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PRINCIPAL GROUPOID C^*-ALGEBRAS WITH BOUNDED TRACE

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ABSTRACT. Suppose G is a second countable, locally compact, Hausdorff, principal groupoid with a fixed left Haar system. We define a notion of integrability for groupoids and show G is integrable if and only if the groupoid C^*-algebra $C^*(G)$ has bounded trace.

1. INTRODUCTION

Let H be a locally compact, Hausdorff group acting continuously on a locally compact, Hausdorff space X, so that (H, X) is a transformation group. A lovely theorem of Green says that if H acts freely on X, then the associated transformation-group C^*-algebra $C_0(X) \rtimes H$ has continuous trace if and only if the action of H on X is proper [5, Theorem 17]. Muhly and Williams defined a notion of proper groupoid and proved that for principal groupoids G, the groupoid C^*-algebra $C^*(G)$ has continuous trace if and only if the groupoid is proper [8, Theorem 2.3]. Of course, when $G = H \times X$ is the transformation-group groupoid, then G is proper if and only if the action of H on X is proper.

In [13] Rieffel introduced a notion of an integrable action of a group H on a C^*-algebra A. This notion of integrability for $A = C_0(X)$ turned out to characterize when $C_0(X) \rtimes H$, arising from a free action of H on X, has bounded trace [6, Theorem 4.8]. In this paper we define a notion of integrability for groupoids (see Definition 3.1) which, when $G = H \times X$ is the transformation-group groupoid, reduces to an integrable action of H on X (see Example 3.3). We then prove that for principal groupoids G, $C^*(G)$ has bounded trace if and only if the groupoid is integrable (see Theorem 4.4). This theorem is thus very much in the spirit of [8, Theorem 2.4], [4, Theorem 7.9], [4, Theorem 4.1] (see also [3, Corollary 5.9]) and [4, Theorem 5.3], which characterize when principal-groupoid C^*-algebras are, respectively, continuous-trace, Fell, CCR and GCR C^*-algebras. The key technical tools used to prove Theorem 4.4 are, first, a homeomorphism of the spectrum of $C^*(G)$ onto the orbit space [4, Proposition 5.1] and, second, a generalisation to groupoids of the notion of k-times convergence in the orbit space of a transformation group from [1].

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2. Preliminaries

Let A be a C^*-algebra. An element a of the positive cone A^+ of A is called a bounded-trace element if the map $\pi \mapsto \operatorname{tr}(\pi(a))$ is bounded on the spectrum $\hat{A}$ of A; the linear span of the bounded-trace elements is a two-sided $*$-ideal in A. We say A has *bounded trace* if the ideal of (the span of) the bounded-trace elements is dense in A.

Throughout, G is a locally compact, Hausdorff groupoid; in our main results G is assumed to be second-countable and principal. We denote the unit space of G by G^0, and the range and source maps $r, s : G \to G^0$ are $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, respectively. We let $\pi : G \to G^0 \times G^0$ be the map $\pi(\gamma) = (r(\gamma), s(\gamma))$; recall that G is principal if π is injective. In order to define the groupoid C^*-algebra, we also assume that G is equipped with a fixed left Haar system: a set $\{\lambda^x : x \in G^0\}$ of non-negative Radon measures on G such that

(1) $\operatorname{supp} \lambda^x = r^{-1}(\{x\})$;
(2) for $f \in C_c(G)$, the function $x \mapsto \int f \, d\lambda^x$ on G^0 is in $C_c(G^0)$; and
(3) for $f \in C_c(G)$ and $\gamma \in G$, the following equation holds:

$$\int f(\gamma\alpha) \, d\lambda^{s(\gamma)}(\alpha) = \int f(\alpha) \, d\lambda^{r(\gamma)}(\alpha).$$

Condition (3) implies that $\lambda^{s(\gamma)}(\gamma^{-1}E) = \lambda^{r(\gamma)}(E)$ for measureable sets E. The collection $\{\lambda_x : x \in G^0\}$, where $\lambda_x(E) := \lambda^x(E^{-1})$, gives a fixed right Haar system such that the measures are supported on $s^{-1}(\{x\})$ and

$$\int f(\gamma\alpha) \; d\lambda_{r(\alpha)} = \int f(\gamma) \; d\lambda_{s(\alpha)}$$

for $f \in C_c(G)$ and $\gamma \in G$. We will move freely between these two Haar systems.

If $N \subseteq G^0$, then the *saturation* of N is $r(s^{-1}(N)) = s(r^{-1}(N))$. In particular, we call the saturation of $\{x\}$ the *orbit* of $x \in G^0$ and denote it by $[x]$.

If G is principal and all the orbits are locally closed, then by [4, Proposition 5.1] the orbit space $G^0/G = \{[x] : x \in G^0\}$ and the spectrum $C^*(G)^\wedge$ of the groupoid C^*-algebra $C^*(G)$ are homeomorphic. This homeomorphism is induced by the map $x \mapsto L^x : G^0 \to C^*(G)^\wedge$, where $L^x : C^*(G) \to B(L^2(G, \lambda_x))$ is given by

$$L^x(f)\xi(\gamma) = \int f(\gamma\alpha)\xi(\alpha^{-1}) d\lambda^x(\alpha)$$

for $f \in C_c(G)$ and $\xi \in L^2(G, \lambda_x)$.

3. Integrable groupoids and convergence in the orbit space of a groupoid

The following definition is motivated by the notion of an integrable action of a locally compact, Hausdorff group on a space from [6, Definition 3.2].

