PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 2, February 2008, Pages 577–585
S 0002-9939(07)09080-6
Article electronically published on November 3, 2007

MAPPING PROPERTIES OF ANALYTIC FUNCTIONS ON THE UNIT DISK

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(Communicated by Juha M. Heinonen)

ABSTRACT. Let f be analytic on the unit disk $\mathbb{D}$ with $f(0) = 0$. In 1989, D. Marshall conjectured the existence of the universal constant $r_0 > 0$ such that $f(r_0\mathbb{D}) \subset \mathbb{D}_M := \{w : |w| < M\}$ whenever the area, counting multiplicity, of a portion of $f(\mathbb{D})$ over $\mathbb{D}_M$ is $< \pi M^2$. Recently, P. Poggi-Corradini (2007) proved this conjecture with an unspecified constant by the method of extremal metrics. In this note we show that such a universal constant r_0 exists for a much larger class consisting of analytic functions omitting two values of a certain doubly-sheeted Riemann surface. We also find a numerical value, $r_0 = .03949\ldots$, which is sharp for the problem in this larger class but is not sharp for Marshall's problem.

Let f be analytic on $\mathbb{D} = \{z : |z| < 1\}$. For $M > 0$, let $\Omega(M) = \{z \in \mathbb{D} : |f(z)| < M\}$ and let $A(M) = \int_{\Omega(M)} |f'|^2\, dA$ be the area of $f(\Omega(M))$ counting multiplicity.

Theorem 1. *Let f be analytic on $\mathbb{D}$ with $f(0) = 0$. If $M > 0$ is such that $A(M) < \pi M^2$, then $f(r_0\mathbb{D}) \subset \mathbb{D}_M := \{w : |w| < M\}$, where $r_0 = .03949\ldots$ is defined by equation* (18) *below.*

This theorem, conjectured by D. Marshall in [5], was recently proved with an unspecified constant r_0 by P. Poggi-Corradini [7] by the method of extremal metrics. In Theorem 2 below, which is the main result of this note, we prove that a universal constant $r_0 > 0$, such that $f(r_0\mathbb{D}) \subset \mathbb{D}$, exists for a much larger class of analytic functions omitting two values of a doubly-sheeted Riemann surface of the square root function. Then Theorem 1 will follow from Theorem 2 as a corollary.

Now we introduce some notation for future use. If D is a domain on $\overline{\mathbb{C}}$ having at least three boundary points, then $\lambda_D(\zeta_1, \zeta_2)$ will denote the hyperbolic distance between points ζ_1 and ζ_2 in D. For the definition and necessary properties of the hyperbolic metric we refer to [2], [4], and [9]. For any $a \in \mathbb{C}\setminus\{0, 1\}$, let $\varphi_a(z)$ denote the universal covering of $\mathbb{C}_a = \mathbb{C} \setminus \{1, a\}$ by $\mathbb{D}$ such that $\varphi_a(0) = 0$ and $\varphi_a'(0) > 0$ and let $\lambda(\zeta_1, \zeta_2; a) = \lambda_{\mathbb{C}_a}(\zeta_1, \zeta_2)$. For $0 \le \rho < \infty$, let $\mathcal{R}_\rho$ be the doubly-sheeted Riemann surface of $\sqrt{w - \rho}$. For $w_0 \in \mathbb{C}$, let $\hat{w}_0^1$ and $\hat{w}_0^2$ denote the points on $\mathcal{R}_\rho$ having affix w_0. For $0 \le x \le 1$, let $\mathcal{K}(x)$ and $\mathcal{K}'(x) = \mathcal{K}(\sqrt{1 - x^2})$ denote the complete elliptic integrals of the first kind.

Received by the editors October 26, 2006.
2000 *Mathematics Subject Classification.* Primary 30C55, 30F45.
Key words and phrases. Analytic function, growth theorem, hyperbolic metric.
This research was supported in part by NSF grant DMS-0525339.

Theorem 2. *Let f be analytic in $\mathbb{D}$ with values in $\mathcal{R}_\rho$, $0<\rho<\infty$, such that $f(0)=0$. If f omits the point $w=\rho$ and for some $w_0\in\mathbb{C}$ such that $|w_0|\le\rho$, $\Re w_0\le 0$, covers at most one of the points $\hat{w}_0^1$ and $\hat{w}_0^2$ on $\mathcal{R}_\rho$, then*

$$|f(z)|\le 1 \tag{1}$$

for all $|z|\le r_0(\rho)$, where

$$r_0(\rho)=\frac{\mathcal{K}\left(\sqrt{\frac{s}{s+1}}\right)-\mathcal{K}'\left(\sqrt{\frac{s}{s+1}}\right)}{\mathcal{K}\left(\sqrt{\frac{s}{s+1}}\right)+\mathcal{K}'\left(\sqrt{\frac{s}{s+1}}\right)} \tag{2}$$

