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BIORTHOGONAL EXPONENTIAL SEQUENCES WITH WEIGHT FUNCTION $\exp(ax^2 + ibx)$ ON THE REAL LINE AND AN ORTHOGONAL SEQUENCE OF TRIGONOMETRIC FUNCTIONS

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ABSTRACT. Some orthogonal functions can be mapped onto other orthogonal functions by the Fourier transform. In this paper, by using the Fourier transform of Stieltjes–Wigert polynomials, we derive a sequence of exponential functions that are biorthogonal with respect to a complex weight function like $\exp(q_1(ix + p_1)^2 + q_2(ix + p_2)^2)$ on $(-\infty, \infty)$. Then we restrict these introduced biorthogonal functions to a special case to obtain a sequence of trigonometric functions orthogonal with respect to the real weight function $\exp(-qx^2)$ on $(-\infty, \infty)$.

1. INTRODUCTION

Using a suitable integral transform, many orthogonal sets (or preferably orthogonal bases) can be mapped onto other orthogonal sets (bases) in the weighted L^2-space. For example, in [12, Prop. 3.1] Koornwinder has shown that classical Laguerre polynomials are mapped onto Meixner–Pollaczek polynomials by the Mellin transform. He has also proved that classical Jacobi polynomials can be mapped onto Wilson polynomials by using the Jacobi function transform [11]. In this way, Koelink in [9] has shown that the continuous Hahn polynomials are in fact the Fourier transform of Jacobi orthogonal polynomials. Since in Askey's scheme of hypergeometric orthogonal polynomials [8], Wilson polynomials are located in the top with the most degrees of freedom; all mentioned references can be generalized by the integral transform of Wilson polynomials. This line of thought has been followed in [6] by Groenevelt. Recently in [10] by using this approach and applying the Fourier transform of some finite sequences of classical orthogonal polynomials investigated in [13, 14], we have obtained two new classes of rational hypergeometric functions which are orthogonal with respect to two weight functions of Beta Ramanujan type. However, we should note that all these mentioned examples are dense in the weighted L^2-space.

On the other hand, it may be interesting to know that there are some biorthogonal functions that are not dense in the weighted L^2-space though they are generated

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by a specific integral transform of a sequence of orthogonal polynomials. In this paper, we introduce one of these examples, i.e., the Fourier transform of Stieltjes–Wigert polynomials corresponding to an indeterminate moment problem. Hence, we should start our discussion with a summary of these polynomials.

In 1895 Stieltjes [15] proved that

$$\int_0^\infty x^n x^{-\ln x} \sin(2\pi \ln x) dx = 0, \qquad n \in \mathbf{Z}. \tag{1}$$

Thus independent of λ we have

(2)
$$\frac{1}{\sqrt{\pi}} \int_0^\infty x^n x^{-\ln x}(1 + \lambda \sin(2\pi \ln x)) dx = \frac{1}{\sqrt{\pi}} \int_0^\infty x^n x^{-\ln x} dx = \exp\left(\frac{1}{4}(n+1)^2\right).$$

This shows that for any $\lambda \in [-1, 1]$ the densities

$$W_\lambda(x) = \frac{1}{\sqrt{\pi}} x^{-\ln x}(1 + \lambda \sin(2\pi \ln x)), \qquad x > 0, \tag{3}$$

have the same moments. By using this approach, one can conclude that there exist many different weight functions for the orthogonal polynomials corresponding to the positive measures (3), known later as Stieltjes–Wigert polynomials. In this sense, it was Wigert [16] in 1923 who first succeeded in finding the explicit form of orthogonal polynomials corresponding to the log-normal weight function

$$\rho(x; q) = \frac{\gamma}{\sqrt{\pi}} \exp(-\gamma^2 \ln^2 x) = \frac{\gamma}{\sqrt{\pi}} x^{(-\gamma^2 \ln x)}, \qquad -\gamma^2 = \frac{1}{\ln q^2}, \tag{4}$$

which is readily a more general case of (3) for $\lambda = 0$. The polynomials are defined as

$$W_n(x; q) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+\frac{1}{2})} x^k \quad \text{for } 0 < q < 1, \tag{5}$$

in which

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_i (q;q)_{n-i}} = \prod_{j=0}^{i-1} \frac{1 - q^{n-j}}{1 - q^{j+1}} \quad \text{and} \quad (a;q)_n = \prod_{i=0}^{n-1} 1 - aq^i. \tag{5.1}$$

