

FREE RESOLUTIONS OF PARAMETER IDEALS FOR SOME RINGS WITH FINITE LOCAL COHOMOLOGY

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ABSTRACT. Let R be a d -dimensional local ring, with maximal ideal \mathfrak{m} , containing a field and let x_1, \dots, x_d be a system of parameters for R . If $\text{depth } R \geq d - 1$ and the local cohomology module $H_{\mathfrak{m}}^{d-1}(R)$ is finitely generated, then there exists an integer n such that the modules $R/(x_1^i, \dots, x_d^i)$ have the same Betti numbers, for all $i \geq n$.

INTRODUCTION

Throughout this note R is a local noetherian ring with maximal ideal \mathfrak{m} .

Let M be a finitely generated R -module and let F be a minimal free resolution of M . The Poincaré series of M is the formal power series $P_M^R(t) = \sum_{i=0}^{\infty} (\text{rank } F_i) t^i$. We let $\Omega_R^i(M)$ denote the i th syzygy of M , that is to say, $\text{Coker } \partial_{i+1}^F$. Let $\mathbf{x} = x_1, \dots, x_d$ be a system of parameters for R . For each $n \in \mathbb{N}$, let \mathbf{x}^n denote the sequence x_1^n, \dots, x_d^n .

Several classical results in local algebra establish that ideals contained in the large powers of the maximal ideal exhibit a similar behavior. In his thesis, [7], Y. H. Lai considers the following question, which he attributes to D. Katz: Do the Poincaré series of the modules $R/(\mathbf{x}^n)$ behave uniformly, for n large enough?

Clearly, the answer is yes if R is Cohen-Macaulay: In this case \mathbf{x}^n is a regular sequence, for each $n \geq 1$, so $R/(\mathbf{x}^n)$ is resolved by a Koszul complex on d elements. It is also not difficult to obtain a positive answer when $\dim R = 1$.

Lai proves that if $\dim R = 2$, $\text{depth } R = 1$ and the local cohomology module $H_{\mathfrak{m}}^1(R)$ is finitely generated, then for n large enough one has

$$P_{R/(\mathbf{x}^n)}^R(t) = 1 + 2t + t^2 + t^2 P_{H_{\mathfrak{m}}^1(R)}^R(t).$$

Now we state our main result. It evidently covers the first two cases mentioned above. Also part (ii) of our Main Theorem generalizes Lai's result, since the Canonical Element Conjecture holds for 2-dimensional rings.

Main Theorem. *Let R be a d -dimensional local ring with maximal ideal \mathfrak{m} . If $d - \text{depth } R \leq 1$ and the R -module $H = H_{\mathfrak{m}}^{d-1}(R)$ is finitely generated, then there exists an integer n , such that for each system of parameters \mathbf{x} for R contained in*

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\mathfrak{m}^n the following assertions hold:

- (i) $\Omega_R^{d+1}(R/(\mathbf{x})) \cong \Omega_R^{d-1}(H)$.
- (ii) If in addition, the Canonical Element Conjecture holds for R , then one has

$$P_{R/(\mathbf{x})}^R(t) = (1 + t)^d + t^2 P_H^R(t).$$

The Cohen-Macaulay defect of R is the number $\text{cmd } R = \dim R - \text{depth } R$. One always has $\text{cmd } R \geq 0$, and equality characterizes Cohen-Macaulay rings. On the other hand, Cohen-Macaulay rings are also characterized by the equality $H_{\mathfrak{m}}^i(R) = 0$ for all $i \neq d$. Comparing these conditions with the hypotheses on R in the theorem, one may say that these hypotheses, in some sense, give the least possible extension of Cohen-Macaulay rings.

1. FREE RESOLUTIONS OF ALMOST COMPLETE INTERSECTION IDEALS

Let $\mathbf{y} = y_1, \dots, y_n$ be a sequence in R . We let $K(\mathbf{y}; M)$ denote the Koszul complex on \mathbf{y} with coefficients in M . Set

$$H_i(\mathbf{y}; M) = H_i(K(\mathbf{y}; M)).$$

If $P(t) = \sum_{i=0}^{\infty} a_i t^i$ and $Q(t) = \sum_{i=0}^{\infty} b_i t^i$ are formal power series, we write $P \preceq Q$ to indicate that $a_i \leq b_i$ holds for all $i \geq 0$.

