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HYPERBOLIC SETS EXHIBITING C^1-PERSISTENT HOMOCLINIC TANGENCY FOR HIGHER DIMENSIONS

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Abstract. For any manifold of dimension at least three, we give a simple construction of a hyperbolic invariant set that exhibits C^1-persistent homoclinic tangency. It provides an open subset of the space of C^1-diffeomorphisms in which generic diffeomorphisms have arbitrary given growth of the number of attracting periodic orbits and admit no symbolic extensions.

1. Introduction

Homoclinic tangency is one of the most important phenomena in the study of non-hyperbolic dynamical systems. In [9] and [11] (see also [12]), Newhouse showed the existence of a hyperbolic invariant set that exhibits C^r-persistent homoclinic tangency for any $r \geq 2$ and any manifold of dimension at least two. Persistent homoclinic tangency implies very complicated dynamics generically. For example, Newhouse [10] showed the genericity of the existence of infinitely many attracting or repelling periodic points in the two-dimensional case. Kaloshin [8] showed the genericity of the arbitrary given growth of the number of periodic points.

In the C^1-category, Newhouse's idea does not work. However, for manifolds of dimension at least three, Bonatti and Díaz [2] constructed a diffeomorphism with *a wild homoclinic class* and showed that generic diffeomorphisms in its small C^1-neighborhood have infinitely many attractors. In [3], they also showed that a wild homoclinic class generates *the universal dynamics*. In particular, one can show that diffeomorphisms exhibiting a homoclinic tangency are dense in a small C^1-neighborhood of a diffeomorphism with a wild homoclinic class (see Remark 1.5 below). However, it was unclear whether a wild homoclinic class generates a hyperbolic set exhibiting C^1-persistent homoclinic tangency or not.

The main aim of this paper is to give a simple example of a hyperbolic set exhibiting C^1-persistent homoclinic tangency. The definitions we need will be given in the next section.

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Theorem 1.1. *For any smooth manifold of dimension at least three, there exists a C^∞-diffeomorphism such that it admits a hyperbolic basic set that contains a sectionally dissipative saddle and exhibits C^1-persistent homoclinic tangency.*

Remark 1.2. As we see later, we can construct the above diffeomorphism so that its support is contained in a ball. So, by the local genericity of the universal dynamics at a wild homoclinic class[1] shown in [3], any diffeomorphism having a wild homoclinic class can be C^1-approximated by a diffeomorphism with a hyperbolic basic set that exhibits C^1-persistent homoclinic tangency.

We also give two applications of the main theorem. By $\mathrm{Diff}^r(M)$, we denote the space of C^r-diffeomorphisms of a manifold M with the C^r-topology. The first application is a proof of the theorems of Newhouse [10] and Kaloshin [8] for $r = 1$ and $\dim M \geq 3$ along their original strategy.

Corollary 1.3. *For any smooth manifold M with $\dim M \geq 3$, there exists an open subset $\mathcal{U}_1$ of $\mathrm{Diff}^1(M)$ that has the following property: For any given sequence $\{a_n\}_{n\geq 1}$ of positive numbers, generic diffeomorphisms F in $\mathcal{U}_1$ satisfy*

$$\limsup_{n\to+\infty} \frac{\#\mathrm{Per}_0(F,n)}{a_n} = +\infty,$$

where $\#\mathrm{Per}_0(F,n)$ is the number of attracting periodic points of period n.

Remark 1.4. The proof in Kaloshin's paper [8] shows the corresponding statement for C^r-diffeomorphisms with $r \geq 2$ and $\dim M \geq 2$. It contains Newhouse's result [10] on infinitely many attracting periodic points.

Remark 1.5. The corollary can be shown by the local genericity of the universal dynamics, which is shown by Bonatti and Díaz in [3]. In fact, they showed that there exists an open subset $\mathcal{U}'$ of $\mathrm{Diff}^1(M)$ in which any diffeomorphism can be C^1-approximated by a diffeomorphism G such that $G^n|_U$ is the identity for some $n \geq 1$ and some open subset U of M. It allows us to show the corollary by the argument in [8, Section 2.7].

