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STABLE INDECOMPOSABILITY OF LOOP SPACES ON SYMPLECTIC GROUPS

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Dedicated to the memory of Professor Masahiro Sugawara

ABSTRACT. We prove that $\Omega Sp(n)$ is stably indecomposable if $n \geq 2$ or $n = \infty$.

1. INTRODUCTION

A spectrum X is said to be decomposable if X is homotopy equivalent to a wedge sum $X_1 \vee X_2$ of non-trivial spectra X_1 and X_2. Otherwise X is said to be indecomposable. A CW complex X is said to be stably decomposable if the suspension spectrum $\Sigma^\infty X$ is decomposable as a spectrum. Otherwise it is said to be stably indecomposable. We are considering the following problem.

Question. Let G be a compact connected Lie group. Is the loop space ΩG stably indecomposable?

If G is not simply connected, then ΩG is not connected and, therefore, is stably decomposable. If $G = G_1 \times G_2$, then ΩG is stably homotopy equivalent to $\Omega G_1 \vee \Omega G_2 \vee \Omega G_1 \wedge \Omega G_2$ and, therefore, is stably decomposable, too. Thus, to solve the problem above, it is sufficient to consider a simply connected, simple Lie group. Hopkins [2] proved that $\Omega Sp(2)$ and $\Omega Sp(3)$ are stably indecomposable. Later Hubbuck [3] added ΩG_2 and ΩF_4 to the list of such spaces. We [4] also proved that ΩE_6 and ΩE_7 are stably indecomposable. In contrast to these results $\Omega SU(n)$ is known to be stably decomposable [1].

In this paper we will show that $\Omega Sp(n)$ are stably indecomposable for $n \geq 2$, which was conjectured by Hubbuck.

Theorem 1.1. *$\Omega Sp(n)$ is stably indecomposable if $n \geq 2$ or $n = \infty$.*

Needless to say, $\Omega Sp(1) = \Omega S^3$ is stably decomposable.

To prove the theorem we will investigate $\tilde{H}_*(\Omega Sp(n); \mathbb{F}_2)$. We are *not* showing that it is indecomposable as a module over the Steenrod algebra. For $n \geq 4$, $\tilde{H}_*(\Omega Sp(n); \mathbb{F}_2)$ is a sum of three indecomposable modules over the Steenrod algebra, and these modules are linked by higher-order operations as showed by Hubbuck.

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2. Proofs

We are really going to prove the following localized version of Theorem 1.1.

Theorem 2.1. *$\Omega Sp(n)$ is stably indecomposable at the prime 2 if $n \geq 2$ or $n = \infty$.*

From now on until the end of this paper all spaces and spectra are assumed to be localized at the prime 2, and $H_*(X)$ and $H^*(X)$ stand for $H_*(X;\mathbb{F}_2)$ and $H^*(X;\mathbb{F}_2)$, respectively.

The Steenrod operation acts on homology groups via the following formula:

$$\langle \alpha, xSq^i \rangle = \langle Sq^i \alpha, x \rangle$$

for $\alpha \in H^*(X)$ and $x \in H_*(X)$, where $\langle \ , \ \rangle$ denotes the Kronecker pairing of cohomology with homology.

First we recall the ring structure of $H_*(\Omega Sp(n))$ and the action of the Steenrod algebra on them by Kono-Kozima [5]:

$$H_*(\Omega Sp(n)) \cong \mathbb{F}_2[z_1, z_3, \ldots, z_{2n-1}]$$

and $|z_{2i-1}| = 4i-2$, where $|x|$ denotes the degree of an element of x. To state the action of the Steenrod algebra on z_{2i-1}, for a positive integer i we define

$$z_i = (z_{b(i)})^{2^{a(i)}},$$

where $a(i)$ and $b(i)$ are unique non-negative integers such that $i = 2^{a(i)}b(i)$ and $b(i)$ is odd. Then we have

$$z_{2i-1}Sq^{2j} = \binom{2i-2-j}{j} z_{2i-1-j}.$$

In particular, $z_{2i-1}Sq^2 = z_{2i-2}$. We also have

$$z_iSq^{2j} = \binom{i-1-j}{j} z_{i-j}$$

for any positive integer i such that $z_i \in H_*(\Omega Sp(n))$.