Definition 3.1. A locally compact, Hausdorff groupoid G is *integrable* if for every compact subset N of G^0,

$$\sup_{x \in N} \{\lambda^x(s^{-1}(N))\} < \infty, \tag{3.1}$$

or, equivalently, $\sup_{x \in N} \{\lambda_x(r^{-1}(N))\} < \infty$.

Remark 3.2. (1) Suppose that G is a principal groupoid. Then $\lambda^x(s^{-1}(E)) = \lambda^y(s^{-1}(E))$ for all $x, y \in G^0$ such that $y \in [x]$. The map $\lambda^x \mapsto s * \lambda^x$, where $s * \lambda^x(E) = \lambda^x(s^{-1}(E))$, gives a family of measures $\{\alpha_{[x]} : [x] \in G^0/G\}$ such that $\alpha_{[x]}$ is a measure on $[x]$ supported on $[x]$, and, for any $f \in C_c(G)$, the function

$$x \mapsto \int_{y\in[x]} f(\pi^{-1}(x,y))\, d\alpha_{[x]}(y)$$

is continuous. (Recall that $\pi : \gamma \mapsto (r(\gamma), s(\gamma))$ is injective by definition of principality.) In fact, the existence of the Haar system $\{\lambda^x\}$ is equivalent to the existence of the family $\{\alpha_{[x]}\}$ [12, Examples 2.5(c)]. Thus a principal groupoid G is integrable if and only if for every compact subset M of G^0/G, the function $[x] \mapsto \alpha_{[x]}(M)$ is bounded.

(2) We could have taken the supremum in (3.1) over the whole unit space, that is,

$$\sup_{x\in G^0}\{\lambda^x(s^{-1}(N))\} = \sup_{x\in N}\{\lambda^x(s^{-1}(N))\}.$$

To see this, first note that if y is not in the saturation $r(s^{-1}(N)) = s(r^{-1}(N))$ of N, then $s^{-1}(N) \cap r^{-1}(\{y\}) = \emptyset$, and hence $\lambda^y(s^{-1}(N)) = 0$. Second, if y is in the saturation of N, then there exists a $\gamma \in G$ such that $s(\gamma) = y$ and $r(\gamma) \in N$. Then

$$r^{-1}(\{y\}) \cap s^{-1}(N) = \gamma^{-1}\gamma\big(r^{-1}(\{y\}) \cap s^{-1}(N)\big) = \gamma^{-1}\big(r^{-1}(\{r(\gamma)\}) \cap s^{-1}(N)\big),$$

and now

$$\begin{aligned}\lambda^y(s^{-1}(N)) &= \lambda^y\big(r^{-1}(\{y\}) \cap s^{-1}(N)\big) = \lambda^{r(\gamma^{-1})}\big(r^{-1}(\{r(\gamma)\}) \cap s^{-1}(N)\big)\\ &= \lambda^{s(\gamma^{-1})}\big(r^{-1}(\{r(\gamma)\}) \cap s^{-1}(N)\big) = \lambda^{r(\gamma)}(s^{-1}(N))\end{aligned}$$

with $r(\gamma) \in N$.

Example 3.3. Let (H, X) be a locally compact, Hausdorff transformation group with H acting on the left of the space X. Then $G = H \times X$ with

$$G^2 = \{((h,x),(k,y)) \in G \times G \colon y = h^{-1}\cdot x\}$$

and operations $(h,x)(k,h^{-1}\cdot x) = (hk,x)$ and $(h,x)^{-1} = (h^{-1}, h^{-1}\cdot x)$ is called the *transformation-group groupoid*. We identify the unit space $\{e\} \times X$ with X, and then the range and source maps $r, s : G \to X$ are $s(h,x) = h^{-1}\cdot x$ and $r(h,x) = x$. If δ_x is the point-mass measure on X and μ is a left Haar measure on H, then $\{\lambda^x := \mu \times \delta_x \colon x \in X\}$ is a left Haar system for G. Now

$$\lambda^x(s^{-1}(N)) = \mu(\{h \in H : h^{-1}\cdot x \in N\})$$

and hence

$$\sup_{x\in N}\{\lambda^x(s^{-1}(N))\} = \sup_{x\in N}\{\mu(\{h \in H : h^{-1}\cdot x \in N\})\};$$

that is, Definition 3.1 reduces to [6, Definition 3.2].

Example 3.4. In [5, pp. 95-96] Green describes an action as follows: the space X is a closed subset of $\mathbb{R}^3$ and consists of countably many orbits, with orbit representatives $x_0 = (0,0,0)$ and $x_n = (2^{-2n}, 0, 0)$ for $n = 1, 2, \ldots$. The action of the group $H = \mathbb{R}$ on X is given by $s \cdot x_0 = (0, s, 0)$ for all s; and for $n \geq 1$,

$$s\cdot x_n = \begin{cases} (2^{-2n}, s, 0) & \text{if } s \le n;\\ (2^{-2n} - (\frac{s-n}{\pi})2^{-2n-1}, n\cos(s-n), n\sin(s-n)) & \text{if } n < s < n+\pi;\\ (2^{-2n-1}, s-\pi-2n, 0) & \text{if } s \ge n+\pi.\end{cases}$$

So the orbit of each x_n ($n \geq 1$) consists of two vertical lines joined by an arc of a helix situated on a cylinder of radius n; the action moves x_n along the vertical lines at unit speed and along the arc at radial speed. This action is free, non-proper and integrable (see [13, Example 1.18] or [6, Example 3.3]). So the associated transformation-group groupoid $G = H \times X$ is principal and integrable by Example 3.3.