The second application of Theorem 1.1 is non-genericity of diffeomorphisms with a symbolic extension. Let σ be the shift map on the space $\{1,\ldots,k\}^{\mathbb{Z}}$ of bi-infinite sequences of k-symbols. We say a homeomorphism H of a topological space X admits *a symbolic extension* if there exists an integer $k \geq 1$, a σ-invariant closed subset Y of $\{1,\ldots,k\}^{\mathbb{Z}}$, and a surjective continuous map π from Y to X such that $\pi \circ \sigma|_Y = H \circ \pi$. In [5], Downarowicz and Newhouse showed that any C^1-generic area-preserving diffeomorphism of a closed surface is either an Anosov diffeomorphism or has no symbolic extension. One of the main ingredients of their proof is the C^1-persistence of homoclinic tangency for non-Anosov area-preserving diffeomorphisms. Hence, we can show the following theorem by essentially the same argument.

Corollary 1.6. *For any smooth manifold M with $\dim M \geq 3$, there exists an open subset $\mathcal{U}_2$ of $\mathrm{Diff}^1(M)$ in which generic diffeomorphisms have no symbolic extension.*

As mentioned in [5], it is still unknown whether there exists a C^r-diffeomorphism without symbolic extension for $r \geq 2$. It is also unclear whether Corollary 1.6 can be obtained as a direct consequence of the local genericity of the universal dynamics.

[1] We do not give the definitions of a wild homoclinic class and the universal dynamics here because they belong to another topic. See [2] for the definitions.

2. Preliminaries

In this section, we review some basic results on hyperbolic dynamics. We refer to Robinson's textbook [14] for the proofs. By $\mathbb{Z}$, we denote the set of integers.

Let M be a Riemannian manifold and F a C^r-diffeomorphism of M with $r \geq 1$. We define *the stable set* $W^s(p;F)$ of a point $p \in M$ by

$$W^s(p;F) = \{q \in M \mid \lim_{n\to+\infty} d_M(F^n(p), F^n(q)) = 0\},$$

where d_M is a distance on M induced by the Riemannian metric. We also define *the unstable set* $W^u(p;F)$ by $W^u(p;F) = W^s(p;F^{-1})$.

Let Λ be a compact invariant set of F and $TM|_\Lambda$ the restriction of the tangent bundle TM to Λ. Suppose that Λ is *a hyperbolic set*, that is, there exists a continuous DF-invariant splitting $TM|_\Lambda = E^u \oplus E^s$ that satisfies

$$\sup_{z\in\Lambda}\{\|DF^N|_{E^s(z)}\|, \|DF^{-N}|_{E^u(z)}\|\} < 1$$

for some $N \geq 1$. For any small $\delta > 0$ and $p \in \Lambda$, we define *the local stable manifold* $W^s_\delta(p;F)$ by

$$W^s_\delta(p;F) = \{q \in M \mid d_M(F^n(p), F^n(q)) \leq \delta \ \text{ for any } \ n \geq 0\}.$$

By the local stable manifold theorem (see *e.g.* Theorem 2.1 in Section X of [14]), if δ is sufficiently small, then $W^s_\delta(p;F)$ is a C^r-embedded disk satisfying $T_pW^s_\delta(p;F) = E^s(p)$ and $W^s(p;F) = \bigcup_{n\geq 0} F^{-n}(W^s_\delta(F^n(p);F))$. In particular, $W^s(p;F)$ is a C^r-injectively immersed manifold for any $p \in \Lambda$. It is known that $W^s_\delta(p;F)$ depends continuously on p and F as a C^r-embedded disk (see *e.g.* [6]). The corresponding statement also holds for *the local unstable set* $W^u_\delta(p;F) = W^s_\delta(p;F^{-1})$.

We define *the stable set* $W^s(\Lambda;F)$ of Λ by

$$W^s(\Lambda;F) = \{z \in M \mid \lim_{n\to+\infty} \inf_{z'\in\Lambda} d_M(F^n(z), z') = 0\}.$$

As Corollary 3.2 in Section X of [14], it satisfies

$$W^s(\Lambda;F) = \bigcup_{z\in\Lambda} W^s(z;F).$$

We put $W^u(\Lambda;F) = W^s(\Lambda;F^{-1})$. If $W^u(\Lambda;F) \subset \Lambda$, then Λ is a *topological attractor*; that is, there exists a compact neighborhood V of Λ such that $F(V) \subset \operatorname{Int} V$ and $\bigcap_{n\geq 0} F^n(V) = \Lambda$. In particular, $W^s(\Lambda;F)$ is a neighborhood of Λ in this case.

