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GLOBAL COEFFICIENT RING IN THE NILPOTENCE CONJECTURE

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ABSTRACT. In this note we show that the nilpotence conjecture for toric varieties is true over any regular coefficient ring containing $\mathbb{Q}$.

In [G] we showed that for any additive submonoid M of a rational vector space with the trivial group of units and a field $\mathbf{k}$ with $char\mathbf{k} = 0$ the multiplicative monoid $\mathbb{N}$ acts nilpotently on the quotient $K_i(\mathbf{k}[M])/K_i(\mathbf{k})$ of the ith K-groups, $i \geq 0$. In other words, for any sequence of natural numbers $c_1, c_2, \ldots \geq 2$ and any element $x \in K_i(\mathbf{k}[M])$ we have $(c_1 \cdots c_j)_*(x) \in K_i(\mathbf{k})$ for all $j \gg 0$ (potentially depending on x). Here c_* refers to the group endomorphism of $K_i(\mathbf{k}[M])$ induced by the monoid endomorphism $M \to M$, $m \mapsto m^c$, writing the monoid operation multiplicatively.

The motivation of this result is that it includes the known results on (stable) triviality of vector bundles on affine toric varieties and higher K-homotopy invariance of affine spaces. Here we show how the mentioned nilpotence extends to all regular coefficient rings containing $\mathbb{Q}$, thus providing the last missing argument in the long project spread over many papers. See the introduction of [G] for more details.

Using Bloch-Stienstra's actions of the big Witt vectors on the NK_i-groups [St] (which has already played a crucial role in [G], but in a different context), Lindel's technique of étale neighborhoods [L], van der Kallen's étale localization [K], and Popescu's desingularization [Sw], we show

Theorem 1. *Let M be an additive submonoid of a $\mathbb{Q}$-vector space with a trivial group of units. Then for any regular ring R with $\mathbb{Q} \subset R$ the multiplicative monoid $\mathbb{N}$ acts nilpotently on $K_i(R[M])/K_i(R)$, $i \geq 0$.*

Conventions. All our monoids and rings are assumed to be commutative. X is a variable. The monoid operation is written multiplicatively, denoting by e the neutral element. $\mathbb{Z}_+$ is the additive monoid of nonnegative integers. For a sequence of natural numbers $\mathfrak{c} = c_1, c_2, \ldots \geq 2$ and an additive submonoid N of a rational space V we put

$$N^{\mathfrak{c}} = \lim_{\to} \left(N \xrightarrow{-^{c_1}} N \xrightarrow{-^{c_2}} \cdots \right) = \bigcup_{j=1}^{\infty} N^{\frac{1}{c_1 \cdots c_j}} \subset V.$$

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Lemma 2. *Let M be a finitely generated submonoid of a rational vector space with the trivial group of units. Then M embeds into a free commutative monoid $\mathbb{Z}_+^r$.*

For the stronger version of Lemma 2 with $r = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes M)$ see, for instance, [BG, Proposition 2.15(e)]. (In [BG] the monoids as in Lemma 2 are called the *affine positive* monoids.)

Lemma 3. *Let F be a functor from rings to abelian groups, H be a monoid with the trivial group of units, and $\Lambda = \bigoplus_H \Lambda_h$ be an H-graded ring (i.e. $\Lambda_h\Lambda_{h'} \subset \Lambda_{hh'}$). Then we have the implication*

$$F(\Lambda) = F(\Lambda[H]) \implies F(\Lambda_e) = F(\Lambda).$$

The special case of Lemma 3 when $H = \mathbb{Z}_+$ is known as the *Swan-Weibel homotopy trick*, and the proof of the general cases makes no real difference; see [G, Proposition 8.2].

Lemma 4. *Theorem* 1 *is true for any coefficient ring of the form $S^{-1}\mathbf{k}[\mathbb{Z}_+^r]$ where $\mathbf{k}$ is a field of characteristic* 0 *and $S \subset \mathbf{k}[\mathbb{Z}_+^r]$ is a multiplicative subset.*

In the special case when $\mathbf{k}$ is a number field, Lemma 4 is proved in Step 2 in [G, §8], but word-by-word the same argument goes through for a field $\mathbf{k}$ provided the nilpotence conjecture is true for the monoid rings with coefficients in $\mathbf{k}$.
Notice. The reason we state the result in [G] only for number fields is that the preceding result in [G] is the validity of the nilpotence conjecture for such coefficient fields. Actually, the proof of Theorem 1 is an étale version of the idea of interpreting the globalization problem for coefficient rings in terms of the K-homotopy invariance, used for Zariski topology in [G, §8].