The following characterization of integrability will be important later. In the case of a transformation-group groupoid, Lemma 3.5 reduces to a special case of [1, Lemma 3.5].

Lemma 3.5. *Let G be a locally compact, Hausdorff groupoid. Then G is integrable if and only if, for each $z \in G^0$, there exists an open neighborhood U of z in G^0 such that*

$$\sup_{x \in U} \{\lambda^x(s^{-1}(U))\} < \infty.$$

Proof. The proof is exactly the same as the proof of [1, Lemma 3.5]. □

If a groupoid fails to be integrable, there exists a $z \in G^0$ such that

$$\sup_{x \in U} \{\lambda^x(s^{-1}(U))\} = \infty$$

for every open neighborhood U of z; we then say that the *groupoid fails to be integrable at z*.

It is evident from [1, 2] that integrability and k-times convergence in the orbit space of a transformation group are closely related. Moreover, Lemma 2.6 of [8] says that, if a principal groupoid fails to be proper and the orbit space G^0/G is Hausdorff, then there exists a sequence that converges 2-times in G^0/G in the sense of Definition 3.6.

Definition 3.6. A sequence $\{x_n\}$ in the unit space of a groupoid G converges k-times in G^0/G to $z \in G^0$ if there exist k sequences

$$\{\gamma_n^{(1)}\},\ \{\gamma_n^{(2)}\}, \dots, \{\gamma_n^{(k)}\} \subseteq G$$

such that

(1) $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leq i \leq k$;
(2) $s(\gamma_n^{(i)}) = x_n$ for $1 \leq i \leq k$;
(3) if $1 \leq i < j \leq k$, then $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$ as $n \to \infty$, in the sense that $\{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\}$ admits no convergent subsequence.

Remarks 3.7. (a) Condition (2) in Definition 3.6 is needed so that the composition in (3) makes sense.

(b) Definition 3.6 does not require that $x_n \to z$, but as in the transformation-group case ([2, Definition 2.2]), this can be arranged by changing the sequence which converges k-times: replace x_n by $r(\gamma_n^{(1)})$ and replace $\gamma_n^{(j)}$ by $\gamma_n^{(j)}(\gamma_n^{(1)})^{-1}$.

(c) Part (3) of Definition 3.6 means $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}$ is eventually outside every compact set. In particular, if LL^{-1} is compact, $L\gamma_n^{(i)} \cap L\gamma_n^{(j)} = \emptyset$ eventually.

Example 3.8. Let $G = H \times X$ be a transformation-group groupoid (see Example 3.3) and suppose that $\{x_n\} \subseteq G^0$ is a sequence converging 2-times in G^0/G to $z \in G^0$. Then there exist two sequences

$$\{\gamma_n^{(1)}\} = \{(s_n, y_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, z_n)\}$$

in G such that (1) $y_n \to z$ and $z_n \to z$; (2) $s_n^{-1} \cdot y_n = x_n$ and $t_n^{-1} \cdot z_n = x_n$; and (3) $(t_n s_n^{-1}, z_n) \to \infty$ as $n \to \infty$. To see that the sequence $\{x_n\}$ converges 2-times in X/H to z in the sense of [2, §4], consider the two sequences $\{s_n\}$ and $\{t_n\}$ in H. We have $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ using (1) and (2). Also, since $z_n \to z$ by (1), (3) implies that $t_n s_n^{-1} \to \infty$ in H.

Conversely, if $\{x_n\} \subseteq X$ converges 2-times in X/H to z, then there exist two sequences $\{s_n\}$, $\{t_n\}$ in H such that (1) $s_n \cdot x_n \to z$ and $t_n \cdot x_n \to z$ and (2) $t_n s_n^{-1} \to \infty$. It is easy to check that

$$\{\gamma_n^{(1)}\} = \{(s_n, s_n \cdot x_n)\} \quad \text{and} \quad \{\gamma_n^{(2)}\} = \{(t_n, t_n \cdot x_n)\}$$

witness the 2-times convergence in G^0/G of $\{x_n\} \subset G^0$ to $z \in G^0$.

In the transformation-group groupoid of Example 3.4, the sequence $\{x_n = (2^{-2n}, 0, 0)\}$ converges 2-times in G^0/G to $z_0 = (0,0,0)$; to see this, just take $s_n = e$ and $t_n = 2n + \pi$ for each n.

In §4 we will prove that a principal groupoid G is integrable if and only if $C^*(G)$ has bounded trace. For the "only if" direction we will need to know that the orbits are locally closed so that [4, Proposition 5.1] applies and $x \mapsto L^x$ induces a homeomorphism of G^0/G onto $C^*(G)^\wedge$; Lemma 3.9 below establishes that if G is integrable, then the orbits are in fact closed, hence locally closed. We will prove the contrapositive of the "if" direction, and a key observation for the proof is Proposition 3.11: if a groupoid fails to be integrable at some z, then there is a non-trivial sequence $\{x_n\}$ which converges k-times in G^0/G to z, for every $k \in \mathbb{N} \setminus \{0\}$.

We thank an anonymous referee for providing the proof of Lemma 3.9.