if $0<\rho\le 1$ and

$$r_0(\rho)=\frac{\mathcal{K}'\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)-\mathcal{K}\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}'\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)}{\mathcal{K}'\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)+\mathcal{K}\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}'\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)} \tag{3}$$

if $1<\rho<\infty$ with

$$s=s(\rho)=\sqrt{1+\rho^{-1}}. \tag{4}$$

Equality occurs in (1) *for some z_0 such that $|z_0|\le r_0(\rho)$ if and only if $|z_0|=r_0(\rho)$ and $f(z)=\rho\varphi_a(e^{i\alpha}z)(2-\varphi_a(e^{i\alpha}z))$ with $\alpha=\pi-\arg z_0$, where $a=2$ if $0<\rho<1$, $a=1+\sqrt{2}$ if $1<\rho<\infty$, and $a=2$ or $a=1+\sqrt{2}$ if $\rho=1$.*

In addition, $r_0(\rho)$ defined by (2) *for $0<\rho\le 1$ and by* (3) *for $1<\rho<\infty$ is a continuous strictly decreasing function from $(0,\infty)$ onto $(0,1)$.*

Proof. First we consider an auxiliary doubly-valued function ψ defined by

$$\psi(w)=1-\sqrt{1-w/\rho}.$$

Then ψ maps the circle $C_\rho=\{w:\,|w|=\rho\}$ one-to-two onto Bernoulli's lemniscate $l=\{\zeta:\,|\zeta(2-\zeta)|=1\}$ with foci at $\zeta=0$ and $\zeta=2$. Similarly, ψ maps the unit circle $\mathbb{T}$ one-to-two onto the Cassini oval $l_\rho=\{\zeta:\,|\zeta(2-\zeta)|=\rho^{-1}\}$. Of course, $l=l_1$. For convenience of the reader, we review some elementary properties of the Cassini ovals and related properties of the function ψ. For $\rho>1$, l_ρ consists of two loops, say l_ρ^1 and l_ρ^2, such that $\zeta=2$ is in the interior of l_ρ^1. For $0<\rho<1$, the curve l_ρ is a single loop. In this case we put $l_\rho^1=\{\zeta\in l_\rho:\,\Re\zeta\ge 1\}$ and $l_\rho^2=\{\zeta\in l_\rho:\,\Re\zeta\le 1\}$. For $k=1,2$, let $l_{\rho,k}^+=\{\zeta\in l_\rho^k:\,\Im\zeta\ge 0\}$ and $l_{\rho,k}^-=\{\zeta\in l_\rho^k:\,\Im\zeta\le 0\}$.

In the proof below, we will use the following monotonicity property of the Cassini oval distances: The modulus $|\zeta|$ strictly decreases and the modulus $|\zeta-1|$ strictly increases when ζ runs along $l_{\rho,2}^+$ in such a way that $\Re\zeta$ decreases.

The function ψ maps the imaginary axis $\{w:\,\Re w=0\}$ one-to-two onto two branches of the hyperbola $\Gamma=\{\zeta=\xi+i\eta:\,(\xi-1)^2-\eta^2=1\}$. It maps the set $\{w:\,|w|>1\}$ one-to-two onto the domain G_0 exterior to l_ρ.

Finally, we note that ψ has two single-valued branches in $\mathbb{D}_\rho^-=\{w:\,|w|<\rho,\,\Re w<0\}$. One of these branches, say ψ_1, maps $\mathbb{D}_\rho^-$ conformally and one-to-one onto the domain G_1 in the half-plane $\{\zeta:\,\Re\zeta>2\}$, which is bounded by an arc of the hyperbola Γ and an arc on the right petal of the lemniscate l. Then the second branch, ψ_2, maps $\mathbb{D}_\rho^-$ conformally and one-to-one onto the domain $G_2=\{\zeta:\,2-\zeta\in G_1\}$. Let $\zeta_k=\psi_k(w_0)$, $k=1,2$. Then $\zeta_1\in\overline{G}_1$ and $\zeta_2\in\overline{G}_2$.

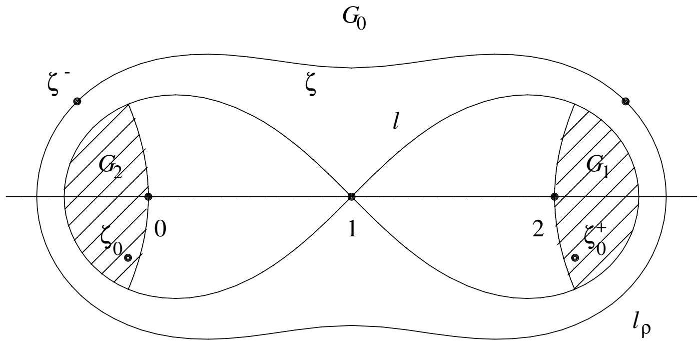

FIGURE 1. Cassini ovals.