Moreover, according to [1, pp. 172–174], they satisfy the orthogonality relation

$$\int_0^\infty \rho(x;q) W_n(x;q) W_m(x;q) dx = (q^{-(n+1/2)}(q;q)_n)\delta_{n,m} \tag{6}$$

$$\text{s.t. } \delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

Although Stieltjes considered the weight function (4) only for $\gamma = 1$ (or equivalently $\lambda = 0$ in (3)) as an example leading to an indeterminate Stieltjes moment problem [15, pp. 507–508], Christiansen [2] has considered this problem associated with Stieltjes–Wigert polynomials in a general case and presented all possible classical solutions. See also [3] in this regard. Consequently, as we pointed out, there should be various weight functions for the Stieltjes–Wigert orthogonal polynomials. For example, according to [8, p. 116], the shifted polynomials

$$S_n(x; q) = \frac{1}{(q;q)_n} W_n(q^{-\frac{1}{2}} x; q) \tag{7}$$

satisfy the orthogonality relation

$$\int_0^\infty \rho^*(x;q)S_n(x;q)S_m(x;q)dx = \left(-\frac{\ln q}{q^n}\frac{(q;q)_\infty}{(q;q)_n}\right)\delta_{n,m}, \tag{8}$$

where

$$\rho^*(x;q) = \frac{1}{(-x;q)_\infty(-qx^{-1};q)_\infty} = \prod_{i=0}^{\infty}\frac{1}{(1+xq^i)(1+x^{-1}q^{i+1})}. \tag{8.1}$$

Now if equality (7) is replaced in (8), then

$$\int_0^\infty \rho^*(q^{1/2}x;q)W_n(x;q)W_m(x;q)dx = \left(-\frac{\ln q}{q^{n+1/2}}(q;q)_n(q;q)_\infty\right)\delta_{n,m}. \tag{9}$$

It is clear that the shape of $\rho^*(q^{1/2}x;q)$, by noting definition (8.1), is completely different from the log-normal function shape though both of them play the role of a weight function for Stieltjes–Wigert polynomials.

After this note, let us come back to the main subject and define the shifted polynomials

$$V_n(x;p,q) = W_n(q^{p-\frac{1}{2}}x;q) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+p)}x^k, \tag{10}$$

in which p is a free real parameter and $q \in (0,1)$ again. If this definition is replaced in the orthogonality relation (6), then we have

$$\int_0^\infty \rho(q^{p-\frac{1}{2}}x;q)V_n(x;p,q)V_m(x;p,q)dx = (q^{-(p+n)}(q;q)_n)\delta_{n,m}. \tag{11}$$

But the weight function of relation (11), corresponding to (4), can be simplified as

$$\begin{aligned} \rho(q^{p-\frac{1}{2}}x;q) &= \frac{1}{\sqrt{(-2\pi)\ln q}}\exp\left(\frac{1}{2\ln q}\left(\left(p-\frac{1}{2}\right)\ln q+\ln x\right)^2\right) \\ &= \frac{1}{\sqrt{(-2\pi)\ln q}}q^{\frac{(2p-1)^2}{8}}x^{p-\frac{1}{2}}\exp\left(\frac{1}{2\ln q}\ln^2 x\right). \end{aligned} \tag{12}$$

Hence, (12) simplifies (11) in the form

$$\begin{aligned} &\int_0^\infty x^{p-\frac{1}{2}}\exp\left(\left(\frac{1}{\ln q^2}\right)\ln^2 x\right)V_n(x;p,q)V_m(x;p,q)dx \\ &\qquad = \left(\sqrt{-2\pi\ln q}\, q^{-n-\frac{1}{8}(2p+1)^2}(q;q)_n\right)\delta_{n,m}, \end{aligned} \tag{13}$$

which has a key role in the next section.

2. Fourier transform of polynomials (10) and its orthogonality property

In this section, we compute the Fourier transform of the polynomials $V_n(x;p,q)$ and substitute it in the Parseval identity by applying the orthogonality relation (13) to generate a new sequence of biorthogonal exponential functions. For this purpose, some introductory definitions should first be stated.