Let Λ_1 be a possibly non-hyperbolic invariant set of F. We say that Λ_1 is *topologically transitive* if $U \cap \bigcup_{n\in\mathbb{Z}} F^n(V)$ is non-empty for any pair (U,V) of non-empty open subsets of Λ_1. We also say that Λ_1 is *locally maximal* if there exists a compact neighborhood V of Λ_1 such that $\bigcap_{n\in\mathbb{Z}} F^n(V) = \Lambda_1$. The neighborhood V is called *an isolating neighborhood* of Λ_1. A subset of M is called *a basic set* of F if it is a topologically transitive and locally maximal invariant set. It is known that if Λ_1 is a hyperbolic basic set and V is its isolating neighborhood, then there exists a C^1-neighborhood $\mathcal{U}$ of F such that $\bigcap_{n\in\mathbb{Z}} G^n(V)$ is also a hyperbolic basic set for any $G \in \mathcal{U}$.

Let Λ_h be a hyperbolic basic set. We say that Λ_h has *a homoclinic tangency* if $W^s(p;F)$ and $W^u(q;F)$ have a tangential intersection point for some $p, q \in \Lambda_h$. We also say that Λ_h *exhibits C^r-persistent homoclinic tangency* if there exists a C^r-neighborhood $\mathcal{U}_1$ of F and an isolating neighborhood V of Λ_h such that $\bigcap_{n\in\mathbb{Z}} G^n(V)$ is a hyperbolic basic set with a homoclinic tangency for any $G \in \mathcal{U}_1$.

Let p be a periodic point of F with period n. We say p is *hyperbolic* if its orbit $\{p, F(p), \dots, F^{n-1}(p)\}$ is hyperbolic. It is equivalent to not all eigenvalues of DF_p^n being of absolute value one. It is known that there exists a C^1-neighborhood $\mathcal{U}_p$ of F and a continuous map $p(\cdot)$ from $\mathcal{U}_p$ to M such that $p(F) = p$ and $p(G)$ is a hyperbolic periodic point of G for any $G \in \mathcal{U}_p$. We call the map $p(\cdot)$ *a hyperbolic continuation* of p.

The periodic point p is called *attracting* or *repelling* if all eigenvalues of DF_p^n are of absolute value less than one or greater than one, respectively. *A saddle* is a hyperbolic periodic point that is neither attracting nor repelling. We say that p is *sectionally dissipative* if $|\lambda_1 \lambda_2| < 1$ for any two distinct eigenvalues λ_1 and λ_2 of DF_p^n.

The following lemma is useful to control the topology of the stable manifolds.

The Inclination Lemma (Theorem 11.1 in Section V of [14]). *Let p be a hyperbolic fixed point of F and L a submanifold of M with $\dim W^u(p;F) = \dim L = m$. Suppose that L and $W^s(p;F)$ have a transverse intersection point q. Then, for any given m-dimensional closed disk D in $W^u(p;F)$, there exists a sequence $(D_n)_{n\geq 1}$ of m-dimensional closed disks in L such that D_n converges to $\{q\}$ in the Hausdorff topology and $F^n(D_n)$ converges to D as C^r-embedded closed disks.*

For $r \geq 1$ and another compact manifold M', let $C^r(M', M)$ be the space of C^r-maps from M' to M with the C^r-topology. We denote the boundary of M' by $\partial M'$ and put Int $M' \setminus M' \partial M'$. We say a compact submanifold L of M is *an r-normally repelling forward-invariant manifold* of F if $F(L) \subset \operatorname{Int} L$ and there exists a continuous splitting $TM|_L = E \oplus TL$ and $N \geq 1$ such that $DF(E) \subset E$ and

$$\|DF_z^N|_{T_z L}\|^r \cdot \|DF^{-N}_{F^N(z)}|_{E(F^N(z))}\| \leq \frac{1}{2}, \quad \|DF^{-N}_{F^N(z)}|_{E(F^N(z))}\| \leq \frac{1}{2}$$

for any $z \in L$.

Proposition 2.1 (Theorem 4.2 in Section XII of [14]). *Suppose that L is an r-normally repelling forward-invariant manifold. Then, there exists a C^r-neighborhood $\mathcal{U}_r$ of F and a continuous function Φ from $\mathcal{U}_r$ to $C^r(L, M)$ such that $\Phi(F)$ is the inclusion of L into M and $\Phi(G)(L)$ is an r-normally repelling forward-invariant manifold of G for any $G \in \mathcal{U}_r$.* □

A compact submanifold of M is called *an r-normally attracting backward-invariant manifold* if it is an r-normally repelling forward-invariant manifold for F^{-1}.