As $Sq^2Sq^2 = 0$ on $H_*(\Omega Sp(n))$ we can define

$$H_*(H_*(\Omega Sp(n)); Sq^2) = \mathrm{Ker} Sq^2 / \mathrm{Im} Sq^2.$$

To compute this group we put

$$\tilde{z}_{2i-1} = \begin{cases} z_{2^j+1} + z_1^{2^j-2}z_3 & \text{if } 2i-1 = 2^j+1 \text{ for some } j \geq 2, \\ z_1^2 z_{2i-1} + z_3 z_{2i-2} & \text{otherwise.} \end{cases}$$

Lemma 2.2. *If $2n-1 = 2^m+1$ for some $m \geq 3$, then*

$$H_*(H_*(\Omega Sp(n)); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2^{m-2}+1}) \otimes \mathbb{F}_2[\tilde{z}_{2^{m-1}+1}, \tilde{z}_{2^m+1}].$$

If $2^m+1 < 2n-1 < 2^{m+1}+1$ for some $m \geq 2$, then

$$H_*(H_*(\Omega Sp(n)); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2^{m-1}+1}) \otimes \mathbb{F}_2[\tilde{z}_{2^m+1}].$$

For $n = \infty$, $H_*(H_*(\Omega Sp); Sq^2) \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \ldots, \tilde{z}_{2^m+1}, \ldots)$.

Proof. If z_1 is inverted, then

$$H_*(\Omega Sp(n))[z_1^{-1}] = \mathbb{F}_2[z_1, z_3, \tilde{z}_5, \ldots, \tilde{z}_{2n-1}][z_1^{-1}].$$

As $\tilde{z}_{2i-1}Sq^2 = 0$ and $z_3Sq^2 = z_1^2$, it is easy to see that

$$\mathrm{Ker}(Sq^2 : H_*(\Omega Sp(n))[z_1^{-1}] \to H_*(\Omega Sp(n))[z_1^{-1}]) = \mathbb{F}_2[z_1, z_3^2, \tilde{z}_5, \ldots, \tilde{z}_{2n-1}][z_1^{-1}]$$

and that

$$\mathrm{Ker}(Sq^2 : H_*(\Omega Sp(n)) \to H_*(\Omega Sp(n))) = \mathbb{F}_2[z_1, z_3^2, \tilde{z}_5, \dots, \tilde{z}_{2n-1}].$$

Let m be the unique integer such that $2^m + 1 \leq 2n - 1 < 2^{m+1} + 1$. Since $(z_3 z_{2i-1})Sq^2 = \tilde{z}_{2i-1}$ for i such that $2i - 1 \neq 2^j + 1$ for any $j \geq 2$, and

$$(z_{2^{j+1}+3} + z_1^{2^{j+1}-4} z_7)Sq^2 = \tilde{z}_{2^j+1}^2$$

for $j \geq 2$, $z_3 Sq^2 = z_1^2$ and $z_7 Sq^2 = z_3^2$, there is an epimorphism

$$\mathbb{F}_2[z_1, \tilde{z}_5, \tilde{z}_9, \dots, \tilde{z}_{2^m+1}]/I \to H_*(H_*(\Omega Sp(n)); Sq^2),$$

where I is the ideal generated by z_1^2 and $\{\tilde{z}_{2^i+1}^2 | 11 \leq 2^{i+1} + 3 \leq 2n - 1\}$. Here we remark that $2n - 1 \geq 7$.

If $2n - 1 = 2^m + 1$ for some $m \geq 3$, then

$$\mathbb{F}_2[z_1, \tilde{z}_5, \tilde{z}_9, \dots, \tilde{z}_{2^m+1}]/I \cong \Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \dots, \tilde{z}_{2^{m-2}+1}) \otimes \mathbb{F}_2[\tilde{z}_{2^{m-1}+1}, \tilde{z}_{2^m+1}].$$