The Fourier transform of a function, say g, is defined as [5]

$$\mathsf{F}g(s) = \int_{-\infty}^{\infty} e^{-isx}g(x)dx = \int_0^\infty t^{-is-1}g(\ln t)dt, \tag{14}$$

and its inverse transform as

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\mathsf{F}g(s)ds \quad \text{with } i=\sqrt{-1}. \tag{14.1}$$

Also, the Parseval identity related to a Fourier transform is denoted by

$$\int_{-\infty}^{\infty} g(x)\overline{h(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathsf{F}g(s)\overline{\mathsf{F}h(s)}\,ds, \tag{15}$$

for $g, h \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Now, let us define the following specific functions:

$$\begin{aligned} g(x) &= \exp(p_1 x)\exp\left(\frac{1}{\ln q_1}x^2\right)V_n(e^x;a_1,b_1), \quad 0<b_1,q_1<1,\ a_1,p_1\in\mathbf{R},\\ h(x) &= \exp(p_2 x)\exp\left(\frac{1}{\ln q_2}x^2\right)V_m(e^x;a_2,b_2), \quad 0<b_2,q_2<1,\ a_2,p_2\in\mathbf{R}, \end{aligned} \tag{16}$$

in terms of the polynomials (10). For both functions defined in (16), the Fourier transform exists. However, for example, for the function g we have

$$\begin{aligned} \mathsf{F}g(s) &= \int_{-\infty}^{\infty} e^{-isx}\exp(p_1x)\exp\left(\frac{1}{\ln q_1}x^2\right)V_n(e^x;a_1,b_1)dx\\ &= \int_0^{\infty} t^{-is+p_1-1}\exp\left(\frac{1}{\ln q_1}\ln^2 t\right)\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{b_1} b_1^{k(k+a_1)}t^k dt\\ &= \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{b_1} b_1^{k(k+a_1)}\int_0^{\infty} t^{-is+p_1-1+k}\exp\left(\frac{1}{\ln q_1}\ln^2 t\right)dt\\ &= \sqrt{-\pi\ln q_1}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{b_1} b_1^{k(k+a_1)}q_1^{\frac{-(-is+p_1+k)^2}{4}}. \end{aligned} \tag{17}$$

To derive the above relation, we have generally used the definite integral

$$\int_0^{\infty} x^{\alpha}\exp(\beta\ln^2 x)dx = \sqrt{-\frac{\pi}{\beta}}\exp\left(-\frac{(\alpha+1)^2}{4\beta}\right), \qquad \beta<0. \tag{18}$$

These (i.e., the explicit solution of the definite integral (18) and also computing the Fourier transform of functions (16) explicitly) are the reasons why we have chosen the log-normal distribution as the main weight function.

Now, by substituting transformation (17) in the Parseval identity (15) and considering the defined functions in (16), we have

(19)

$$\begin{aligned} &2\pi\int_{-\infty}^{\infty}\exp((p_1+p_2)x)\exp\left(\left(\frac{1}{\ln q_1}+\frac{1}{\ln q_2}\right)x^2\right)V_n(e^x;a_1,b_1)V_m(e^x;a_2,b_2)dx\\ &\quad = 2\pi\int_0^{\infty} t^{-1+p_1+p_2}\exp\left(\left(\frac{1}{\ln q_1}+\frac{1}{\ln q_2}\right)\ln^2 t\right)V_n(t;a_1,b_1)V_m(t;a_2,b_2)dt\\ &\quad = \sqrt{\pi^2\ln q_1\ln q_2}\int_{-\infty}^{\infty} q_1^{-(p_1-is)^2/4}\overline{q_2^{-(p_2-is)^2/4}}\\ &\qquad\qquad \times J_n(is;a_1,b_1,p_1,q_1)\overline{J_m(is;a_2,b_2,p_2,q_2)}ds, \end{aligned}$$

in which $0<q_1,q_2<1$; $p_1,p_2\in\mathbf{R}$ and

$$J_n(x;a,b,c,d) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_b b^{k(k+a)}d^{-\frac{1}{4}k(k+2c)}(d^{\frac{x}{2}})^k. \tag{19.1}$$