Lemma 3.9. *Let G be a second countable, locally compact, Hausdorff, principal groupoid. If G is integrable, then all orbits are closed.*

Proof. Let $\{\alpha_{[x]} : [x] \in G^0/G\}$ be the family of measures from Remark 3.2(1). We claim that, for fixed $h \in C_c(G^0/G)$, the function $[x] \mapsto \int_{y\in[x]} h(y)\, d\alpha_{[x]}(y)$ is continuous. To see this, choose $g_n \in C_c(G^0 \times G^0)$ such that, for all $u \in G^0$, the function $g_n(u, \cdot)$ increases to the function $v \mapsto 1$. Then

$$\int_{y\in[x]} h(y)\, d\alpha_{[x]}(y) = \lim_n \int_{y\in[x]} g_n(x,y) h(y)\, d\alpha_{[x]}(y) = \lim_n \int_G f_n(\gamma)\, d\lambda^x(\gamma),$$

where $f_n(\gamma) = g_n(\pi(\gamma)) h(s(\gamma))$. Since $f_n \in C_c(G)$, the function

$$x \mapsto \int_G f_n(\gamma)\, d\lambda^x(\gamma)$$

is continuous for each n. Note that $x \mapsto \int_{y\in[x]} g_n(x,y) h(y)\, d\alpha_{[x]}(y)$ is compactly supported for each n. Since limits of uniformly continuous functions are continuous, $x \mapsto \int_{y\in[x]} h(y)\, d\alpha_{[x]}(y)$ is continuous; this function is constant on orbits, which proves the claim.

Fix $x_0 \in G^0$ and suppose that G is integrable. Since G is principal, for each compact subset M of G^0/G, the function $[x] \mapsto \alpha_{[x]}(M)$ is bounded. In particular, for each $h \in C_c(G^0/G)^+$, $\int h\, d\alpha_{[x_0]} \in \mathbb{R}$. Since the the support of $\alpha_{[x]}$ is $[x]$, we have

$$\{x_0\} = \bigcap_{h \in C_c(G^0/G)^+} \Big\{ x : \int h\, d\alpha_{[x]} \le \int h\, d\alpha_{[x_0]} \Big\}. \tag{3.2}$$

But the function $[x] \mapsto \int_{y\in[x]} h(y)\, d\alpha_{[x]}(y)$ is continuous, hence lower semi-continuous, so the left-hand side of (3.2) is an intersection of closed sets. Thus $\{x_0\}$ is closed in G^0/G, and hence $[x_0]$ is closed in G^0. □

The transformation group of [13, Example 1.18] provides an example of a non-integrable free action with closed orbits (by choosing repetition numbers with infinite supremum). Thus there are non-integrable principal groupoids with closed orbits.

Recall that a neighborhood W of G^0 is called *conditionally compact* if the sets WV and VW are relatively compact for every compact set V in G. The following lemma will be used repeatedly.

Lemma 3.10. *Let G be a second countable, locally compact, Hausdorff groupoid.*

(1) *Let $z \in G^0$ and let K be a relatively compact neighborhood of z in G. There exist $a \in \mathbb{R}$ and a neighborhood U of z in G^0 such that $0 < a \le \lambda_x(K)$ for all $x \in U$.*
(2) *Let Q be a conditionally compact neighborhood in G. Given any relatively compact neighborhood V in G^0 such that $QV \neq \emptyset$, there exists $c \in \mathbb{R}$ such that $c > 0$ and $\lambda_x(Q) \le c$ for all $x \in V$.*

Proof. (1) Suppose not. Let $\{U_i\}$ be a decreasing sequence of open neighborhoods of z in G^0. There exists an increasing seqence $i_1 < i_2 < \cdots < i_n < \cdots$ and $x_n \in U_{i_n}$ such that $\lambda_{x_n}(K) < 1/n$ for each $n \ge 1$. Note that $x_n \to z$.

Let $f \in C_c(G)$ such that $0 \le f \le 1$, $f(z) = 1$ and $\operatorname{supp} f \subseteq K$; note that $\int f(\gamma)\, d\lambda_z(\gamma) > 0$. By the continuity of the Haar system,

$$\frac{1}{n} > \lambda_{x_n}(K) \ge \int f(\gamma)\, d\lambda_{x_n}(\gamma) \to \int f(\gamma)\, d\lambda_z(\gamma) \text{ as } n \to \infty,$$

which is impossible since the left-hand side converges to 0 and $\int f(\gamma)\, d\lambda_z(\gamma) > 0$.

(2) Let V be any relatively compact neighborhood in G^0 such that $QV \neq \emptyset$. Let $f \in C_c(G)$ such that $0 \le f \le 1$ and f is identically one on the relatively compact subset QV. The function $w \mapsto \int f(\gamma)\, d\lambda_w(\gamma)$ is in $C_c(G^0)$, so it achieves a maximum $c > 0$. Then, for $x \in V$,

$$\lambda_x(Q) = \lambda_x(Qx) \le \int f(\gamma)\, d\lambda_x(\gamma) \le c. \qquad \square$$

Proposition 3.11. *Let G be a locally compact, Hausdorff groupoid. Let $z \in G^0$ and suppose that G fails to be integrable at z. Then there exists a sequence $\{x_n\}$ in G^0 such that $x_n \to z$, and $\{x_n\}$ converges k-times in G^0/G to z, for every $k \in \mathbb{N} \setminus \{0\}$. In addition, if G is second countable, principal and the orbits are locally closed, then $x_n \neq z$ eventually.*

Proof. Suppose the groupoid fails to be integrable at z. Fix $k \in \mathbb{N} \setminus \{0\}$. Let $\{U_n\}$ be a decreasing sequence of open relatively compact neighborhoods of z in G^0. By Lemma 3.5

$$\sup_{y \in U_n} \{\lambda^y(s^{-1}(U_n))\} = \infty$$

for each n. So we can choose a sequence $\{x_n\}$ such that $x_n \in U_n$ and $\lambda^{x_n}(s^{-1}(U_n)) > n$. Note that $x_n \to z$ as $n \to \infty$.