Since $z_{2i-1} Sq^2 = z_{i-1}^2$ for $i \leq 2^{m-1} + 1$, ySq^2 is a sum of monominals

$$z_1^{k_1} z_3^{k_3} z_5^{k_5} \cdots z_{2^m-1}^{k_{2^m-1}} z_{2^m+1}^{k_{2^m+1}}$$

with $k_{2i-1} > 1$ for some $2i - 1 \leq 2^{m-1} - 1$. Therefore, the epimorphism

$$\Lambda(z_1, \tilde{z}_5, \tilde{z}_9, \dots, \tilde{z}_{2^{m-2}+1}) \otimes \mathbb{F}_2[\tilde{z}_{2^{m-1}+1}, \tilde{z}_{2^m+1}] \to H_*(H_*(\Omega Sp(n)); Sq^2)$$

is monomorphic, and therefore, isomorphic.

The other cases are proved similarly. □

Lemma 2.3. *Let $n \geq 4$ and $x \in H_*(\Omega Sp(n))$ with $|x| > 2$. If $xSq^i = 0$ for all $i > 0$, then $x \in (H_{|x|+2}(\Omega Sp(n)))Sq^2$.*

Proof. We prove the lemma only when $2n - 1 = 2^m + 1$ for some $m \geq 3$ since the other cases are proved similarly.

First we will show that without loss of generality we may assume that x is in the subring $\mathbb{F}_2[z_1^2, z_3, \dots, z_{2^m+1}]$. We write $x = z_1 x' + x''$ where x', x'' are in the subring $\mathbb{F}_2[z_1^2, z_3, \dots, z_{2^m+1}]$. If $xSq^i = 0$ for all $i > 0$, then we have

$$0 = xSq^i = z_1(x' Sq^i) + x'' Sq^i.$$

Since $x' Sq^i, x'' Sq^i$ are in the subring $\mathbb{F}_2[z_1^2, z_3, \dots, z_{2^m+1}]$, the equation above implies that $x' Sq^i = x'' Sq^i = 0$. If x', x'' are in the Sq^2 image, then so is x. Thus we assume that x is in the subring $\mathbb{F}_2[z_1^2, z_3, \dots, z_{2^m+1}]$.

For a sequence of non-negative integers $I = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{m-2}, j_{m-1}, j_m)$ with $\varepsilon_i = 0$ or 1, we define

$$\tilde{z}_I = \tilde{z}_5^{\varepsilon_2} \cdots \tilde{z}_{2^{m-2}+1}^{\varepsilon_{m-2}} \tilde{z}_{2^{m-1}+1}^{j_{m-1}} \tilde{z}_{2^m+1}^{j_m}.$$

Then by Lemma 2.2 x is written as

$$x = \sum_I a_I \tilde{z}_I + ySq^2 \tag{2.1}$$

for some $y \in H_{|x|+2}(\Omega Sp(n))$, where $a_I \in \mathbb{F}_2$ and the sum is taken over sequences of non-negative integers $I = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{m-2}, j_{m-1}, j_m)$ such that $\varepsilon_i = 0$ or 1 and $|z_I| = |x|$. Now we are breaking the argument up into three steps.

Step I). We will show that $a_I = 0$ for all I such that $\varepsilon_i = 1$ for some $2 \leq i \leq m-2$.

Let Λ_1 be the ideal of $\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}]$ generated by the elements z_{2i-1}^2 for $1 \le 2i-1 \le 2^{m-1}-1$. Since $ySq^2 \in \Lambda_1$ and the ideal Λ_1 is stable under the action of the Steenrod algebra, by applying Sq^4 to equation (2.1) we have

$$0 = xSq^4 = \sum_I a_I(\tilde{z}_I Sq^4) \tag{2.2}$$

in $\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}]/\Lambda_1 \cong \Lambda(z_1, z_3, \ldots, z_{2^{m-1}-1}) \otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]$.

For an integer ℓ such that $2 \le \ell \le m-2$, we define a map

$$\begin{aligned}\phi_\ell : \Lambda(z_1, z_3, \ldots, z_{2^{m-1}-1}) &\otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}] \\ &\to \Lambda(z_1, z_3, \ldots, z_{2^\ell-3}, z_{2^\ell+1}, \ldots, z_{2^{m-1}-1}) \otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]\end{aligned}$$

as follows: If $\alpha \in \Lambda(z_1, z_3, \ldots, z_{2^{m-1}-1}) \otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]$ is written as

$$\alpha = z_{2^\ell-1}\beta + \gamma$$

with $\beta, \gamma \in \Lambda(z_1, z_3, \ldots, z_{2^\ell-3}, z_{2^\ell+1}, \ldots, z_{2^{m-1}-1}) \otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]$, then we define $\phi_\ell(\alpha) = \beta$.