On the other hand, if in the left-hand side of (19) we assume

$$a_1 = a_2 = p_1 + p_2 - \frac{1}{2} \quad \text{and} \quad b_1 = b_2 = \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right), \tag{20}$$

then according to orthogonality (13), the relation (19) changes:

$$\begin{aligned} &2\pi \int_0^\infty t^{-1+p_1+p_2} \exp\left(\left(\frac{1}{\ln q_1} + \frac{1}{\ln q_2}\right) \ln^2 t\right) \\ &\qquad \times V_n\left(t; p_1 + p_2 - \frac{1}{2}, \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right)\right) \\ &\qquad \times V_m\left(t; p_1 + p_2 - \frac{1}{2}, \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right)\right) dt \\ &= 2\pi \left(\sqrt{-\pi \frac{\ln q_1 \ln q_2}{\ln q_1 q_2}} \exp\left(-\left(n + \frac{1}{2}(p_1+p_2)^2\right) \frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right)\right. \\ &\qquad \times \left.\left(\exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right); \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right)\right)_n \right) \delta_{n,m} \\ &= \sqrt{\pi^2 \ln q_1 \ln q_2} \int_{-\infty}^{\infty} q_1^{-(p_1 - is)^2/4} \overline{q_2^{-(p_2 - is)^2/4}} \\ &\qquad \times J_n\left(is; p_1 + p_2 - \frac{1}{2}, \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right), p_1, q_1\right) \\ &\qquad \times \overline{J_m\left(is; p_1 + p_2 - \frac{1}{2}, \exp\left(\frac{\ln q_1 \ln q_2}{2 \ln q_1 q_2}\right), p_2, q_2\right)} ds. \end{aligned} \tag{21}$$

This result leads to the main theorem of the paper. But before stating it, let us (for convenience) suppose in (21) that $q_1 \to \exp(-4q_1), q_2 \to \exp(-4q_2)$ and $p_1 \to -p$. Hence, by noting the definition (19.1), a modified type of exponential function will appear in the theorem of the next section.

2.1. **Theorem.** The *exponential* functions

$$\begin{aligned} &E_n(x; q_1, p_1; q_2, p_2) \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{\exp\left(\frac{-2q_1 q_2}{q_1+q_2}\right)} \\ &\qquad \times \exp\left(\frac{q_1 k}{q_1 + q_2}((q_1 - q_2)k + q_2 - 2(q_1 p_1 + q_2 p_2))\right) \exp(-2q_1 kx), \end{aligned} \tag{22}$$

satisfy a biorthogonality relation as

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp(q_1(ix + p_1)^2 + q_2(ix + p_2)^2) \\ &\qquad \times E_n(ix; q_1, p_1; q_2, p_2) E_m(-ix; q_2, -p_2; q_1, -p_1) dx \\ &= \left(\sqrt{\frac{\pi}{q_1 + q_2}} \exp\left((2n + (p_2 - p_1)^2) \frac{q_1 q_2}{q_1 + q_2}\right)\right. \\ &\qquad \times \left.\left(\exp\left(\frac{-2q_1 q_2}{q_1 + q_2}\right); \exp\left(\frac{-2q_1 q_2}{q_1 + q_2}\right)\right)_n \right) \delta_{n,m}, \end{aligned} \tag{23}$$

provided that

$$q_1, q_2 > 0 \quad \text{and} \quad p_1, p_2 \in \mathbf{R}. \tag{23.1}$$

2.2. **Some remarks on the given theorem.**

i) The equality (23) is in fact a biorthogonality relation in the complex weighted space and $E_n(ix;q_1,p_1;q_2,p_2)$ in this relation are the corresponding complex valued functions, which are biorthogonal with respect to $\exp(q_1(ix+p_1)^2+q_2(ix+p_2)^2)$ on $(-\infty,\infty)$. In general, the complex orthogonal functions are defined on a rectifiable curve like C such that

$$\int_C W(z)\Phi_n(z)\overline{\Phi_m(z)}dz = \left(\int_C W(z)|\Phi_n(z)|^2 dz\right)\delta_{n,m}, \tag{24}$$

where

$$|\Phi_n(z)|^2 = \Phi_n(z)\overline{\Phi_n(z)}, \tag{24.1}$$

and $W(z)$ is a positive function of the complex variable $z = x+iy$ defined on C.