The main result of this section is the following theorem:

Theorem 1.1. *Let R be a d -dimensional ring and let \mathbf{x} be a system of parameters for R . Set $H_i = H_i(\mathbf{x}; R)$ for $i = 1, \dots, d$. One then has*

$$P_{R/(\mathbf{x})}^R(t) \preceq (1 + t)^d + \sum_{i=1}^{\text{cmd } R} t^{i+1} P_{H_i}^R(t).$$

Moreover, if $\text{cmd } R \leq 1$ and the Canonical Element Conjecture holds for R , then one has

$$P_{R/(\mathbf{x})}^R(t) = (1 + t)^d + t^2 P_{H_1}^R(t).$$

Under the hypotheses of the second part of the theorem, \mathbf{x} generates an almost complete intersection ideal. Some of the discussion below is carried out in this more general framework. Recall that an ideal I is called an *almost complete intersection* if $\text{grade } I \geq \mu(I) - 1$, where $\text{grade } I$ is the maximal length of an R -regular sequence in I and $\mu(I)$ is the number of minimal generators of I .

Now we give an example that shows the inequality in Theorem 1.1 can be strict if $\text{depth } R < d - 1$.

Example 1.2. Set $R = k[[a, b, c]]/(ac, bc, c^2)$, where k is a field of characteristic zero; then $\dim R = 2$ and $\text{depth } R = 0$. Consider the system of parameters $\mathbf{x} = a, b$. Using Macaulay 2 we get:

$$\begin{aligned} P_{H_2}^R(t) &= 1 + 3t + 6t^2 + 13t^3 + 28t^4 + \dots, \\ P_{H_1}^R(t) &= 3 + 7t + 12t^2 + 26t^3 + 56t^4 + \dots, \\ P_{R/(\mathbf{x})}^R(t) &= 1 + 2t + 3t^2 + 7t^3 + 15t^4 + \dots. \end{aligned}$$

Thus one has

$$\begin{aligned} P_{R/(\mathbf{x})}^R(t) &< 1 + 2t + 4t^2 + 8t^3 + 15t^4 + \dots \\ &= (1 + t)^2 + t^2 P_{H_1}^R(t) + t^3 P_{H_2}^R(t). \end{aligned}$$

Let X be a complex of R -modules and let ∂^X denote its differential, set

$$\sup H(X) = \sup\{n \in \mathbb{Z} \mid H_n(X) \neq 0\}.$$

Let Σ^t denote the shift functor defined by

$$(\Sigma^t X)_n = X_{n-t} \quad \text{and} \quad \partial_n^{\Sigma^t X} = (-1)^t \partial_{n-t}^X.$$

Let $\alpha: X \rightarrow Y$ be a morphism of complexes. Recall that the mapping cone of α is defined to be the complex C such that $C_n = X_{n-1} \oplus Y_n$ and

$$\partial_n^C((x, y)) = (-\partial_{n-1}^X(x), \partial_n^Y(y) + \alpha_{n-1}(x)) \quad \text{for all } (x, y) \in C_n.$$

A quasi-isomorphism is a morphism of complexes that induces an isomorphism in homology in all degrees.

The following lemma is elementary and well known, but we could not find a reference with a desired amount of detail.

Lemma 1.3. *Let X be a complex with $s = \sup H(X) < \infty$ and let F be a free resolution of $H_s(X)$. There exists a morphism of complexes $\alpha: \Sigma^s F \rightarrow X$ such that the mapping cone X' of α satisfies*

$$H_i(X') \cong \begin{cases} 0, & i \geq s, \\ H_i(X), & i \leq s - 1. \end{cases}$$