For $\rho = 5/7$, the curves l and l_ρ and domains G_1 and G_2 are shown in Figure 1, which also illustrates some other notation of this proof.

Since $\rho \notin f(\mathbb{D})$, the function $g = \psi \circ f$ can be considered as a single-valued function that is analytic in $\mathbb{D}$ such that $g(0) = 0$. We note that g omits the point $\zeta = 1$ and at least one of the points ζ_1 and ζ_2. Let ζ_0 denote this second omitted point.

Since analytic maps are contractions in the hyperbolic metric (see [2, Ch. 9.4]), we have

$$\lambda(0, g(z); \zeta_0) \leq \lambda_{\mathbb{D}}(0, z), \qquad \text{for all } z \in \mathbb{D}, \tag{5}$$

with the equality sign for some $0 < |z_0| < 1$ if and only if $g(z) = \varphi_{\zeta_0}(e^{i\alpha} z)$ with some $\alpha \in \mathbb{R}$.

To estimate $\lambda(0, g(z); \zeta_0)$, we consider the following minimization problem: Find

$$\inf \ \lambda(0, \zeta; \zeta_0) \tag{6}$$

over all $\zeta \in \overline{G}_0$ and all $\zeta_0 \in \overline{G}_1 \cup \overline{G}_2$. Since l_ρ separates 0 from G_0, it follows that every continuous curve from 0 to $\zeta \in G_0$ meets l_ρ. This shows that we may assume in (6) that $\zeta \in l_\rho$. The hyperbolic distance $\lambda(0, \zeta; \zeta_0)$ is a continuous function of $\zeta \in l_\rho$ and $\zeta_0 \in \overline{G}_1 \cup \overline{G}_2$ if $\zeta \neq \zeta_0$. Continuity in ζ is obvious since the universal covering map $\zeta = \varphi_{\zeta_0}(z)$ is an analytic function of $z \in \mathbb{D}$. The continuous dependence of $\lambda(0, \zeta; \zeta_0)$ on the puncture ζ_0 is also well known and follows from Theorem 1 in [4].

For a given $\zeta \in l_\rho$, let $\zeta^- = \zeta$ if $\Re\zeta \leq 1$ and let $\zeta^- = 2 - \bar{\zeta}$ if $\Re\zeta > 1$. Also let $\zeta_0^+ = \zeta_0$ if $\Re\zeta_0 \geq 1$ and let $\zeta_0^+ = 2 - \bar{\zeta}_0$ if $\Re\zeta_0 < 1$. It follows from Theorem 9 in [8] on the change of the hyperbolic density under polarization that

$$\lambda(0, \zeta^-; \zeta_0^+) \leq \lambda(0, \zeta; \zeta_0) \qquad \text{for all } \zeta \in l_\rho \text{ and } \zeta_0 \in \overline{G}_1 \cup \overline{G}_2, \tag{7}$$

with the sign of equality if and only if $\zeta^- = \zeta$ and $\zeta_0^+ = \zeta_0$. This shows that considering the minimization problem (6) we may assume that $\zeta \in l_\rho^1$ and $\zeta_0 \in \overline{G}_1$. Then, of course, $\zeta \neq \zeta_0$ and therefore $\lambda(0, \zeta; \zeta_0)$ achieves its minimum at some points $\zeta = \zeta^*$ and $\zeta_0 = \zeta_0^*$ such that $\zeta^* \in l_\rho^1$ and $\zeta_0^* \in \overline{G}_1$. Using Minda's inequality for hyperbolic density (see [9, Theorem 2.12]) or applying polarization with respect to the real axis, we may assume that $\Im\,\zeta^* \geq 0$ and $\Im\,\zeta_0^* \leq 0$.

We claim that $\zeta^* = 1 - s$, where $s = \sqrt{1 + \rho^{-1}}$, and $\zeta_0^* = 2$ if $0 < \rho < 1$, $\zeta_0^* = 1 + \sqrt{2}$ if $1 < \rho < \infty$, and $\zeta_0^* = 2$ or $\zeta_0^* = 1 + \sqrt{2}$ if $\rho = 1$. To prove this, we first show that ζ^* and ζ_0^* are real. Then we will have

$$\zeta^* = 1 - s \qquad \text{and} \qquad 2 \leq \zeta_0^* \leq 1 + \sqrt{2}. \tag{8}$$

Assume that $\alpha = \arg(\zeta_0^* - 1) < 0$. For $\varepsilon \geq 0$ small enough, let $\gamma_\varepsilon = \{\zeta = 1 + te^{i(\alpha+\varepsilon)} : -\infty < t < \infty\}$ and let H_ε^+ and H_ε^- be the half-planes of $\mathbb{C} \setminus \gamma_\varepsilon$ such that $2 \in H_\varepsilon^+$.