Finally, in order to explain one formula we now summarize very briefly the Bloch-Stienstra action of the ring of big Witt vectors $\mathrm{W}(\Lambda)$ on

$$NK_i(\Lambda) = \mathrm{Coker}(K_i(\Lambda) \to K_i(\Lambda[X])).$$

For the details the reader is referred to [St].

The additive group of $\mathrm{W}(R)$ can be thought of as the multiplicative group of formal power series $1 + X\Lambda[[X]]$. It has the decreasing filtration by the ideals $I_p(R) = (1 + X^{p+1}\Lambda[[X]])$, $p = 1, 2, \ldots$, and every element $\alpha(X) \in \mathrm{W}(\Lambda)$ admits a convergent series expansion in the corresponding additive topology $\alpha(X) = \Pi_{\mathbb{N}}(1 - \lambda_m X^m)$, $\lambda_m \in \Lambda$. To define a continuous $\mathrm{W}(\Lambda)$-module structure on $NK_i(\Lambda)$ it is enough to define the appropriate action of the Witt vectors of type $1 - \lambda X^m$, satisfying the condition that every element of $NK_i(\Lambda)$ is annihilated by some ideal $I_p(\mathrm{W}(\Lambda))$. Finally, such an action of $1 - \lambda X^m$ on $NK_i(\Lambda)$ is provided by the composite map in the upper row of the following commutative diagram with exact vertical columns:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
NK_i(\Lambda[X]) & \longrightarrow & NK_i(\Lambda[X]) & \longrightarrow & NK_i(\Lambda[X]) & \longrightarrow & NK_i(\Lambda[X]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_i(\Lambda[X]) & \xrightarrow{m^*} & K_i(\Lambda[X]) & \xrightarrow{\lambda_*} & K_i(\Lambda[X]) & \xrightarrow{m_*} & K_i(\Lambda[X]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_i(\Lambda) & \xrightarrow{m\cdot -} & K_i(\Lambda) & \xrightarrow{1} & K_i(\Lambda) & \xrightarrow{1} & K_i(\Lambda) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

where:

(1) m_* corresponds to scalar extension through the Λ-algebra endomorphism $\Lambda[X] \to \Lambda[X]$, $X \mapsto X^m$,
(2) m^* corresponds to *scalar restriction* through the same endomorphism $\Lambda[X] \to \Lambda[X]$,
(3) λ_* corresponds to scalar extension through the Λ-algebra endomorphism $\Lambda[X] \to \Lambda[X]$, $X \mapsto \lambda X$.
(4) $m \cdot -$ is multiplication by m.

A straightforward check of the commutativity of the appropriate diagrams, based on the description above, shows that for a ring homomorphism $f : \Lambda_1 \to \Lambda_2$ we have

$$f_*(\alpha z) = f_*(\alpha) f_*(z), \quad \alpha \in \mathrm{W}(\Lambda_1), \quad z \in NK_i(\Lambda_1), \tag{1}$$

where the same f_* is used for the both induced homomorphisms

$$\mathrm{W}(\Lambda_1) \to \mathrm{W}(\Lambda_2) \quad \text{and} \quad NK_i(\Lambda_1) \to NK_i(\Lambda_2).$$

Proof of Theorem 1. Since K-groups commute with filtered colimits there is no loss of generality in assuming that M is finitely generated. Then by Lemma 2, $R[M]$ admits a $\mathbb{Z}_+$-grading

$$R[M] = \bigoplus_{\mathbb{Z}_+} R_j, \qquad R_e = R.$$

In particular, by the Quillen local-global patching for higher K-groups [V], we can without loss of generality assume that R is local.
Notice. Actually, the local-global patching proved in [V] is for the special case of polynomial extensions. However, the more general version for graded rings is a straightforward consequence via the Swan-Weibel homotopy trick, discussed above.