3. Construction of a diffeomorphism exhibiting C^1-persistent homoclinic tangency

In this section, we prove Theorem 1.1. Our construction is close to a classical example given by Abraham and Smale [1], which exhibits C^1-persistence of heterodimensional cycles.

By $\mathbb{R}^n$, we denote the n-dimensional Euclidean space. For $n \geq 1$ and $1 \leq r \leq \infty$, let $\mathrm{Diff}^r_c(\mathbb{R}^n)$ be the subset of $\mathrm{Diff}^r(\mathbb{R}^n)$ that consists of diffeomorphisms with compact support.

The construction of a Plykin-like attractor (see *e.g.* Section 8.8 of [14]) gives a diffeomorphism f_0 in $\mathrm{Diff}^\infty_c(\mathbb{R}^2)$ and a compact subset V_0 of $\mathbb{R}^2$ that satisfy the following properties:

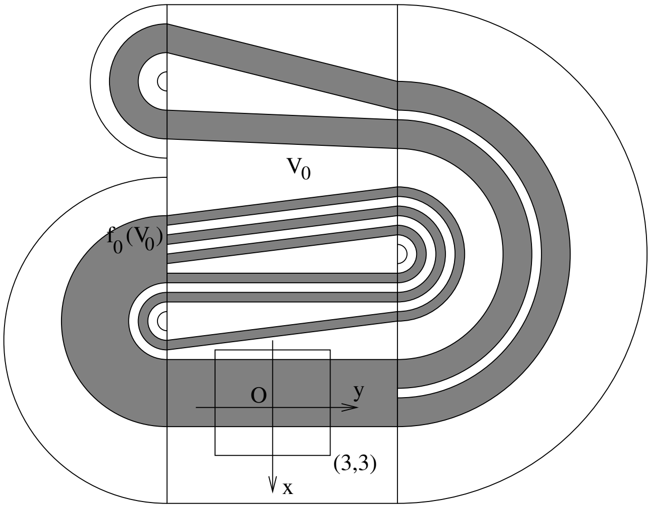

FIGURE 1. A Plykin-like attractor Λ_0

(1) $0 \times [-1,1] \subset f_0(V_0) \subset \mathrm{Int}\, V_0$ and $[1,3] \times [-3,3] \subset V_0 \setminus f_0(V_0)$.
(2) $\Lambda_0 = \bigcap_{n \geq 0} f_0^n(V_0)$ is a one-dimensional hyperbolic basic set with $V_0 \subset W^s(\Lambda_0)$.
(3) The origin $(0,0)$ is a fixed saddle of f_0 with $\det(Df_0)_{(0,0)} > 1$.
(4) $0 \times [-1,1] \subset W^u((0,0); f_0)$ and $[-3,3] \times y \subset W^s((0,y); f_0)$ for any $y \in [-1,1]$.

See Figure 1. We also take a diffeomorphism g_0 in $\mathrm{Diff}_c^\infty(\mathbb{R})$ so that $g_0([-1,1]) = [-3,3]$, $g_0(0) = 0$, $(dg_0/dy)(0) > \sup_{q \in \mathbb{R}^2} \|(Df_0)_q\|$, and $[-3,3] \subset W^u(0; g_0)$. We define a diffeomorphism F_0 in $\mathrm{Diff}_c^\infty(\mathbb{R}^3)$ by $F_0(x,y,z) = (f_0(x,y), g_0(z))$. Remark that the origin $(0,0,0)$ is a sectionally dissipative saddle of *the inverse of* F_0 and that $V_0 \times 0$ is a 1-normally repelling forward-invariant manifold for F_0. It is easy to see that the set $\Lambda = \Lambda_0 \times 0$ is a hyperbolic basic set of F_0 and that it satisfies

- $0 \times [-1,1] \times [-3,3] \subset W^u((0,0,0); F_0)$ and
- $[-3,3] \times y \times 0 \subset W^s((0,y,0); F_0)$ for any $y \in [-1,1]$.

By the construction of F_0, we can isotope F_0 to another diffeomorphism F in $\mathrm{Diff}_c^\infty(\mathbb{R}^3)$ so that

- $F|_{V_0 \times [-1,1]} = F_0|_{V_0 \times [-1,1]}$, and
- $F(0,y,z) = (y+2, 2y^2, 2-z)$ for $(y,z) \in [-1/2, 1/2] \times [3/2, 5/2]$.