Suppose that $\zeta^* \in H_0^-$. Let $\varepsilon > 0$ be small enough such that $\zeta^* \in H_\varepsilon^-$. Let ζ_0^ε be the reflection of ζ_0^* in γ_ε. It follows from the monotonicity property of the lemniscate distances mentioned in the beginning of this proof that $\zeta_0^\varepsilon \in G_1$ if $\zeta_0 \in \overline{G}_1$. In addition, $\Im\,\zeta_0^\varepsilon < 0$ if $\varepsilon > 0$ is small enough. Applying polarization (Theorem 9 of [8]) once more, we obtain the strict inequality

$$\lambda(0, \zeta^*; \zeta_0^\varepsilon) < \lambda(0, \zeta^*; \zeta_0), \tag{9}$$

contradicting the minimality property of $\lambda(0, \zeta^*; \zeta_0^*)$. Thus, we must have $\zeta^* \in H_0^+ \cup \gamma_0$.

Suppose now that $\beta = \arg \zeta^* > 0$. For $\varepsilon \geq 0$ small enough, let $\nu_\varepsilon = \{\zeta = te^{i(\beta+\varepsilon)} : -\infty < t < \infty\}$. Then by Minda's inequality for the hyperbolic density, see [9, Theorem 2.12] (or by Theorem 9 in [8] for polarization with respect to ν_ε), we obtain

$$\lambda(0, e^{2i\varepsilon}\zeta^*; \zeta_0^*) < \lambda(0, \zeta^*; \zeta_0^*). \tag{10}$$

Using the monotonicity property of the Cassini ovals mentioned above, we find that $e^{2i\varepsilon}\zeta^* \in \overline{G}_0$, whenever $\zeta^* \in l_\rho^1$. Hence, (10) contradicts the minimality property of $\lambda(0, \zeta^*; \zeta_0^*)$.

Therefore, we must have $\Im\,\zeta^* = 0$. Then we also have $\Im\,\zeta_0^* = 0$. Now, using the monotonicity property of the lemniscate distances, we obtain (8).

Let $\tau = \psi_1(\zeta)$ with

$$\psi_1(\zeta) = -\frac{\zeta - (1 - \sqrt{s})}{\zeta - (1 + \sqrt{s})},$$

where s is defined by (4). Then ψ_1 maps $\mathbb{C}_b$ with $b = \zeta_0^*$ conformally onto $\overline{\mathbb{C}}_t = \overline{\mathbb{C}} \setminus \{-1, 1, t\}$, where $t = \psi_1(\zeta_0^*)$, $1 < |t| \leq \infty$. Let $a = \psi_1(0) = (\sqrt{s} - 1)/(\sqrt{s} + 1)$. Then $0 < a < 1$ and $\psi_1(1 - s) = -a$. Since ψ_1 is conformal we have $\lambda(0, 1 - s; \zeta_0^*) = \lambda_{\overline{\mathbb{C}}_t}(-a, a)$.

Let $t_1 = \psi_1(2) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$, $t_2 = \psi_1(1 + \sqrt{2}) = \frac{\sqrt{s}+\sqrt{2}}{\sqrt{s}-\sqrt{2}}$, and let $t_0 = \min\{t_1, |t_2|\}$. Then $1 < t_0 \leq |t| \leq \infty$. Now, using the monotonicity property of $\lambda_{\overline{\mathbb{C}}_t}(-a, a)$ proved in Lemma 1 below, we obtain

$$\lambda_{\overline{\mathbb{C}}_{t_0}}(-a, a) < \lambda_{\overline{\mathbb{C}}_t}(-a, a) = \lambda(0, 1 - s; \zeta_0^*) \tag{11}$$

for all ζ_0^* such that $2 < \zeta_0^* < 1 + \sqrt{2}$. Using the explicit expressions for t_1 and t_2, one can easily check that $t_0 = t_1$ if $0 < \rho \leq 1$ and $t_0 = t_2$ if $1 \leq \rho < \infty$.