To study the biorthogonal relations with more detail, we refer the reader to [4, 7]. However, in the next section we will restrict the complex valued functions $E_n(ix)$ to a special case to obtain a new sequence of trigonometric functions *orthogonal* with respect to the real weight function $\exp(-qx^2)$ on $(-\infty,\infty)$.

ii) According to the well-known Euler's identity $\exp(it) = \cos(t) + i\sin(t)$, the real and imaginary parts of the sequence $E_n(ix)$ are defined, respectively, as

$$\begin{aligned}\operatorname{Re} E_n(ix;q_1,p_1;q_2,p_2) &= C_n(x;q_1,p_1;q_2,p_2)\\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix}_{\exp(\frac{-2q_1q_2}{q_1+q_2})}\\ &\quad\times \exp\left(\frac{q_1k}{q_1+q_2}((q_1-q_2)k+q_2-2(q_1p_1+q_2p_2))\right)\cos(2q_1kx),\end{aligned} \tag{25}$$

$$\begin{aligned}\operatorname{Im} E_n(ix;q_1,p_1;q_2,p_2) &= S_n(x;q_1,p_1;q_2,p_2)\\ &= \sum_{k=0}^{n}(-1)^{k+1} \begin{bmatrix} n\\ k\end{bmatrix}_{\exp(\frac{-2q_1q_2}{q_1+q_2})}\\ &\quad\times \exp\left(\frac{q_1k}{q_1+q_2}((q_1-q_2)k+q_2-2(q_1p_1+q_2p_2))\right)\sin(2q_1kx),\end{aligned} \tag{26}$$

where $q_1,q_2>0$ and $p_1,p_2\in\mathbf{R}$.

iii) The density of biorthogonality relation (23) is expandable as

$$\begin{aligned}W(z;q_1,p_1;q_2,p_2) &= \exp(q_1(iz+p_1)^2+q_2(iz+p_2)^2)\\ &= \exp(q_1p_1^2+q_2p_2^2)\exp(-(q_1+q_2)z^2)\exp(2i(q_1p_1+q_2p_2)z).\end{aligned} \tag{27}$$

So, by noting that $|\exp(it)|=1$, we have

$$|W(z)| = \exp(q_1p_1^2+q_2p_2^2)\exp(-(q_1+q_2)z^2), \qquad z\in\mathbf{R}, \tag{27.1}$$

in which $\exp(q_1p_1^2+q_2p_2^2)$ is only a constant and has no effect on the weight function. This means that the complex weight function (27) is real in the weighted L^2-space if and only if $q_1p_1+q_2p_2=0$. Moreover, according to the expanded form (27), the aforesaid weight function can be considered as $\exp(ax^2+ibx)$, where $a=-(q_1+q_2)$ and $b=2(q_1p_1+q_2p_2)$.

3. An orthogonal sequence of trigonometric functions

Let us reconsider the relation (23) and suppose $q_1 = q_2 = q/2$, $q > 0$ and $p_1 = -p_2$, $p_1 \in \mathbf{R}$. So the special sequence

$$E_n^{(q)}(x) = E_n\left(-x; \frac{q}{2}, p_1; \frac{q}{2}, -p_1\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{e^{-\frac{q}{2}}} e^{k\frac{q}{4}} e^{kqx}, \tag{28}$$

which is independent of p_1 satisfies

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-qx^2} E_n^{(q)}(ix) E_m^{(q)}(-ix) dx &= \int_{-\infty}^{\infty} e^{-qx^2} E_n^{(q)}(-ix) E_m^{(q)}(ix) dx \\ &= \left(\sqrt{\frac{\pi}{q}} e^{n\frac{q}{2}} (e^{-\frac{q}{2}}; e^{-\frac{q}{2}})_n \right) \delta_{n,m} \end{aligned} \tag{29}$$

for $q > 0$. There is also a direct proof for this relation. Since

$$\int_{-\infty}^{\infty} e^{-qx^2} e^{-imqx} dx = \sqrt{\frac{\pi}{q}} e^{-q\frac{m^2}{4}}, \qquad q > 0, \tag{30}$$

to prove the orthogonality (29), it is sufficient to deal with $0 \le m \le n$. In other words, if $Q = e^{-q/2} \in (0,1)$ in (29), then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-qx^2} E_n^{(q)}(ix) e^{-imqx} dx &= \sqrt{\frac{\pi}{q}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_Q e^{q\frac{k}{4}} e^{-q\frac{(k-m)^2}{4}} \\ &= \sqrt{\frac{\pi}{q}} Q^{\frac{m^2}{2}} \sum_{k=0}^{n} \frac{(Q^{-n}; Q)_k}{(Q; Q)_k} Q^{k(n-m)} \\ &= \sqrt{\frac{\pi}{q}} Q^{\frac{m^2}{2}} (Q^{-m}; Q)_n. \end{aligned} \tag{31}$$

To derive the last equality of (31) we have used the Q-binomial theorem [8]. Furthermore, the last expression of (31) is equal to zero if $0 \le m \le n$. This proves the orthogonality property.