Proof. Let $\tau_{\geq s}(X)$ be the complex

$$\cdots \longrightarrow X_{s+2} \longrightarrow X_{s+1} \longrightarrow \ker \partial_s^X \longrightarrow 0$$

and $\iota: \tau_{\geq s}(X) \rightarrow X$ be the inclusion map. Since F is a bounded below complex of free modules, there is a morphism of complexes $\beta: F \rightarrow \tau_{\geq s}(X)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \Sigma^s F & \xrightarrow{\beta} & \tau_{\geq s}(X) & \xrightarrow{\iota} & X \\ & \searrow \Sigma^s \epsilon & \downarrow \pi & & \\ & & \Sigma^s H_s(X), & & \end{array}$$

where $\epsilon: F \rightarrow H_s(X)$ and π are quasi-isomorphisms. Set $\alpha = \iota\beta$ and let X' be the mapping cone of α . One has an exact sequence

$$0 \longrightarrow X \longrightarrow X' \longrightarrow \Sigma^{s+1} F \longrightarrow 0$$

of complexes. It induces an exact sequence of homology modules

$$\cdots \rightarrow H_{i+1}(X) \rightarrow H_{i+1}(X') \rightarrow H_{i+1}(\Sigma^{s+1} F) \xrightarrow{H_i(\alpha)} H_i(X) \rightarrow \cdots .$$

From the construction of α one sees that $H_s(\alpha)$ is an isomorphism, and since $H_i(\Sigma^{s+1} F) \cong H_{i-s-1}(F) = 0$ for all $i \neq s + 1$, we get the desired result. \square

Remark 1.4. Assume X is a complex of free modules such that $X_i = 0$, for $i < 0$, and $H_i(X) = 0$, for $i > s$, where s is some positive integer. Let F^i be a free resolution of $H_i(X)$, for $i = 1, \dots, s$. Applying Lemma 1.3 s times, one gets a free resolution G of $H_0(X)$ such that $G_i = X_i \oplus F_{i-2}^1 \oplus \cdots \oplus F_{i-s-1}^s$. However, even if the complexes X, F^1, \dots, F^s are minimal, G need not be minimal; see Example 1.2.

Now we show the relation between the Poincaré series of an almost complete intersection ideal and the Poincaré series of its first Koszul homology module.

Lemma 1.5. *Let I be an almost complete intersection ideal of R and let $\mathbf{y} = y_1, \dots, y_r$ be a minimal set generators for I . Set $H = H_1(\mathbf{y}; R)$. One then has*

$$\begin{aligned} \Omega_R^{r+1}(R/I) &= \Omega_R^{r-1}(H), \\ P_{R/I}^R(t) &= Q(t) + t^{r+1} P_N^R(t), \end{aligned}$$

where $Q(t)$ is a polynomial of degree r and $N = \Omega_R^{r-1}(H)$.

Proof. Let K denote the Koszul complex of \mathbf{y} with coefficients in R . Since grade $I \geq r - 1$, one has $H_i(\mathbf{y}; R) = 0$ for $i \neq 0, 1$. Let F be a minimal free resolution of H . Let $\alpha: \Sigma F \rightarrow K$ be the map from Lemma 1.3 and let C be the mapping cone of α . The complex C is a complex of free modules and has only one non-vanishing homology module, namely $H_0(C) \cong H_0(\mathbf{y}; R) \cong R/I$, so C is a free resolution of R/I .

By construction of mapping cones the complex C is minimal if and only if $\alpha F \subseteq \mathfrak{m}K$. Since F is minimal and $C_m = F_m$ for all $m \geq r + 1$, non-minimality can only happen in the first $r + 1$ degrees; therefore

$$\Omega_R^{r+1}(R/I) = \Omega_R^r(H).$$

This completes the proof of the first equality. The second equality follows from the first one. □

1.6. For a system of parameters \mathbf{x} for R , let $F_{\mathbf{x}}$ be a free resolution of $R/(\mathbf{x})$ and let $\gamma_{\mathbf{x}}: K_{\mathbf{x}} = K(\mathbf{x}; R) \rightarrow F_{\mathbf{x}}$ be a lifting of the map $K_{\mathbf{x}} \rightarrow R/(\mathbf{x})$. The following are equivalent:

- (i) The Canonical Element Conjecture holds for R .
- (ii) For every system of parameters \mathbf{x} for R and every free resolution $F_{\mathbf{x}}$ of $R/(\mathbf{x})$, the map $H(k \otimes_R \gamma_{\mathbf{x}}): H(k \otimes_R K_{\mathbf{x}}) \rightarrow H(k \otimes_R F_{\mathbf{x}})$ is injective.