By Popescu's desingularization [Sw] and the same filtered colimit argument we can further assume that R is a regular localization of an affine $\mathbf{k}$-algebra for a field $\mathbf{k}$ with $char\mathbf{k} = 0$. In this situation Lindel has shown [L, Proposition 2] that there is a subring $A \subset R$ of the form $\mathbf{k}[\mathbb{Z}_+^d]_\mu$, $\mu \in \max(\mathbf{k}[\mathbb{Z}_+^d])$, $d = \dim R$, such that

$$R \text{ is étale over } A. \tag{2}$$

Notice. Lindel's result is valid in arbitrary characteristic under the conditions that the residue field of R is a simple separable extension of $\mathbf{k}$, which is automatic in our situation because $char\mathbf{k} = 0$.

Using again that K-groups commute with filtered colimits, the validity of Theorem 1 for R is easily seen to be equivalent to the equality

$$K_i(R) = K_i(R[M^{\mathfrak{c}}]) \tag{3}$$

for every sequence of natural numbers $\mathfrak{c} = c_1, c_2, \ldots \geq 2$.

Next we show that (3) follows from the condition

$$NK_i(R[M^{\mathfrak{c}}]) = 0. \tag{4}$$

In fact, by the filtered colimit argument we have

$$\begin{aligned} &K_i(R[M^{\mathfrak{c}}]) = K_i(R[M^{\mathfrak{c}}])[X] \implies \\ &K_i(R[M^{\mathfrak{c}}]) = K_i(R[M^{\mathfrak{c}}])[\mathbb{Z}_+^{\mathfrak{c}}] \ (= \varinjlim K_i(R[M^{\mathfrak{c}}])[\mathbb{Z}_+]). \end{aligned}$$

On the other hand, by Lemma 2 the ring $R[M^{\mathfrak{c}}]$ has a $\mathbb{Z}_+^{\mathfrak{c}}$-grading:

$$R[M^{\mathfrak{c}}] = \bigoplus_{\mathbb{Z}_+^{\mathfrak{c}}} S_j, \qquad S_e = R.$$

So by Lemma 3 we have (3).

To complete the proof it is enough to show (4) assuming (2).

By the base change property, the ring extension $A[M] \subset R[M] = A[M] \otimes_A R$ is étale. Then by van der Kallen's result [K, Theorem 3.2] we have the isomorphism of $\mathrm{W}(R[M])$-modules

$$NK_i(R[M]) = \mathrm{W}(R[M]) \otimes_{\mathrm{W}(A[M])} NK_i(A[M]). \tag{5}$$

Notice. Van der Kallen proves his formula (the *étale localization*) for a modified tensor product that takes care of the filtrations on $\mathrm{W}(A[M])$ and $\mathrm{W}(R[M])$ by the ideals $I_p(-)$. But the presence of the characteristic 0 subfield $\mathbf{k}$ yields (via the *ghost map*) the infinite product presentations $\mathrm{W}(A[M]) = \prod_{\mathbb{N}} A[M]$ and $\mathrm{W}(R[M]) = \prod_{\mathbb{N}} R[M]$ and, in particular, makes checkable the appropriate compatibility of the two filtrations: $I_p(R[M]) = I_p(A[M])\mathrm{W}(R[M])$; see the discussion before [K, Theorem 3.2].

Pick an element $z \in NK_i(R[M])$. By (5) it admits a representation of the form

$$z = \sum_q \alpha_q \bar{y}_q, \quad \alpha_q \in \mathrm{W}(R[M]), \quad y_q \in NK_i(A[M]),$$

where the bar refers to the image in $NK_i(R[M])$.

By Lemma 4 we know that Theorem 1 is true for A. Therefore, $(c_1 \cdots c_j)_*(y_q) = 0$ for all q provided $j \gg 0$. In particular, (1) implies

$$(c_1 \cdots c_j)_*(z) = \sum_q (c_1 \cdots c_j)_*(\alpha_q)\overline{(c_1 \cdots c_j)_*(y_q)} = 0, \qquad j \gg 0.$$

Since z was an arbitrary element, the filtered colimit argument shows (4). □

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