facts that k does not contain μ_3 and that the unit group of k is finite that there is an isomorphism

$$\mu_3 \oplus U \simeq G_k^{\mathrm{ab}} \tag{4}$$

by class field theory. Hence, if it is necessary, we may assume $\mathfrak{f}$ to be in $\mathcal{K} = \mathrm{Ker}(G_K \to G_k)$. Indeed, since $\mathrm{H}_2(G_k, \mathbb{Z}_3) = 0$, we then have the following exact sequence:

$$0 \longrightarrow \mathcal{K}/[\mathcal{K}, G_K] \longrightarrow G_K^{\mathrm{ab}} \longrightarrow G_k^{\mathrm{ab}} \longrightarrow 0. \tag{5}$$

By the isomorphisms (2) and (4), we find that there is $\mathfrak{f}' \in \mathcal{K}$ such that $\mathfrak{f}' \bmod G_K' = (3, M_K/K)$ and that $\mathcal{K}$ is generated by $\mathfrak{f}'$ as a closed normal subgroup of G_K. Since $\mathfrak{f}' \in H_{K^{(2)}}(\supseteq \mathcal{K})$ and $\mathfrak{f} \equiv \mathfrak{f}' \bmod G_K'$, we may assume $\mathfrak{f}' = \mathfrak{f}$.

Put $H_{\widetilde{k}} = \mathrm{Ker}(G_k \to \Gamma)$, so that $X_{\widetilde{k}} = H_{\widetilde{k}}^{\mathrm{ab}}$ as mentioned before. Because $\mathrm{H}_2(G_k, \mathbb{Z}_3) = 0$, one sees that $\mathrm{H}_2(H_{\widetilde{k}}, \mathbb{Z}_3) = 0$. Note that the sequence $1 \to \mathcal{K} \to H_{K^{(2)}} \to H_{\widetilde{k}} \to 1$ is exact. It follows that $\mathcal{K}/[\mathcal{K}, H_{K^{(2)}}] \simeq \mathrm{Ker}(X_{K^{(2)}} \to X_{\widetilde{k}}) = \Lambda\mathfrak{f}H'_{K^{(2)}}$. Here we need the following two results:

Theorem B (Proposition 2.6 of [15] and Example 5.2(c) of [10]). *Let p be a prime number, $E/\mathbb{Q}_p$ a finite extension and $E^{(d)}/E$ a $\mathbb{Z}_p^{\oplus d}$-extension. Then the following two assertions hold:*

(1) $\mathrm{rank}_{\Lambda_{E^{(d)}/E}} X_{E^{(d)}} = [E : \mathbb{Q}_p]$.

(2) *If $\mu_{p^\infty} \subseteq E^{(d)}$ and if $\mathrm{Gal}(E^{(d)}/E) \simeq \mathbb{Z}_p^{\oplus 2}$, then $X_{E^{(d)}}$ has no non-trivial $\Lambda_{E^{(d)}/E}$-torsion.*

Property (2) of Theorem B is deduced from the explicit structure of $X_{K^{(2)}}$ as a Λ-module given in [10]. What we did in this article was to argue the structure with given generators of $X_{K^{(2)}}$.

From Theorem B, we find that $\mathrm{rank}_\Lambda X_{K^{(2)}} = 2$, whence

$$\mathrm{rank}_\Lambda(\Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}}) = 1.$$

Furthermore, since $X_{K^{(2)}}$ has no Λ-torsion, we get $\Lambda\mathfrak{f}H'_{K^{(2)}} \simeq \Lambda$. This shows that

$$(\Lambda\mathfrak{f}H'_{K^{(2)}} + \Lambda\zeta H'_{K^{(2)}})/\Lambda\mathfrak{f}H'_{K^{(2)}}$$

is a non-trivial torsion Λ-submodule of $X_{\widetilde{k}}$. This completes the proof of Theorem 1. □

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Department of Mathematical Sciences, Graduate School of Science and Engineering, Keio University, Hiyoshi, Kohoku-ku, Yokohama City, Kanagawa, 223-8522, Japan
E-mail address: moph@a2.keio.jp