See Figure 2. Then, Λ is a hyperbolic basic set of F with an isolating neighborhood $V_0 \times [-1,1]$. The two-dimensional unstable manifold $W^u((0,0,0); F)$ has a quadratic tangency with a one-dimensional manifold $W^s((0,0,0); F)$, but it is transverse to a two-dimensional set $W^s(\Lambda; F)$ at $(1,0,0)$. Put $\Lambda(G) = \bigcap_{n \in \mathbb{Z}} G(V_0 \times [-1,1])$ for $G \in \mathrm{Diff}^1(\mathbb{R}^3)$. If G is sufficiently C^1-close to F, then $\Lambda(G)$ is a hyperbolic basic set.

Proposition 3.1. *There exists a neighborhood* $\mathcal{U}$ *of* F *in* $\mathrm{Diff}^1(\mathbb{R}^3)$ *such that* $\Lambda(G)$ *has a homoclinic tangency for any* $G \in \mathcal{U}$.

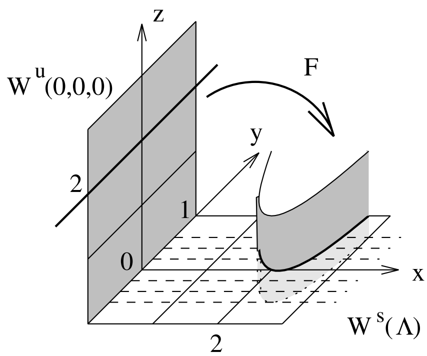

FIGURE 2. The diffeomorphism F

Proof. Proposition 2.1 implies that there exists a C^1-neighborhood $\mathcal{U}_0$ of F and a continuous map Φ from $\mathcal{U}_0$ to $C^1(V_0, \mathbb{R}^3)$ such that $\Phi(F)(q) = (q, 0)$ for any $q \in V_0$ and the following holds for any $G \in \mathcal{U}_0$:

- $G(\Phi(G)(V_0)) \subset \Phi(G)(\mathrm{Int}\ V_0) \cap (V_0 \times [-1, 1])$.
- $\Phi(G)(0, 0)$ is a sectionally dissipative saddle of the inverse of G.

Notice that $\Phi(\cdot)(0, 0)$ is a hyperbolic continuation of the fixed point $(0, 0, 0)$ of F on $\mathcal{U}_0$. By the continuous dependence of the local unstable manifolds on diffeomorphisms, $W^u_\delta(\Phi(G)(0, 0); G)$ depends continuously on G as a C^1-embedded disk. Since $V_0 \times 0$ intersects $W^u((0, 0, 0); F)$ transversely at $0 \times [-1, 1] \times 0$, we may replace $\mathcal{U}_0$ and Φ so that $\Phi(G)(0 \times [-1, 1]) \subset W^u(\Phi(G)(0, 0); G) \cap \Phi(G)(V_0)$ for any $G \in \mathcal{U}_0$. It is easy to see

$$\Phi(G)(0 \times [-1, 1]) \subset \bigcap_{n \geq 0} G^n(\Phi(V_0)) \subset \Lambda(G), \qquad \Phi(G)(V_0) \subset W^s(\Lambda(G)).$$

By the continuous dependence of the local stable and unstable manifolds on points and diffeomorphisms, we can take a C^1-neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of F and continuous maps $Q : \mathcal{U} \to C^1([-1/2, 1/2], \mathbb{R})$ and $h : \mathcal{U} \times [-1/2, 1/2] \to C^1([-3, 3], \mathbb{R})$ such that $(Q(F))(y) = y^2$ and $h(F, y)(x) = y$ for any $(x, y) \in [-3, 3] \times [-1/2, 1/2]$, and the following holds for any $G \in \mathcal{U}_1$:

(1) $|h(G, y)(x) - y| \leq 1/100$ and $|Q(G)(y) - y^2| \leq 1/100$ for any $(x, y) \in [-3, 3] \times [-1/2, 1/2]$.
(2) $\Phi(G)(\{(x, h(G, y)(x)) \mid x \in [-3, 3]\}) \subset W^s(\Phi(G)(0, y); G)$ for any $y \in [-1/2, 1/2]$.
(3) $\Phi(G)(\{(2 + y, Q(G)(y)) \mid y \in [-1/2, 1/2]\}) \subset W^u(\Phi(G)(0, 0); G)$.