It is shown in [1, (1.6)] that the equivalence of (i) and (ii) follows from a theorem of P. Roberts [8]. A proof of Roberts' theorem is given in [6, (1.3)].

The Canonical Element Conjecture holds for R provided R is equicharacteristic or $\dim R \leq 3$: Hochster has proved that the Canonical Element Conjecture is equivalent to the Direct Summand Conjecture, and that the conjectures hold if R is equicharacteristic or $\dim R \leq 2$; see [5]. In [3], R. Heitmann shows that the Direct Summand Conjecture, hence the Canonical Element Conjecture, holds for every 3-dimensional ring.

Proof of Theorem 1.1. The inequality follows from Remark 1.4.

For the rest of the proof, we keep the notation in the proof of Lemma (1.5). To prove the equality it suffices to show that C is minimal. Assume not, and let $0 \leq n \leq d - 1$ be such that $\partial(C_n) \not\subseteq \mathfrak{m}C_{n-1}$. Since $\text{Im}(\partial^K)$ and $\text{Im}(\partial^F)$ are in $\mathfrak{m}C$, there exists an element f such that $\partial(f) = e \in K_n \setminus \mathfrak{m}K_n$. It follows that f is not in $\mathfrak{m}F_n$, hence Rf is a direct summand of C_n , and Re is a direct summand of K_n . Let G be the complex $0 \rightarrow Rf \xrightarrow{\lambda} Re \rightarrow 0$ where Re is in degree n and λ is the restriction of ∂_n^C . Set $\overline{C} = C/G$. Since λ is an isomorphism, \overline{C} is exact, hence is a free resolution of $R/(\mathbf{x})$. Let $\pi: C \rightarrow \overline{C}$ be the natural surjection.

The inclusion map $\iota: K \rightarrow C$ is a lifting of the augmentation map $\alpha: K \rightarrow R/(\mathbf{x})$. Thus $\pi\iota$ is a lifting of α , but $H_n(k \otimes_R \pi\iota)(1 \otimes e) = 0$. This contradicts, by section 1.6, the hypothesis that Canonical Element Conjecture holds for R . Therefore, C is minimal; this implies the theorem. □

2. STANDARD SYSTEM OF PARAMETERS

Let M be a finitely generated R -module. A system of parameters \mathbf{x} for M is said to be *standard* if

$$(\mathbf{x}) H_{\mathfrak{m}}^i(M/(x_1, \dots, x_j)M) = 0$$

holds, for all non-negative integers i, j with $i + j < d$. For information about standard system of parameters we refer the reader to [10] and [11].

An R -module M is said to have *finite local cohomology* if for each integer $i \leq \dim M - 1$ the local cohomology module $H_{\mathfrak{m}}^i(M)$ is of finite length. Modules with finite local cohomology are also called *generalized Cohen-Macaulay modules*.

The following statement is a consequence of [11, (2.1) and (3.1)] and [9, (3.7)].

2.1. An R -module M has finite local cohomology if and only if there exists a positive integer n such that every system of parameters in \mathfrak{m}^n is standard.

For an R -module M we set $H^i(\mathbf{y}; M) = H_{-i}(\text{Hom}_R(K(\mathbf{y}; R), M))$. One then has $H_i(\mathbf{y}; M) \cong H^{d-i}(\mathbf{y}; M)$, for all i ; see [2, 1.6.10].

Theorem 2.2. *Let M have finite local cohomology. Set $g = \text{depth } M$. If \mathbf{x} is a standard system of parameters for M , then for every integer $n \geq 1$, the canonical map $\lambda_n: H^g(\mathbf{x}^n; M) \rightarrow H_{\mathfrak{m}}^g(M)$ is an isomorphism.*

To prove this theorem we need to recall some facts about standard systems of parameters and modules with finite local cohomology.

The next statement follows from [11, (3.1)] and [9, (3.3)].

2.3. Let \mathbf{x} be a system of parameters for M . If M has finite local cohomology and $\text{depth } M = g$, then the subsequence x_1, \dots, x_g of \mathbf{x} is M -regular.