Let Q be an open symmetric conditionally compact neighborhood of z in G and let V be an open relatively compact neighborhood of z in G^0. By Lemma 3.10(2) there exists $c > 0$ such that $\lambda_v(Q^2) \leq c$ whenever $v \in V$. Choose n_0 such that $n_0 > (k-1)c$ and $U_{n_0} \subseteq V$. Temporarily fix $n > n_0$. Set $\gamma_n^{(1)} = x_n$. For $k \geq 2$ choose $k-1$ elements $\gamma_n^{(2)}, \dots, \gamma_n^{(k)}$ as follows. Note that since $x_n = r(\gamma_n^{(1)}) \in V$, we have

$$\begin{aligned}\lambda_{x_n}\big(r^{-1}(U_n) \setminus Q^2\gamma_n^{(1)}\big) &\geq \lambda_{x_n}\big(r^{-1}(U_n)\big) - \lambda_{x_n}(Q^2\gamma_n^{(1)}) \\ &= \lambda_{x_n}\big(r^{-1}(U_n) \cap s^{-1}(\{x_n\})\big) - \lambda_{r(\gamma_n^{(1)})}(Q^2) \\ &> (k-1)c - c = (k-2)c \geq 0.\end{aligned}$$

So there exists

$$\gamma_n^{(2)} \in \big(r^{-1}(U_n) \cap s^{-1}(\{x_n\})\big) \setminus Q^2\gamma_n^{(1)};$$

note that $r(\gamma_n^{(2)}) \in U_n \subset V$ and $s(\gamma_n^{(2)}) = x_n$. Next,

$$\begin{aligned}&\lambda_{x_n}\big((r^{-1}(U_n) \setminus (Q^2\gamma_n^{(1)} \cup Q^2\gamma_n^{(2)})\big) \\ &\quad\geq \lambda_{x_n}(r^{-1}(U_n)) - \lambda_{x_n}(Q^2\gamma_n^{(1)}) - \lambda_{x_n}(Q^2\gamma_n^{(2)}) \\ &\quad\geq \lambda_{x_n}(r^{-1}(U_n) \cap s^{-1}(\{x_n\})) - \lambda_{r(\gamma_n^{(1)})}(Q^2) - \lambda_{r(\gamma_n^{(2)})}(Q^2) \\ &\quad> (k-3)c \geq 0.\end{aligned}$$

Continue until $\gamma_n^{(1)}, \dots, \gamma_n^{(k)}$ have been chosen in this way.

If $n > n_0$, then by construction $s(\gamma_n^{(i)}) = x_n$ and $r(\gamma_n^{(i)}) \in U_n$ for each n; so $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leq i \leq k$. Moreover $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin Q^2$ for $1 \leq i < j \leq k$ and $n > n_0$. To see that $\{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\}$ tends to infinity, suppose that it doesn't. Then, $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \gamma$ by passing to a subsequence and relabelling. But then $s(\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}) = r(\gamma_n^{(i)}) \to z$ and $r(\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}) = r(\gamma_n^{(j)}) \to z$ implies $\gamma = z$, which is impossible because $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin Q^2$ and Q contains G^0. Hence $\{x_n\}$ converges k-times in G^0/G to z.

We claim that if G is second countable and principal, then $x_n \neq z$ eventually. To see this, suppose $x_n = z$ frequently. Then $\lambda^z(s^{-1}(U_n)) > n$ frequently, and hence

$$\lambda^z(s^{-1}(U_1)) = \infty. \tag{3.3}$$

The orbits are locally closed and G is second countable and principal, so the source map restricts to a homeomorphism $s| : r^{-1}(\{z\}) \to [z]$. Since U_1 is relatively compact, $s^{-1}([z] \cap U_1)$ is relatively compact in $r^{-1}(\{z\})$ because $s| : r^{-1}(\{z\}) \to [z]$ is a homeomorphism. But now $\lambda^z(s^{-1}([z] \cap U_1)) = \lambda^z(s^{-1}(U_1)) < \infty$, contradicting (3.3). □

4. Integrability of G and trace properties of $C^*(G)$

Proposition 4.1. *Let G be a second-countable, locally compact, Hausdorff, principal groupoid. If $C^*(G)$ has bounded trace, then G is integrable.*

The proof of Proposition 4.1 is based on that of [8, Theorem 2.3]. There, Muhly and Williams choose a sequence $\{x_n\} \subseteq G^0$ with $x_n \to z$ which witnesses the failure of the groupoid to be proper. They then carefully construct a function $f \in C_c(G)$ to obtain an element d of the Pedersen ideal of $C^*(G)$ such that $\mathrm{tr}(L^{x_n}(d))$ does not converge to $\mathrm{tr}(L^z(d))$. Since the Pedersen ideal is the minimal dense ideal [9, Theorem 5.6.1], the ideal of continuous-trace elements cannot be dense, so $C^*(G)$ does not have continuous trace. We adopt the same strategy, use exactly the same function f, but adapt the proof of [8, Theorem 2.3] using ideas from [6, Proposition 3.5].