For $2 \le \ell \le m-2$ and $I = (\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{m-2}, j_{m-1}, j_m)$ we define

$$z_{I_\ell} = z_5^{\varepsilon_2} \cdots z_{2^{\ell-1}+1}^{\varepsilon_{\ell-1}} z_{2^{\ell+1}+1}^{\varepsilon_{\ell+1}} \cdots z_{2^{m-2}+1}^{\varepsilon_{m-2}} z_{2^{m-1}+1}^{j_{m-1}} z_{2^m+1}^{j_m}.$$

Since $\tilde{z}_{2^k+1}Sq^4 = \binom{2^k+1-1-2}{2} z_{2^k-1} = z_{2^k-1}$ for $k > 1$, it is easy to see that

$$\phi_\ell(\tilde{z}_I Sq^4) = \begin{cases} z_{I_\ell} & \text{if } \varepsilon_\ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus equation (2.2) implies that

$$0 = \phi_\ell(xSq^4) = \sum_{I \text{ such that } \varepsilon_\ell = 1} a_I z_{I_\ell}$$

in $\Lambda(z_1, z_3, \ldots, z_{2^\ell-3}, z_{2^\ell+1}, \ldots, z_{2^{m-1}-1}) \otimes \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]$ and we proved that $a_I = 0$ for all I such that $\varepsilon_i = 1$ for some $2 \le i \le m-2$.

Step II). We proved that x is written as

$$x = \sum a_{(j_{m-1}, j_m)} \tilde{z}_{2^{m-1}+1}^{j_{m-1}} \tilde{z}_{2^m+1}^{j_m} + ySq^2 \tag{2.3}$$

for some $y \in H_{|x|+2}(\Omega Sp(n))$. In the second step we will show that $a_{(j_{m-1}, j_m)} = 0$ if $j_m > 0$. This is done by downward induction on j_m. Let k be a positive integer and assume that $a_{(j_{m-1}, j_m)} = 0$ if $j_m > k$.

Let Λ_2 be the ideal generated by z_{2i-1} for $1 \le 2i-1 \le 2^{m-1}-1$. Since $ySq^2 \in \Lambda_2$ and the ideal Λ_2 is stable under the action of the Steenrod algebra, by applying $Sq^{2^m k}$ to equation (2.3) we have

$$0 = xSq^{2^m k} = \sum a_{(j_{m-1}, j_m)} (\tilde{z}_{2^{m-1}+1}^{j_{m-1}} \tilde{z}_{2^m+1}^{j_m}) Sq^{2^m k}$$

in $\mathbb{F}_2[z_1, \ldots, z_{2^m+1}]/\Lambda_2 \cong \mathbb{F}_2[z_{2^{m-1}+1}, \ldots, z_{2^m+1}]$.

To compute $(\tilde{z}_{2^{m-1}+1}^{j_{m-1}} \tilde{z}_{2^m+1}^{j_m}) Sq^{2^m k}$ we remark that

$$\tilde{z}_{2^{m-1}+1} Sq^i \in \Lambda_2 \quad \text{unless } i = 0$$

and that

$$\tilde{z}_{2^m+1} Sq^{2i} = \binom{2^m - i}{i} z_{2^m+1-i} = 0 \quad \text{if } i > 2^{m-1}.$$

Thus in $\mathbb{F}_2[z_1,\ldots,z_{2^m+1}]/\Lambda_2 \cong \mathbb{F}_2[z_{2^{m-1}+1},\ldots,z_{2^m+1}]$ we have

$$\begin{aligned} 0 &= xSq^{2^m k} = \sum a_{(j_{m-1},j_m)}(\tilde{z}^{j_{m-1}}_{2^{m-1}+1}\tilde{z}^{j_m}_{2^m+1})Sq^{2^m k} \\ &= a_{(j,k)}z^j_{2^{m-1}+1}(z^k_{2^m+1}Sq^{2^m k}) + \sum_{j_m<k} a_{(j_{m-1},j_m)}\tilde{z}^{j_{m-1}}_{2^{m-1}+1}(z^{j_m}_{2^m+1}Sq^{2^m k}) \\ &= a_{(j,k)}z^{j+k}_{2^{m-1}+1}, \end{aligned}$$

which implies that $a_{(j,k)} = 0$ and completes the induction argument.