In [4, (1)], standard systems of parameters are characterized in terms of Koszul homology:

2.4. Assume M has finite local cohomology. Let \mathbf{x} be a system of parameters for M . The following are then equivalent:

- (i) \mathbf{x} is standard.
- (ii) $\ell(H_p(x_1, \dots, x_r; M)) = \ell(H_p(x_1^2, \dots, x_r^2; M))$, for $p \geq 1$ and $1 \leq r \leq d$.
- (iii) $\ell(H_p(x_1, \dots, x_r; M)) = \sum_{i=0}^{r-p} \binom{r}{i+p} \ell(H_{\mathfrak{m}}^i(M))$, for all $p \geq 1$ and $1 \leq r \leq d$.

In particular, if \mathbf{x} is standard, then

$$\ell(H_{d-g}(\mathbf{x}; M)) = \ell(H_{\mathfrak{m}}^g(M)), \quad \text{where } g = \text{depth } M \quad \text{and} \quad d = \dim M.$$

Here $\ell(M)$ denotes the length of M .

The next result is [4, (4)].

2.5. Let M have finite local cohomology, and let x_1, \dots, x_d be a system of parameters for M . Let n_1, \dots, n_d be positive integers. For all $p > 0$ and $1 \leq r \leq d$ and for all positive integers m_1, \dots, m_r satisfying $n_1 \leq m_1, \dots, n_r \leq m_r$, one has

$$\ell(H_p(x_1^{n_1}, \dots, x_r^{n_r}; M)) \leq \ell(H_p(x_1^{m_1}, \dots, x_r^{m_r}; M)).$$

Lemma 2.6. *Let M have finite local cohomology and let \mathbf{x} be a standard system of parameters for M . For all positive integers p and m , one has*

$$\ell(H_p(\mathbf{x}; M)) = \ell(H_p(\mathbf{x}^m; M)).$$

Proof. For every $i > 0$, from section 2.4, one obtains:

$$\ell(H_p(\mathbf{x}; M)) = \ell(H_p(\mathbf{x}^{2i}; M)).$$

For each integer $m > 0$ there exists an integer $i > 0$ such that $2i \geq m$. Using section 2.5, we get inequalities

$$\ell(H_p(\mathbf{x}; M)) \leq \ell(H_p(\mathbf{x}^m; M) \leq \ell(H_p(\mathbf{x}^{2i}; M)),$$

which imply the desired statement. □

Next we recall the relation between Koszul homology and local cohomology . Let \mathbf{x} be a system of parameters for M and set $K^{(n)} = K(\mathbf{x}^n; R)$ and let $e_1^{(n)}, \dots, e_d^{(n)}$ denote the standard basis of $K_1^{(n)} \simeq R^d$. For all $n \geq 1$, there is a commutative diagram

$$\begin{CD} K_1^{(n+1)} @>\varphi_1^{(n)}>> K_1^{(n)} \\ @VVV @VVV \\ K_0^{(n+1)} @= K_0^{(n)} \end{CD}$$

where $\varphi_1^{(n)}(e_j^{(n+1)}) = x_j e_j^{(n)}$ for $j = 1, \dots, d$. This defines a morphism of complexes $\varphi^{(n)} = \bigwedge \varphi_1^{(n)}: K^{(n+1)} \rightarrow K^{(n)}$. Set $\psi^{(n)} = \text{Hom}_R(\varphi^{(n)}, M)$; then $\psi^{(n)}$ induces a map $\alpha_{(n)}^i: H^i(\mathbf{x}^n; M) \rightarrow H^i(\mathbf{x}^{(n+1)}; M)$. By [2, 3.5.6], one has

$$H_m^i(M) = \varinjlim H^i(\mathbf{x}^n; M) \quad \text{for all } i \geq 0.$$

Lemma 2.7. *Let $\mathbf{z} = z_1, \dots, z_t$ be a sequence in R and let s be the length of maximal M -regular sequences in (\mathbf{z}) . For each positive integer n , if $\mathbf{y} = y_1, \dots, y_r$ is an M -regular sequence in (\mathbf{z}^{n+1}) , then one has the following commutative diagram:*

$$\begin{CD} H^s(\mathbf{z}^n; M) @>\alpha_{(n)}^s>> H^s(\mathbf{z}^{n+1}; M) \\ @V \simeq VV @VV \simeq V \\ H^{s-r}(\mathbf{z}^n; \overline{M}) @>\overline{\alpha}_{(n)}^{s-r}>> H^{s-r}(\mathbf{z}^{n+1}; \overline{M}) \end{CD}$$

where $\overline{M} = M/(\mathbf{y})M$.