Combining inequalities (7), (9), (10), and (11), we obtain the following solution to problem (6). For all $\zeta \in \overline{G}_0$ and $\zeta_0 \in \overline{G}_1 \cup \overline{G}_2$,

$$\lambda(0, 1 - s; 2) \leq \lambda(0, \zeta; \zeta_0) \qquad \text{if } \ 0 < \rho \leq 1 \tag{12}$$

and

$$\lambda(0, 1 - s; 1 + \sqrt{2}) \leq \lambda(0, \zeta; \zeta_0) \qquad \text{if } \ 1 \leq \rho < \infty. \tag{13}$$

If $0 < \rho < 1$, then equality occurs in (12) if and only if $\zeta = 1 - s$ and $\zeta_0 = 2$. If $1 < \rho < \infty$, then equality occurs in (13) if and only if $\zeta = 1 - s$ and $\zeta_0 = 1 + \sqrt{2}$. For $\rho = 1$, we have

$$\lambda(0, 1-\sqrt{2}; 2) = \lambda(0, 1-\sqrt{2}; 1+\sqrt{2}) \le \lambda(0, \zeta; \zeta_0)$$

with the sign of equality if and only if $\zeta = 1 - \sqrt{2}$ and $\zeta_0^* = 2$ or $\zeta_0^* = 1 + \sqrt{2}$.

To evaluate $\lambda(0, 1-s; 2)$, we make a linear change of variables $\zeta \mapsto 3 - 2\zeta$ to obtain $\lambda(0, 1-s; 2) = \lambda(3, 1+2s; -1, 1)$. Here $\lambda(a, b; -1, 1) = \lambda_D(a, b)$ with $D = \mathbb{C} \setminus \{-1, 1\}$. Now to find $\lambda(a, b; -1, 1)$ for $1 < a < b < \infty$, we apply the following formula (see [9, Lemma 3.10]):

$$\lambda(a, b; -1, 1) = \frac{1}{2} \log \left(\frac{\mu\left(\sqrt{\frac{a-1}{a+1}}\right)}{\mu\left(\sqrt{\frac{b-1}{b+1}}\right)} \right), \tag{14}$$

where $\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}$. Taking $a = 3$, $b = 1 + 2s$ in (14) and simplifying, we find

$$\lambda(0, 1-s; 2) = \frac{1}{2} \log \frac{\mathcal{K}\left(\sqrt{\frac{s}{s+1}}\right)}{\mathcal{K}'\left(\sqrt{\frac{s}{s+1}}\right)}.$$

The hyperbolic distance $\lambda_{\mathbb{D}}(0, r)$ from 0 to $z = r$, $0 < r < 1$, in $\mathbb{D}$ is given by $\lambda_{\mathbb{D}}(0, r) = (1/2)\log((1+r)/(1-r))$. Solving the equation $(1/2)\log((1+r)/(1-r)) = \lambda(0, 1-s; 2)$ for r, we find the solution $r_1 = r_1(\rho)$, where $r_1(\rho)$ is the function given by the right-hand side of equation (2) with s defined by (4) and $0 < \rho < \infty$. It is clear that both sides of the latter equation are monotone functions. Hence, the solution $r_1(\rho)$ is monotone on $(0, \infty)$. Since $\lambda(0, 1-s; 2) \to 0$ as $s \to 1$ and $\lambda(0, 1-s; 2) \to \infty$ as $s \to \infty$, it is easily seen that $r_1(\rho) \to 0$ as $\rho \to \infty$ and $r_1(\rho) \to 1$ as $\rho \to 0^+$.

To evaluate $\lambda(0, 1-s; 1+\sqrt{2})$, we change variables via $\zeta \mapsto 1+\sqrt{2}-\sqrt{2}\zeta$. Then we obtain $\lambda(0, 1-s; 1+\sqrt{2}) = \lambda(1+\sqrt{2}, 1+\sqrt{2}s; -1, 1)$. To find $\lambda(1+\sqrt{2}, 1+\sqrt{2}s; -1, 1)$, we use (14) once more now with $a = 1 + \sqrt{2}$ and $b = 1 + \sqrt{2}s$. After simplification we obtain

$$\lambda(0, 1-s; 1+\sqrt{2}) = \frac{1}{2} \log \frac{\mathcal{K}'\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)}{\mathcal{K}\left(\sqrt{\frac{1}{1+\sqrt{2}}}\right)\mathcal{K}'\left(\sqrt{\frac{s}{s+\sqrt{2}}}\right)}. \tag{15}$$

Solving the equation $(1/2)\log((1+r)/(1-r)) = \lambda(0, 1-s; 1+\sqrt{2})$ with $\lambda(0, 1-s; 1+\sqrt{2})$ given by (15) for r, we find its solution $r_2 = r_2(\rho)$, where $r_2(\rho)$ is the function given by the right-hand side of equation (3) with s defined by (4) and $0 < \rho < \infty$. As in the case of $r_1(\rho)$, it is easy to see that $r_2(\rho)$ strictly decreases from 1 to 0 as ρ runs from 0 to ∞.

Since

$$\lambda(0, 1-\sqrt{2}; 2) = \lambda(0, 1-\sqrt{2}; 1+\sqrt{2}) = (1/2)\log \frac{\mathcal{K}'(\sqrt{\sqrt{2}-1})}{\mathcal{K}(\sqrt{\sqrt{2}-1})},$$

the monotonicity properties of $r_1(\rho)$ and $r_2(\rho)$ imply that the function $r_0(\rho)$ defined by (2) for $0 < \rho \le 1$ and by (3) for $1 < \rho < \infty$ is continuous and strictly decreasing from $(0, \infty)$ onto $(0, 1)$.