Put $C_Q = \{(2 + y', Q(G)(y')) \mid y' \in [-1/2, 1/2]\}$ and $L(y) = \{(x, h(G, y)(x)) \mid x \in [-3, 3]\}$. They intersect for $y = 1/8$ and do not intersect for $y = -1/8$. Hence, there exists $y_0 \in [-1/8, 1/8]$ such that C_Q and $L(y_0)$ have tangential intersection. This implies that $W^u(\Phi(G)(0, 0); G)$ is tangent to $W^s(\Phi(G)(0, y_0); G)$. Since $\Phi(G)(0 \times [-1, 1]) \subset \Lambda(G)$, the proof is completed. □

Proof of Theorem 1.1. As mentioned in Section 4.3 of [7], a standard argument using a normally attracting invariant manifold allows us to reduce the theorem to the three-dimensional case. Suppose that M is a three-dimensional manifold. Take a C^∞-diffeomorphism F_M of M that is smoothly conjugate to the inverse of the above F on its support. Then, Proposition 3.1 implies the theorem. $\square$

4. Consequences of C^1-persistent homoclinic tangency

In this section, we prove Corollaries 1.3 and 1.6.

Let F be a diffeomorphism of a manifold M. We say that two hyperbolic periodic points p and q are *homoclinically related* if both $W^u(p;F)$ and $W^s(q;F)$, and $W^s(p;F)$ and $W^u(q;F)$ have transverse intersection points. By the inclination lemma, this defines an equivalent relation on the set of hyperbolic periodic points. If a hyperbolic basic set Λ contains a fixed point p_0, then any periodic point in Λ is homoclinically related to p_0 since $W^u(p_0;F)$ and $W^s(p_0;F)$ are dense in Λ. In particular, any two periodic points in Λ are homoclinically related in this case.

We say that a hyperbolic periodic point p *has a homoclinic tangency* if $W^u(p;F)$ and $W^s(p,F)$ have a tangential intersection. For a property $\mathcal{P}$ of a diffeomorphism, we say a diffeomorphism F is *C^r-approximated by* a diffeomorphism satisfying $\mathcal{P}$ if any C^r-neighborhood of F contains a diffeomorphism satisfying $\mathcal{P}$.

Lemma 4.1. *Let Λ be a hyperbolic basic set of F that contains a fixed point p_0, and let p be a hyperbolic periodic point of F. Suppose that Λ has a homoclinic tangency and p is homoclinically related to p_0. Then, for any $r \geq 1$, F can be C^r-approximated by a diffeomorphism G such that p is a hyperbolic periodic point of G with a homoclinic tangency.*

Proof. By the density of periodic orbits in Λ, and by the continuous dependence of the local stable and unstable manifolds on points and diffeomorphisms, F can be C^r-approximated by a diffeomorphism F_1 such that Λ is a hyperbolic basic set of F_1, p and p_0 are hyperbolic periodic points of F_1 that are homoclinically related, and $W^u(q_1;F_1)$ and $W^s(q_2;F_1)$ have tangential intersection for some periodic points $q_1, q_2 \in \Lambda$. Since p_0 is homoclinically related to q_1 and q_2, so is p. Now, the lemma is an easy consequence of the inclination lemma. $\square$

Now, we prove Corollaries 1.3 and 1.6. Let M be a smooth manifold. By Theorem 1.1, there exists a C^∞-diffeomorphism F_0 that admits a hyperbolic basic set Λ_0 containing a sectionally dissipative fixed saddle p_0 and exhibiting C^1-persistent homoclinic tangency. Let $\mathcal{U}_0$ be a C^1-neighborhood of F_0 and V an isolating neighborhood of Λ such that

- $\Lambda(G) = \bigcap_{n\in\mathbb{Z}} G^n(V)$ is a hyperbolic basic set exhibiting homoclinic tangency for any $G \in \mathcal{U}_0$,
- p_0 admits a hyperbolic continuation $p_0(\cdot)$ on $\mathcal{U}_0$, and
- $p_0(G)$ is a sectionally dissipative saddle in $\Lambda(G)$ for any $G \in \mathcal{U}_0$.

Corollary 1.3 is an almost direct consequence of the existence of renormalization at a homoclinic tangency associated with a sectionally dissipative saddle, which is investigated by Palis and Viana [13, Section 6].