Proof. If $r = 1$, then one has a short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/y_1M \longrightarrow 0.$$

It induces a commutative diagram

$$\begin{CD} 0 @>>> H^{s-1}(\mathbf{z}^n; M/y_1M) @>\partial_n^{s-1}>> H^s(\mathbf{z}^n; M) @>y_1>> H^s(\mathbf{z}^n; M) \\ @. @VV \alpha_{(n)}^{s-1} V @VV \alpha_{(n)}^s V @VV \alpha_{(n)}^s V \\ 0 @>>> H^{s-1}(\mathbf{z}^{n+1}; M/y_1M) @>\partial_{n+1}^{s-1}>> H^s(\mathbf{z}^{n+1}; M) @>y_1>> H^s(\mathbf{z}^{n+1}; M). \end{CD}$$

Since y_1 annihilates both $H^g(\mathbf{z}^n; M)$ and $H^g(\mathbf{z}^{n+1}; M)$, the connecting maps ∂_n^{g-1} and ∂_{n+1}^{g-1} are isomorphisms, and this gives the desired commutative diagram for $r = 1$. The general case follows by iteration. □

Proof of Theorem 2.2. Let $n \geq 1$. The sequence $\mathbf{x}' = x_1, \dots, x_g$ is M -regular; see (2.3). So \mathbf{x}'^{n+1} is an M -regular sequence in the ideals $(\mathbf{x}'^{n+1}) \subseteq (\mathbf{x}'^n)$. Set $\overline{M} = M/(\mathbf{x}')M$. By Lemma 2.7, we get the following commutative diagram:

$$\begin{CD} H^g(\mathbf{x}^n; M) @>\alpha_{(n)}^g>> H^g(\mathbf{x}^{n+1}; M) \\ @V \simeq VV @VV \simeq V \\ H^0(\mathbf{x}^n; \overline{M}) @>\bar{\alpha}_{(n)}^0>> H^0(\mathbf{x}^{n+1}; \overline{M}). \end{CD}$$

The map $\bar{\alpha}_{(n)}^0$ is injective, since it is induced by the identity map in the following commutative diagram:

$$\begin{CD} \text{Hom}_R(K^{(n)}, \overline{M}) = @>>> \overline{M} @>>> \overline{M}^d @>>> \cdots \\ @V \psi^{(n)} VV @V = VV @VV V \\ \text{Hom}_R(K^{(n+1)}, \overline{M}) = @>>> \overline{M} @>>> \overline{M}^d @>>> \cdots \end{CD}$$

Therefore $\alpha_{(n)}^g$ is injective.

The modules $H^g(\mathbf{x}^n; M)$ and $H^g(\mathbf{x}^{n+1}; M)$ have the same length, see (2.6), so $\alpha_{(n)}^g$ is an isomorphism and this implies the desired statement. \square

3. FREE RESOLUTIONS OF PARAMETER IDEALS

In this section we assume that R is a d -dimensional ring with finite local cohomology, and $\text{cmd } R \leq 1$.

Because of the property recalled in section 2.1, the following result contains the Main Theorem stated in the Introduction.

Theorem 3.1. *Let \mathbf{x} be a standard system of parameters for R and set $H = H_{\mathfrak{m}}^{d-1}(R)$. One then has*

- (i) $\Omega_R^{d+1}(R/(\mathbf{x})) \cong \Omega_R^{d-1}(H)$.
- (ii) *If in addition, the Canonical Element Conjecture holds for R , then one has*

$$P_{R/(\mathbf{x})}^R(t) = (1 + t)^d + t^2 P_H^R(t).$$

Proof. One has $\text{grade}(\mathbf{x}) = \text{depth } R$ and $\mu(\mathbf{x}) = \dim R$, so $\text{grade}(\mathbf{x}) \geq \mu(\mathbf{x}) - 1$. Then part (i) follows from Lemma 1.5 and Theorem 2.2, and part (ii) follows from Theorems 1.1 and 2.2. \square

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