Proof of Proposition 4.1. Fix $M \in \mathbb{N} \setminus \{0\}$. We will show that there is an element d of the Pedersen ideal of $C^*(G)$, a sequence of representations $\{L^{x_n}\}$ and $n_0 > 0$ such that $\operatorname{tr}(L^{x_n}(d)) > M$ whenever $n > n_0$. Since M is arbitrary, $C^*(G)$ cannot have bounded trace.

If G is not integrable, then the integrability fails at some $z \in G^0$ by Lemma 3.5. If the orbits are not closed, then $C^*(G)$ cannot be CCR by [4, Theorem 4.1] and hence cannot have bounded trace. So from now on we may assume that the orbits are closed. By Proposition 3.11, there exists a sequence $\{x_n\}$ such that $x_n \neq z$, $x_n \to z$, and $\{x_n\}$ converges k-times in G^0/G to z, for every $k \in \mathbb{N} \setminus \{0\}$.

Since we will use exactly the same function f that was used in the proof of [8, Theorem 2.3], our first task is to briefly outline its construction. Fix a function $g \in C_c(G^0)$ such that $0 \leq g \leq 1$ and g is identically one on a neighborhood U of z. Let $N = \operatorname{supp} g$ and let

$$\begin{aligned} F_z^N &:= s^{-1}(\{z\}) \cap r^{-1}([z] \cap N) = s^{-1}(\{z\}) \cap r^{-1}(N), \\ F_N^z &:= r^{-1}(\{z\}) \cap s^{-1}([z] \cap N) = r^{-1}(\{z\}) \cap s^{-1}(N). \end{aligned}$$

There exist symmetric, open, conditionally compact neighborhoods W_0 and W_1 in G such that

$$G^0 \subseteq W_0 \subseteq \overline{W_0} \subseteq W_1 \quad \text{and} \quad F_N^z \cup F_z^N \subseteq W_0.$$

Thus $\overline{W}_1^7 z \setminus W_0 z \subseteq r^{-1}(G^0 \setminus N)$. (The reason for using $\overline{W_1}^7$ becomes clear at (4.4) below.) By a compactness argument, there exist open, symmetric, relatively compact neighborhoods $V_0 \subseteq G^0$ and V_1 of z in G such that $\overline{V_0} \subset V_1$ and

$$\overline{W}_1^7 \overline{V}_0 \setminus W_0 V_0 \subseteq r^{-1}(G^0 \setminus N). \tag{4.1}$$

Now note that if $\gamma \in \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \setminus W_0 V_0 W_0$, then $r(\gamma) \in r(\overline{W}_1^7 \overline{V}_0 \setminus W_0 V_0) \subseteq G^0 \setminus N$. It follows that the function $g^{(1)} : G \to [0,1]$ defined by

$$g^{(1)}(\gamma) = \begin{cases} g(r(\gamma)) & \text{if } \gamma \in \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7, \\ 0 & \text{if } \gamma \notin W_0 V_0 W_0 \end{cases}$$

is well-defined and continuous with compact support in G. By construction

$$(W_0 V_0 W_0)^2 = W_0 V_0 W_0^2 V_0 W_0 \subseteq W_0^4 V_0 W_0^4 \subseteq \overline{W}_0^4 \overline{V}_0 \overline{W}_0^4 \subseteq W_1^4 V_1 W_1^4 \subseteq \overline{W}_1^4 \overline{V}_1 \overline{W}_1^4.$$

So there exists a function $b \in C_c(G)$ such that $0 \leq b \leq 1$, b is identically one on $W_0 V_0 W_0^2 V_0 W_0$ and it is identically zero on the complement of $\overline{W}_1^4 \overline{V}_1 \overline{W}_1^4$. Further, we can replace b with $(b + b^*)/2$ to ensure that b is self-adjoint. Set

$$f(\gamma) = g(r(\gamma)) g(s(\gamma)) b(\gamma);$$

note that $f \in C_c(G)$ is self-adjoint.

For $\xi \in L^2(G, \lambda_u)$ and $\gamma \in G$ we have

$$\begin{aligned} L^u(f)\xi(\gamma) &= \int f(\gamma\alpha)\xi(\alpha^{-1})\ d\lambda^u(\alpha) \\ &= \int g(r(\gamma))g(s(\alpha))b(\gamma\alpha)\xi(\alpha^{-1})\ d\lambda^u(\alpha) \\ &= g(r(\gamma))\int g(s(\alpha))b(\gamma\alpha)\xi(\alpha^{-1})\ d\lambda^u(\alpha) \\ &= g(r(\gamma))\int g(r(\alpha))b(\gamma\alpha^{-1})\xi(\alpha)\ d\lambda_u(\alpha). \end{aligned} \tag{4.2}$$

By [8, Lemma 2.8], $g^{(1)}$ is an eigenvector for $L^{x_n}(f)$ with eigenvalue

$$\mu_{x_n}^{(1)} = \int g(r(\alpha))g^{(1)}(\alpha)\ d\lambda_{x_n}(\alpha) = \int_{W_0V_0W_0} g(r(\alpha))^2\ d\lambda_{x_n}(\alpha).$$

By [8, Lemma 2.9], there exist an open $V_2 \subseteq V_0$ and a conditionally compact neighborhood Y of G^0 so that $Y \subseteq W_0$ and if $v \in V_2$, then $r(Yv) \subseteq U$. Notice that YV_2Y is a relatively compact subset of $W_0V_0W_0$. By Lemma 3.10(1) there exist an open neighborhood V_3 of z and $a > 0$ such that

$$\lambda_v(YV_2Y) \geq a \text{ whenever } v \in V_3. \tag{4.3}$$

Now, if $\alpha \in YV_2Y$, then $r(\alpha) \in U$ and hence $g(r(\alpha)) = 1$; it follows that

$$\mu_{x_n}^{(1)} \geq \int_{YV_2Y} g(r(\alpha))^2\ d\lambda_{x_n}(\alpha) = \lambda_{x_n}(YV_2Y) \geq a > 0$$

whenever $x_n \in V_3$.