Now, by noting the definitions (25) and (26), let us define the following trigonometric sequences:

$$C_n^{(q)}(x) = C_n\left(-x; \frac{q}{2}, p_1; \frac{q}{2}, -p_1\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{e^{-\frac{q}{2}}} e^{k\frac{q}{4}} \cos(kqx), \tag{32}$$

$$S_n^{(q)}(x) = S_n\left(-x; \frac{q}{2}, p_1; \frac{q}{2}, -p_1\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{e^{-\frac{q}{2}}} e^{k\frac{q}{4}} \sin(kqx). \tag{33}$$

It can be verified that

$$E_n^{(q)}(ix) = C_n^{(q)}(x) + iS_n^{(q)}(x) \quad \text{and} \quad E_n^{(q)}(-ix) = C_n^{(q)}(x) - iS_n^{(q)}(x). \tag{34}$$

If the relations of (34) are replaced in (29), then we get

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-qx^2} (C_n^{(q)}(x) C_m^{(q)}(x) + S_n^{(q)}(x) S_m^{(q)}(x)) dx \\ &\qquad + i \int_{-\infty}^{\infty} e^{-qx^2} (S_n^{(q)}(x) C_m^{(q)}(x) - S_m^{(q)}(x) C_n^{(q)}(x)) dx \\ &= \left(\sqrt{\pi/q}\, e^{nq/2} (e^{-q/2}; e^{-q/2})_n \right) \delta_{n,m}. \end{aligned} \tag{35}$$

But $S_n^{(q)}(x)$ is always an odd sequence while $C_n^{(q)}(x)$ is even. Hence,

$$\int_{-\infty}^{\infty} e^{-qx^2} S_n^{(q)}(x) C_m^{(q)}(x)dx = \int_{-\infty}^{\infty} e^{-qx^2} S_m^{(q)}(x) C_n^{(q)}(x)dx = 0. \tag{36}$$

By using the relations in (36) we can conclude the following corollary:

3.1. **Corollary.** Two trigonometric sequences defined as

$$M_{n,+}^{(q)}(x) = \frac{S_n^{(q)}(x) + C_n^{(q)}(x)}{\sqrt{2}} = \sum_{k=0}^{n} (-e^{q/4})^k \begin{bmatrix} n \\ k \end{bmatrix}_{e^{-\frac{q}{2}}} \sin\left(kqx + \frac{\pi}{4}\right), \tag{37}$$

$$M_{n,-}^{(q)}(x) = \frac{S_n^{(q)}(x) - C_n^{(q)}(x)}{\sqrt{2}} = \sum_{k=0}^{n} (-e^{q/4})^k \begin{bmatrix} n \\ k \end{bmatrix}_{e^{-\frac{q}{2}}} \sin\left(kqx - \frac{\pi}{4}\right), \tag{38}$$

respectively, satisfy the orthogonality relations

$$\int_{-\infty}^{\infty} e^{-qx^2} M_{n,+}^{(q)}(x) M_{m,+}^{(q)} dx = \left(\sqrt{\frac{\pi}{4q}} e^{n\frac{q}{2}} (e^{-\frac{q}{2}}; e^{-\frac{q}{2}})_n\right) \delta_{n,m}, \tag{39}$$

$$\int_{-\infty}^{\infty} e^{-qx^2} M_{n,-}^{(q)}(x) M_{m,-}^{(q)} dx = \left(\sqrt{\frac{\pi}{4q}} e^{n\frac{q}{2}} (e^{-\frac{q}{2}}; e^{-\frac{q}{2}})_n\right) \delta_{n,m}, \tag{40}$$

in which $q > 0$. Note that the proof of above orthogonalities is straightforward if one refers to relations (35) and (36). Also, since $M_{n,-}^{(q)}(-x) = -M_{n,+}^{(q)}(x)$ if $x = -t$ in (40), the orthogonality relation (39) will be derived. Therefore, only one orthogonality relation between (39) and (40) should be considered to be the main relation.

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