To prove inequality (1), we assume that $|f(z_0)| \geq 1$ for some z_0 such that $|z_0| \leq r_0(\rho)$. Then $g(z_0) \in \overline{G}_0$, where the function $g = \psi \circ f$ and domain G_0 are defined above. Since analytic maps are contractions in the hyperbolic metric, from (5), (12), and (13), we obtain

$$(16) \qquad \lambda_{\mathbb{D}}(0, r_0(\rho)) \geq \lambda_{\mathbb{D}}(0, z_0) \geq \lambda(0, g(z_0); \zeta_0) \geq \lambda(0, 1-s; 2) = \lambda_{\mathbb{D}}(0, r_0(\rho))$$

if $0 < \rho \leq 1$ and

$$(17) \quad \lambda_{\mathbb{D}}(0, r_0(\rho)) \geq \lambda_{\mathbb{D}}(0, z_0) \geq \lambda(0, g(z_0); \zeta_0) \geq \lambda(0, 1-s; 1+\sqrt{2}) = \lambda_{\mathbb{D}}(0, r_0(\rho))$$

if $1 \leq \rho < \infty$.

If $0 < \rho < 1$, we must have equality in all the inequalities in (16). From the first of these equalities we find that $z = r_0(\rho)$. The second equality implies that $g(z) = \varphi_{\zeta_0}(e^{i\alpha} z)$ for some $\alpha \in \mathbb{R}$. In case of equality in the third inequality in (16), we have $\zeta_0 = 2$ and $g(z_0) = 1 - s < 0$. Therefore, in this case we must have $f(z) = \rho\varphi_a(e^{i\alpha} z)(2 - \varphi_a(e^{i\alpha} z))$ with $a = 2$ and $\alpha = \pi - \arg z_0$.

If $1 < \rho < \infty$, we must have equality in all the inequalities in (17). Analyzing the cases of equality as in (16), we find that $\zeta_0 = 1 + \sqrt{2}$, $g(z_0) = 1 - s$, and $f(z) = \rho\varphi_a(e^{i\alpha} z)(2 - \varphi_a(e^{i\alpha} z))$ with $a = 1 + \sqrt{2}$ and $\alpha = \pi - \arg z_0$. Finally, if $\rho = 1$, then (16) and (17) each becomes a chain of equalities. This implies that $f(z) = \varphi_a(e^{i\alpha} z)(2 - \varphi_a(e^{i\alpha} z))$ with $\alpha = \pi - \arg z_0$ and $a = 2$ or $a = 1 + \sqrt{2}$. The proof is complete. □

Corollary 1. *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$. If f omits the point $w = \rho$, $0 < \rho < \infty$, and covers a point w_0 such that $|w_0| \leq \rho$, $\Re w_0 \leq 0$, at most once, then $|f(z)| < 1$ for all $|z| \leq r_0(\rho)$, where $r_0(\rho)$ is defined by* (2) *for $0 < \rho \leq 1$ and by* (3) *for $1 < \rho < \infty$.*

In particular, if $\rho \leq 1$, then $f(r_0\mathbb{D}) \subset \mathbb{D}$, where

$$(18) \qquad r_0 = r_0(1) = \frac{\mathcal{K}'(\sqrt{\sqrt{2}-1}) - \mathcal{K}(\sqrt{\sqrt{2}-1})}{\mathcal{K}'(\sqrt{\sqrt{2}-1}) + \mathcal{K}(\sqrt{\sqrt{2}-1})} = .03949\ldots$$

Proof. Since f satisfies the assumptions of Theorem 2, we have $|f(z)| \leq 1$ for $|z| \leq r_0(\rho)$. We note that for every $0 < \rho < \infty$, every extremal function of Theorem 2 covers every point w in $\mathbb{C} \setminus \{\rho\}$ infinitely many times. Thus, if f satisfies the assumptions of this corollary, then it is not extremal for Theorem 2 and therefore $|f(z)| < 1$ for all $|z| \leq r_0(\rho)$.