Proof of Corollary 1.3. Take $F \in \mathcal{U}_0$ and put $m = \dim M$. By Lemma 4.1, we may assume that F is of class C^2 and p_0 has a quadratic homoclinic tangency. For $u \in \mathbb{R}^{m-1}$ and $t \in [0,1]$, we define a self-map $Q_{u,t}$ on $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ by $Q_{u,t}(x,y) = (x^2 + t, xu)$. As shown in [13, Section 6], there exists a C^2-family $(F_\tau)_{\tau\in[-1,1]}$ in $\mathcal{U}_0$, $v \in \mathbb{R}^{m-1}$, $N \geq 1$, sequences $(R_n)_{n\geq 1}$ of embeddings from $[-10,10]^m$ to M and $(\tau_n)_{n\geq 1}$ from $[0,1]$ to $[-1,1]$ such that $F_0 = F$, τ_n converges to 0 uniformly as $n \to \infty$, and the family $(R_n^{-1} \circ F_{\tau_n(t)}^{N+n} \circ R_n|_{[-2,2]^2})_{t\in[0,1]}$ converges to $(Q_{v,t})_{t\in[0,1]}$ as a C^2-family of maps.

Notice that $Q_{v,1/4}$ has a non-hyperbolic fixed point, and it exhibits saddle-node bifurcation. Hence, for any sufficiently large n, there exists $t_n \in [0,1]$ such that $F_{\tau_n(t_n)}^{N+n}$ has a sectionally dissipative fixed point p_n such that $DF_{\tau_n(t_n)}^{N+n}(p_n)$ has an eigenvalue 1. By a C^1-small perturbation at p_n, F can be C^1-approximated by a diffeomorphism G_1 that admits an embedded interval I_n consisting of sectionally dissipative fixed points of F^{N+n}. Hence, another C^1-small perturbation produces a diffeomorphism G with $\#\mathrm{Per}_0(G, n+N) \geq (n+N)a_{n+N}$. Now, the same argument as in Subsection 2.7 of Kaloshin's paper [8] completes the proof. $\square$

Finally, we show Corollary 1.6. The proof is essentially the same as the argument in Section 5 of [5] due to Downarowicz and Newhouse.

Before starting the proof, let us give some definitions. For $F \in \mathcal{U}_0$ and an F-invariant set Λ, let $\mathcal{M}(\Lambda, F)$ be the set of F-invariant Borel probability measures on Λ and $\mathcal{M}_e(\Lambda, F)$ be its subset consisting of ergodic measures. By weak convergence, the set $\mathcal{M}(\Lambda, F)$ becomes a compact metric space with a distance ρ. For $\mu \in \mathcal{M}_e(\Lambda, F)$, we denote its maximal Lyapunov exponent by $\chi(\mu)$ and its entropy with respect to F by $h(F, \mu)$. For a periodic point p of F, we denote the unique element of $\mathcal{M}_e(\Lambda, F)$ supported on the orbit of p by μ_p.

Fix a sequence $(\alpha_k)_{k\geq 1}$ of simplicial partitions such that $\alpha_{k+1}(z) \subset \alpha_k(z)$ for any $k \geq 1$ and any $z \in M$, and the diameter of $\alpha_k(z)$ converges uniformly to zero as k approaches infinity. We say that an invariant set Λ of a diffeomorphism F on M is *subordinate to* α_k if there exists a decomposition $\Lambda = \bigcup_{i=0}^{n-1} \Lambda_i$ and a sequence $(A_i)_{i=0}^{n-1}$ of elements of α_k such that $\Lambda_i \subset A_i$ and $F(\Lambda_i) = \Lambda_{i+1}$ for any $i = 0, \ldots, n-1$, where we put $\Lambda_n = \Lambda_0$.[2]

Proof of Corollary 1.6. Recall that $p_0(F)$ is a sectionally dissipative saddle fixed by F. For any $n \geq 1$ and any $F \in \mathcal{U}_0$, let $H_n(F, p_0)$ be the set of all sectionally dissipative saddles p of F whose period is n and which is homoclinically related to $p_0(F)$. We put $H(F, p_0) = \bigcup_{n\geq 1} H_n(F, p_0)$.