So far our set-up is the one from [8]. Now choose $l \in \mathbb{N} \setminus \{0\}$ such that $la^2 > M$. (Note that a is independent of l!) The sequence $\{x_n\}$ converges k-times in G/G^0 to z for every $k \in \mathbb{N} \setminus \{0\}$, so it certainly converges l times. So there exist l sequences

$$\{\gamma_n^{(1)}\},\ \{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(l)}\} \subseteq G$$

such that

(1) $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leq i \leq l$;
(2) $s(\gamma_n^{(i)}) = x_n$ for $1 \leq i \leq k$;
(3) if $1 \leq i < j \leq l$, then $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$.

Moreover, by construction (see Proposition 3.11), we may take $\gamma_n^{(1)} = x_n$. Temporarily fix n. Set $g_n^{(1)} := g^{(1)}$, and for $2 \leq j \leq l$ set

$$\begin{aligned} g_n^{(j)}(\gamma) &:= \begin{cases} g^{(1)}(\gamma(\gamma_n^j)^{-1}), & \text{if } s(\gamma) = s(\gamma_n^j); \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g(r(\gamma)), & \text{if } \gamma \in \overline{W}_1^7\overline{V}_1\overline{W}_1^7\gamma_n^j; \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g(r(\gamma)), & \text{if } \gamma \in \overline{W}_1^7\overline{V}_1\overline{W}_1^7\gamma_n^j; \\ 0, & \text{if } \gamma \notin W_0V_0W_0\gamma_n^j. \end{cases} \end{aligned}$$

Each $g_n^{(i)}$ $(1 \leq j \leq l)$ is a well-defined function in $C_c(G)$ with support contained in $W_0V_0W_o\gamma_n^{(j)}$. For $1 \leq i < j \leq l$, $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \notin (W_0V_0W_0)^2$ eventually, so there exists $n_0 > 0$ such that, for every $0 \le i, j \le l$, $i \ne j$,

$$W_0 V_0 W_0 \gamma_n^{(j)} \cap W_0 V_0 W_0 \gamma_n^{(i)} = \emptyset$$

whenever $n > n_0$.

We now prove a generalization of [8, Lemma 2.8] which, together with (4.2), immediately implies that each $g_n^{(j)}$ is an eigenvector of $L^{x_n}(f)$ for $1 \le j \le l$.

Lemma 4.2. *With the choices made above, for all $\alpha, \gamma \in G$ and $1 \le j \le l$,*

$$g(r(\gamma))g(r(\alpha))b(\gamma\alpha^{-1})g_n^{(j)}(\alpha) = g_n^{(j)}(\gamma)g(r(\alpha))g_n^{(j)}(\alpha).$$

Proof. If $\alpha \notin W_0 V_0 W_0 \gamma_n^{(j)}$, then both sides are zero. So we may assume throughout that $\alpha \in W_0 V_0 W_0 \gamma_n^{(j)}$.

If $\gamma \in W_0 V_0 W_0 \gamma_n^{(j)}$, then $g_n^{(j)}(\gamma) = g(r(\gamma))$ and $\gamma\alpha^{-1} \in W_0 V_0 W_0^2 V_0 W_0$, so $b(\gamma\alpha^{-1}) = 1$ and both sides agree.

If $\gamma \in \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \gamma_n^{(j)} \setminus W_0 V_0 W_0 \gamma_n^{(j)}$, then $g(r(\gamma)) = 0 = g_n^{(j)}(\gamma)$, so both sides are zero.

Finally, if $\gamma \notin \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \gamma_n^{(j)}$, then $g_n^{(j)}(\gamma) = 0$, so the right-hand side is zero. On the other hand, if $\gamma\alpha^{-1} \in \overline{W}_1^4 \overline{V}_1 \overline{W}_1^4 (= \operatorname{supp} b)$, then

$$\gamma \in \overline{W}_1^4 \overline{V}_1 \overline{W}_1^7 \gamma_n^{(j)} \subseteq \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \gamma_n^{(j)}. \tag{4.4}$$

So $\gamma \notin \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \gamma_n^{(j)}$ implies $\gamma\alpha^{-1} \notin \operatorname{supp} b$, so the left-hand side is zero as well. □

Let $\mu_n^{(j)}$ be the eigenvalue corresponding to the eigenvector $g_n^{(j)}$. Using (4.3),

$$\mu_n^{(j)} = \int_{W_0 V_0 W_0 \gamma_n^{(j)}} g(r(\alpha))^2 \, d\lambda_{x_n}(\alpha) \ge \lambda_{x_n}(Y V_2 Y \gamma_n^{(j)}) = \lambda_{r(\gamma_n^{(j)})}(Y V_2 Y) \ge a$$

whenever $r(\gamma_n^{(j)}) \in V_3$. Choose $n_1 > n_0$ such that $n > n_1$ implies $x_n \in V_3$ and $r(\gamma_n^{(j)}) \in V_3$ for $1 \le j \le l$. Then $L^{x_n}(f * f)$ is a positive compact operator with l eigenvalues $\mu_n^{(j)} \ge a^2$ for $1 \le j \le l$. To push $f * f$ into the Pedersen ideal, let $r \in C_c(0, \infty)$ be any function satisfying

$$r(t) = \begin{cases} 0, & \text{if } t < \frac{a^2}{3}; \\ 2t - \frac{2a^2}{3}, & \text{if } \frac{a^2}{3} \le t < \frac{2a^2}{3}; \\ t, & \text{if } \frac{2a^2}{3} \le t \le \|f * f\|. \end{cases}$$