Step III). We proved that x is written as

$$x = a\tilde{z}^j_{2^{m-1}+1} + ySq^2 \tag{2.4}$$

for some $a \in \mathbb{F}_2$, $y \in H_{|x|+2}(\Omega Sp(n))$ and $j > 0$. In the final step we will show that $a = 0$ and complete the proof.

By applying $Sq^{2^{m-1}j}$ to equation (2.4) we have

$$az^j_{2^{m-2}+1} = ySq^2Sq^{2^{m-1}j} = ySq^{2^{m-1}j+2}.$$

For a sequence of integers $S = (2s_1-1, 2s_2-1, \ldots, 2s_t-1)$ such that $2^{m-2}+1 \leq 2s_1-1 \leq 2s_2-1 \leq \cdots \leq 2s_t-1 \leq 2^m+1$, we define

$$z_S = z_{2s_1-1}z_{2s_2-1}\cdots z_{2s_t-1}.$$

Then y is written as

$$y = \sum b_S z_S + y',$$

where $b_S \in \mathbb{F}_2$, S ranges over all sequences of integers $S = (2s_1-1, 2s_2-1, \ldots, 2s_t-1)$ such that $2^{m-2}+1 \leq 2s_1-1 \leq 2s_2-1 \leq \cdots \leq 2s_t-1 \leq 2^m+1$ and $|z_S| = |y|$, and y' is an element of the ideal generated by z_{2i-1} for $1 \leq 2i-1 \leq 2^{m-2}-1$. To prove that $a = 0$ it is sufficient to prove that the coefficient of $z^j_{2^{m-2}+1}$ in the expansion of $z_S Sq^{2^{m-1}j+2}$ as a sum of the standard monomials for $\mathbb{F}_2[z_1, z_3, \ldots, z_{2^m+1}]$ is zero.

Since

$$\begin{aligned} &(z_{2s_1-1}z_{2s_2-1}\cdots z_{2s_t-1})Sq^{2^{m-1}j+2} \\ = &\sum_{k_1+\cdots+k_t=2^{m-2}j+1}(z_{2s_1-1}Sq^{2k_1})(z_{2s_2-1}Sq^{2k_2})\cdots(z_{2s_t-1}Sq^{2k_t}), \end{aligned}$$

we consider the equation

$$z_{2s_i-1}Sq^{2k_i} = \binom{2s_i-2-k_i}{k_i}z_{2s_i-1-k_i} = z^{2^{r_i}}_{2^{m-2}+1} = z_{(2^{m-2}+1)2^{r_i}}.$$

Then $2s_i-1-k_i = (2^{m-2}+1)2^{r_i}$, that is, $2s_i-1 = (2^{m-2}+1)2^{r_i}+k_i$. Since $2s_i-1 = (2^{m-2}+1)2^{r_i}+k_i \leq 2^m+1$, we have $r_i = 0$ or 1.

If $r_i = 0$ and $k_i > 0$, then

$$\binom{2s_i-2-k_i}{k_i} = \binom{2^{m-2}}{k_i} \neq 0$$

implies that $k_i = 2^{m-2}$ and is even. Thus $2s_i-1 = 2^{m-1}+1$.

If $r_i = 1$ and $k_i > 0$, then

$$\binom{2s_i-2-k_i}{k_i} = \binom{2^{m-1}+1}{k_i} \neq 0$$

implies that $k_i = 1$ or $2^{m-1}+1$ since $k_i = 2s_i - 1 - 2(2^{m-2}+1)$ is odd. Thus $2s_i - 1 = 2^{m-1}+3$ or 2^m+3. The last case is impossible since $2s_i - 1 \leq 2^m + 1$. Thus $2s_i - 1 = 2^{m-1}+3$ and $k_i = 1$.