Since $r_0(\rho) \geq r_0 = r_0(1)$ if $\rho \leq 1$, we have $f(r_0\mathbb{D}) \subset f(r_0(\rho)\mathbb{D}) \subset \mathbb{D}$. □

Proof of Theorem 1. Dilating the image if necessary, we may assume that $M = 1$. Since $A(1) < \pi$, we have $\overline{\mathbb{D}} \setminus f(\mathbb{D}) \neq \emptyset$. Let $\rho = \max |w|$, where the maximum is taken over all points $w \in \overline{\mathbb{D}} \setminus f(\mathbb{D})$. Then $0 < \rho \leq 1$ and there is a point $\zeta \in \overline{\mathbb{D}} \setminus f(\mathbb{D})$ such that $|\zeta| = \rho$. Rotating if necessary, we may assume that $\zeta = \rho$. Since $A(1) < \pi$ and $\{w : \rho < |w| < 1\} \subset f(\mathbb{D})$ there is a point w_0 with $|w_0| \leq \rho$ and $\Re w_0 \leq 0$, which is covered by $f(\mathbb{D})$ at most once. Now Theorem 1 with $r_0 = .03949\ldots$ follows from Corollary 1. □

Lemma 1. *For $0 < a < 1$ and $1 < t < \infty$, let $h(t, a) = \lambda_{D(t)}(-a, a)$, where $D(t) = \overline{\mathbb{C}} \setminus \{-1, 1, t\}$. For a fixed a, the hyperbolic distance $h(t, a)$ strictly increases from 0 to $\log\left(\mathcal{K}'(\sqrt{(1-a)/2})/\mathcal{K}(\sqrt{(1-a)/2})\right)$ as t runs from 1 to ∞.*

Proof. Let $\zeta = \varphi(z)$ be a Möbius map such that $\varphi(1) = 1$, $\varphi(t) = -1$, and $\varphi(-1) = \infty$. Then

$$\varphi(z) = 1 + \frac{2}{\tau}\frac{1-z}{1+z}, \qquad \text{where } \tau = \frac{t-1}{t+1}.$$

Since φ is conformal, we have $h(t, a) = \lambda(\varphi(a), \varphi(-a); -1, 1)$. Let $r = (1-a)/(1+a)$. Then

$$\varphi(a) = 1 + \frac{2r}{\tau}, \qquad \varphi(-a) = 1 + \frac{2}{\tau r}.$$

Since $1 < \varphi(a) < \varphi(-a) < \infty$, applying (14), we obtain $h(t, a) = (1/2)\log F(\tau, r)$, where

$$F(\tau, r) = \frac{\mathcal{K}'\left(\frac{1}{\sqrt{1+\tau r^{-1}}}\right)\mathcal{K}\left(\frac{1}{\sqrt{1+\tau r}}\right)}{\mathcal{K}\left(\frac{1}{\sqrt{1+\tau r^{-1}}}\right)\mathcal{K}'\left(\frac{1}{\sqrt{1+\tau r}}\right)}. \tag{19}$$

Next we differentiate (19) with respect to τ and then simplify the result, applying the Legendre identity

$$\mathcal{E}(k)\mathcal{K}'(k) + \mathcal{E}'(k)\mathcal{K}(k) - \mathcal{K}(k)\mathcal{K}'(k) = \frac{\pi}{2}.$$

Then we obtain

$$\frac{\partial F}{\partial \tau}(\tau, r) = \frac{\pi}{4\tau}\,\frac{\mathcal{K}\left(\frac{1}{\sqrt{1+\tau r^{-1}}}\right)\mathcal{K}'\left(\frac{1}{\sqrt{1+\tau r^{-1}}}\right) - \mathcal{K}\left(\frac{1}{\sqrt{1+\tau r}}\right)\mathcal{K}'\left(\frac{1}{\sqrt{1+\tau r}}\right)}{\mathcal{K}^2\left(\frac{1}{\sqrt{1+\tau r^{-1}}}\right)\mathcal{K}'^2\left(\frac{1}{\sqrt{1+\tau r}}\right)}. \tag{20}$$

By Lemma 3.32(2) in [1], for $0 < x, y < 1$, $\mathcal{K}(x)\mathcal{K}'(x) = \mathcal{K}(y)\mathcal{K}'(y)$ if and only if $x = y$ or $x = \sqrt{1-y^2}$. Thus, if $\frac{\partial F}{\partial \tau}(\tau, r) = 0$, we must have

$$(1 + \tau r)^{-1} + (1 + \tau r^{-1})^{-1} = 1,$$

which gives $\tau = 1$ or $t = \infty$, Hence, $\frac{\partial F}{\partial \tau}(\tau, r)$ does not change its sign on $0 < \tau < 1$. By Lemma 3.22(1) in [1], $\mathcal{K}(x)\mathcal{K}'(x)$ strictly decreases in $1/\sqrt{2} < x < 1$. This together with (20) implies that $\frac{\partial F}{\partial \tau}(\tau, r) > 0$ for τ close to 0 and therefore for all τ, $0 < \tau < 1$. Since $\tau = (t-1)/(t+1)$ the latter implies that $h(t, a)$ strictly increases in $1 < t < \infty$.