Set $d := r(f * f)$. Now d is a positive element of the Pedersen ideal of $C^*(G)$ with $\operatorname{tr}(L^{x_n}(d)) \ge la^2 > M$ whenever $n > n_1$. Since M was arbitrary, $L^x \mapsto \operatorname{tr}(L^x(d))$ is unbounded on $C^*(G)^\wedge$. Thus $C^*(G)$ does not have bounded trace. □

Proposition 4.3. *Suppose G is a second countable, locally compact, Hausdorff, principal groupoid. If G is integrable, then $C^*(G)$ has bounded trace.*

Proof. Since G is principal and integrable, the orbits are closed by Lemma 3.9, and $x \mapsto L^x$ induces a homeomorphism of G^0/G onto $C^*(G)^\wedge$ by [4, Proposition 5.1]. To show that $C^*(G)$ has bounded trace, it suffices to see that for a fixed $u \in G^0$ and all $f \in C_c(G)$, $\operatorname{tr}(L^u(f^* * f))$ is bounded independent of u.

Fix $u \in G^0$ and let $\xi \in L^2(G, \lambda_u)$. Since

$$L^u(f)\xi(\gamma) = \int f(\gamma\alpha^{-1})\xi(\alpha) \, d\lambda_u(\alpha),$$

$L^u(f)$ is a kernel operator on $L^2(G, \lambda_u)$ with kernel k_f given by $k_f(\gamma, \alpha) = f(\gamma\alpha^{-1})$. We will show that $k_f \in L^2(G \times G, \lambda_u \times \lambda_u)$ and we will find a bound on k_f independent of u. This will complete the proof since $\operatorname{tr}(L^u(f^* * f)) = \|k_f\|^2$ by, for example, [10, Theorem 3.4.16].

Notice that

$$\|k_f\|^2 = \int_{G\times G} |k_f(\gamma,\alpha)|^2 \, d(\lambda_u \times \lambda_u)(\gamma,\alpha) = \int_G \int_G |f(\gamma\alpha^{-1})|^2 \, d\lambda_u(\gamma) \, d\lambda_u(\alpha)$$

$$= \int_G \int_G |f(\gamma)|^2 \, d\lambda_{r(\alpha)}(\gamma) \, d\lambda_u(\alpha) \tag{4.5}$$

by Tonelli's Theorem and right invariance. For a fixed α, the inner integral

$$\int_G |f(\gamma)|^2 \, d\lambda_{r(\alpha)}(\gamma) \le \|f\|_\infty^2 \lambda_{r(\alpha)}(\operatorname{supp} f)$$

and is zero unless $r(\alpha) \in s(\operatorname{supp} f)$. The outer integral is zero unless $s(\alpha) = u$. Let

$$K = r^{-1}(s(\operatorname{supp} f)) \cap s^{-1}(\{u\}).$$

So

$$\begin{aligned}(4.5) &\le \int_K \|f\|_\infty^2 \lambda_{r(\alpha)}(\operatorname{supp} f) \, d\lambda_u(\alpha) \\ &\le \|f\|_\infty^2 \sup\big\{\lambda_{r(\alpha)}(\operatorname{supp} f) : r(\alpha) \in s(\operatorname{supp} f)\big\}\lambda_u(K) \\ &\le \|f\|_\infty^2 \sup\big\{\lambda_x(\operatorname{supp} f) : x \in s(\operatorname{supp} f)\big\} \sup\big\{\lambda_x(r^{-1}(s(\operatorname{supp} f)) : x \in G^0\big\}.\end{aligned}$$

Since $s(\operatorname{supp} f)$ is a compact subset of G^0, by integrability there exists $N > 0$ such that

$$\sup\big\{\lambda_x(r^{-1}(s(\operatorname{supp} f))) : x \in G^0\big\} < N$$

(see also Remark 3.2). Note that N does not depend on u. By Lemma 3.10(2), applied to the conditionally compact neighborhood $\operatorname{supp} f$ and the relatively compact neighborhood $s(\operatorname{supp} f)$, there exists $M > 0$ such that $\lambda_x(\operatorname{supp} f) < M$ for all $x \in s(\operatorname{supp} f)$; that is,

$$\sup\{\lambda_x(\operatorname{supp} f) : x \in s(\operatorname{supp} f)\} < M.$$

Note that M does not depend on u.

Thus $\|k_f\|^2 < \|f\|_\infty^2 MN$, so $k_f \in L^2(G \times G, \lambda_u \times \lambda_u)$ as claimed, and

$$\operatorname{tr}(L^u(f^* * f)) = \|k_f\|^2 < \|f\|_\infty^2 MN,$$

which is a bound on $\operatorname{tr}(L^u(f^* * f))$ independent of u. □

Combining Propositions 4.1 and 4.3 we have

Theorem 4.4. *Suppose G is a second countable, locally compact, Hausdorff, principal groupoid. Then G is integrable if and only if $C^*(G)$ has bounded trace.*

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