We say that $p \in H_n(F, p_0)$ satisfies the property $(S)_n$ if there exists a zero-dimensional topologically transitive hyperbolic invariant set Λ_p of F such that

(1) $\Lambda_p \cap \partial A = \emptyset$ for any $A \in \alpha_n$,
(2) Λ_p is subordinate to α_n,
(3) there exists $\mu_{\Lambda_p} \in \mathcal{M}_e(\Lambda_p, F)$ such that
$$|h(F, \mu_{\Lambda_p}) - \chi(\mu_p)| < \frac{1}{n}\chi(\mu_p),$$
(4) any $\mu \in \mathcal{M}_e(\Lambda_p, F)$ satisfies
$$\rho(\mu, \mu_p) < \frac{1}{n},\ |\chi(\mu) - \chi(\mu_p)| < \frac{1}{n}\chi(\mu_p),$$
and
(5) any periodic point in Λ_p is contained in $H(F, p_0)$.

A key of the proof is the following claim.

[2] The definition is slightly different from [5], where Downarowicz and Newhouse considered F^n-invariant sets.

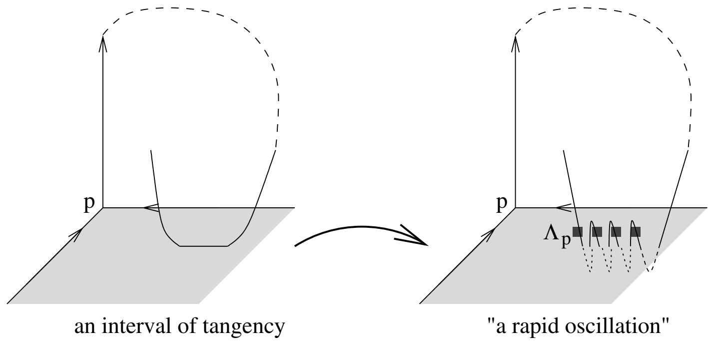

FIGURE 3. A rapid oscillation of the unstable manifold

Claim 4.2. For any $F \in \mathcal{U}_0$, $n \geq 1$, and $p \in H_n(F, p_0)$, the diffeomorphism F can be C^1-approximated by a diffeomorphism G such that p is contained in $H_n(G, p_0)$ and satisfies the property $(S)_n$ for G.

The claim is proved by the same construction as in Section 5 of [5]. In fact, by Lemma 4.1, we may assume that p has a homoclinic tangency after a small C^1-perturbation of F. By another small C^1-perturbation, we may also assume that $W^u(p; F) \cap W^s(p; F)$ contains an interval I. By the construction in [5, Section 5] using a rapid oscillation of $W^u(p; F)$ on I (see Figure 3), we can show that F can be C^1-approximated by a diffeomorphism G that admits an invariant set Λ_p satisfying all the conditions in the property $(S)_n$ except the last one. On the other hand, the construction allows us to take the set Λ_p so that any $q \in \Lambda_p$ is homoclinically related to $p(G)$, where $p(\cdot)$ is a hyperbolic continuation of p. Since $p(G)$ is homoclinically related to $p_0(G)$, the inclination lemma implies that the last condition in the property $(S)_n$ holds for Λ_p. This completes the proof of the claim.

For any $n \geq 1$, let $\mathcal{R}_n$ be the subset of $\mathcal{U}_0$ consisting of diffeomorphisms F such that all periodic points of period n are hyperbolic and any periodic point in $H_n(F, p_0)$ satisfies the property $(S)_n$. It is easy to see that $\mathcal{R}_n$ is an open set. The above claim implies that $\mathcal{R}_n$ is also a dense subset of $\mathcal{U}_0$. Hence, $\mathcal{R} = \bigcap_{n \geq 1} \mathcal{R}_n$ is a residual subset of $\mathcal{U}_0$.

Fix a constant ρ_0 so that $0 < \rho_0 < \chi(\mu_{p_0}; F)/2$. For $F \in \mathcal{R}$, we put

$$\mathcal{E}_1(F) = \{\mu_p \mid p \in H(F, p_0), \chi(\mu_p) > \rho_0\}.$$

Since $W^s(p_0(F); F)$ and $W^u(p_0(F); F)$ have transverse intersection for any $F \in \mathcal{U}_0$, the inclination lemma implies that this is true for any $p \in H(F, p_0)$ also. By the Birkhoff-Smale theorem (see *e.g.* Theorem 4.5 of Chapter VIII of [14]), any $p \in H(F, p_0)$ is an accumulation point of $H(F, p_0)$. Now, we can apply the proof in Section 5 of [5] to the closure of $\mathcal{E}_1(F)$, and this implies that F has no symbolic extension for any $F \in \mathcal{R}$. $\square$

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