Finally, finding the limit values of $(1/2)\log F(\tau, r)$ as $\tau \to 1^-$ and $\tau \to 0^+$, we obtain

$$h(\infty, a) = \log\Big(\mathcal{K}'(\sqrt{(1-a)/2})/\mathcal{K}(\sqrt{(1-a)/2})\Big) \quad \text{and} \quad \lim_{t \to 1^+} h(t, a) = 0.$$

The proof is complete. □

Remarks. (1) As the reader may notice, two main ingredients of the proof of Theorem 2 are estimates of the hyperbolic distances in the punctured plane and usage of a doubly-sheeted covering surface. In the first version of this paper, to prove Theorem 2 (called Lemma 1 there), the logarithmic covering surface was used instead of the square root surface, which gives the value $r_1 = .0005$ instead of $r_0 = .03949$ in Theorem 1. The referee of this paper showed that the logarithmic surface in our proof can be replaced by the thrice-sheeted covering surface of the function $1 - \sqrt[3]{1 - w/\rho}$, which gives a better constant $r_2 = .0033$. He also suggested that the result stated in Theorem 2 may have an independent significance. So, in this version, using a doubly-sheeted covering surface, we obtained the best possible constant that our method can produce.

(2) The constant $r_0 = .03949...$, which is sharp for Theorem 2 with $\rho = 1$, is far from being optimal even in the context of Corollary 1. We conjecture that for every $0 < \rho \leq 1$, the extremal function of Corollary 1 has the form $\rho Q(e^{i\alpha} z)$ with $\alpha \in \mathbb{R}$. Here

$$Q(z) = J(\frac{1}{\pi i}\log z) = 16z \prod_{n=1}^{\infty} \left(\frac{1 + z^{2n}}{1 + z^{2n-1}}\right)^8, \qquad |z| < 1,$$

where J is the elliptic modular function. It is worth mentioning, see [6, Ch. VI], that Q maps $\mathbb{D}$ conformally onto the Riemann surface $\mathcal{R}^*$ without branch points and such that every point $w \in \mathbb{C} \setminus \{0, 1\}$ is covered by $\mathcal{R}^*$ an infinity of times. The only boundary points of $\mathcal{R}^*$ are the points 0, 1, and ∞. The points $w = 1, \infty$ do not belong to $\mathcal{R}^*$, while the point $w = 0$ is covered by one sheet of $\mathcal{R}^*$ only. If true, this conjecture will give one more nice extremal property of the classical elliptic modular function J; see [6, Ch. VI].

(3) In the form stated in Theorem 1, Marshall's problem does not admit an extremal function. Thus, we consider a slightly modified problem of finding the maximal $r_0 > 0$ such that if $M > 0$ is such that $A(M) \leq \pi M^2$ and $\overline{\mathbb{D}}_M \setminus f(\mathbb{D}) \neq \emptyset$, then $f(r_0\mathbb{D}) \subset \mathbb{D}_M$. To find the maximal constant r_0 and to describe all extremal functions of this modified Marshall's problem may be a difficult task.

(4) If f is one-to-one, then the classical covering theorem for the standard class S of univalent functions easily implies that Theorem 1 holds with the maximal constant $r_0 = 3 - 2\sqrt{2}$. The extremal functions for the univalent case are the rotations of the scaled Koebe function $Mk(z)$, where $k(z) = z/(1 - z)^2$.

Let $\mathcal{R}_1$ be the Riemann surface of $\log(1 - z)$ and let $\mathcal{R}$ be the Riemann surface obtained from $\mathcal{R}_1$ by deleting the closed unit disk $\overline{\mathbb{D}}$ from all sheets of $\mathcal{R}_1$ except one, for which $\log 1 = 0$. Then $\mathcal{R}$ is a simply connected hyperbolic Riemann surface that covers the unit disk precisely once. Let $F(z)$ be a conformal mapping from $\mathbb{D}$ onto $\mathcal{R}$ such that $F(0) = 0$, $F'(0) > 0$.

For every $M > 0$, the scaled function $MF(z)$ seems to be a reasonable candidate for the extremal function of the modified Marshall's problem.

In support of this conjecture, we note that some properties of F, relevant to Marshall's problem, resemble corresponding properties of the Koebe function. In addition, using the subordination principle, it is not difficult to show that if f is an extremal function of the modified Marshall's problem, then $f(\mathbb{D})$ must cover the exterior of the disk $\mathbb{D}_M$ infinitely many times and all finite boundary points of $f(\mathbb{D})$ must be on the closed disk $\overline{\mathbb{D}}_M$. The function $MF(z)$ possesses these properties.

Acknowledgments

The author would like to thank the referee for his suggestion to use finitely-sheeted Riemann surfaces in the proof of Theorem 2. This enabled us to prove this theorem with the best possible constant, which in turn leads to a much better constant in Theorem 1.

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