According to the argument above the coefficient of $z^j_{2^{m-2}+1}$ in $Sq^{2^{m-1}j+2}z_S$ is zero unless z_S is

$$z^r_{2^{m-2}+1}z^s_{2^{m-1}+1}z^t_{2^{m-1}+3}$$

for some non-negative integers r, s, t. If the coefficient of $z^j_{2^{m-2}+1}$ in

$$\begin{aligned} & (z^r_{2^{m-2}+1}z^s_{2^{m-1}+1}z^t_{2^{m-1}+3})Sq^{2^{m-1}j+2} \\ = \; & z^r_{2^{m-2}+1}(z^s_{2^{m-1}+1})Sq^{2^{m-1}s}(z^t_{2^{m-1}+3})Sq^{2t} + \cdots \end{aligned}$$

is non-zero, then $2^{m-1}j + 2 = 2^{m-1}s + 2t$, that is, $t = 1 + 2^{m-2}(j-s)$. As $m \geq 3$, this implies that $j - s \geq 0$. Since the degree of $\tilde{z}^j_{2^{m-1}+1}$ is equal to that of $(z^r_{2^{m-2}+1}z^s_{2^{m-1}+1}z^t_{2^{m-1}+3})Sq^2$, we have

$$j(2^{m-1}+1) \quad = \quad r(2^{m-2}+1) + s(2^{m-1}+1) + t(2^{m-1}+3) - 1,$$

that is,

$$2 \leq r + 2 = -(j-s)(2^{m-1}-1) \leq 0,$$

which is impossible. We proved that the coefficient of $z^j_{2^{m-2}+1}$ in $z_S Sq^{2^{m-1}j+2}$ is zero and, therefore, completed the proof of the lemma. □

For a connected space X of finite type we associate a graph $G(X)$ as follows. The vertices of $G(X)$ are non-zero elements of $\tilde{H}_*(X)$ and a pair of vertices $\{x,\ y\}$ is an edge of $G(X)$ if and only if $xSq^i = y$ or $ySq^i = x$ for some $i > 0$.

Lemma 2.4. *Let $n \geq 4$ or $n = \infty$. Every vertex of $G(\Omega Sp(n))$ whose dimension is greater than two is connected to z_1^2 or z_1^3.*

Proof. By induction on the dimension of a vertex we will prove the lemma. Let x be a vertex of $G(\Omega Sp(n))$.

If $|x| = 4$, then $x = z_1^2$. If $|x| = 6$, then $x = z_1^3, z_1^3 + z_3$ or z_3. Since $(z_1^3 + z_3)Sq^2 = z_3 Sq^2 = z_1^2$, the assertion is true.

Let $|x| = 2m \geq 8$ and assume that the assertion is true for vertices whose dimensions are less than $|x|$. If $xSq^i \neq 0$ for some $i > 0$, then the assertion is true by induction. If $xSq^i = 0$ for all $i > 0$, then by Lemma 2.3 there is a vertex y' such that $y'Sq^2 = x$. We put

$$y = \begin{cases} y' & \text{if } y'Sq^4 \neq 0, \\ y' + z_1^{m-4}z_5 + z_1^{m-2}z_3 & \text{if } y'Sq^4 = 0. \end{cases}$$

Then $ySq^2 = x$ and $ySq^4 \neq 0$, and $x \leftarrow y \rightarrow ySq^4$ is a path which connects x and a vertex whose dimension is less than x. Then by induction there is a path which connects ySq^4 and z_1^2 or z_1^3. Therefore there is a path which connects x and z_1^2 or z_1^3, and we complete the proof. □

We remark that Lemma 2.3 is valid for $n = 2$ or 3 and that Lemma 2.4 is valid for $n = 3$. These facts follow Proposition 2.1 of [3].

Proof of Theorem 2.1. As the theorem for $n = 2$ and 3 was proved by Hopkins and Hubbuck, we prove the theorem for $n \geq 4$ or $n = \infty$.

We give CW-decompositions for $\Omega Sp(n)$ without odd dimensional cells for all n. Then $(\Omega Sp(2))_8$ is homotopy equivalent to $(\Omega Sp(n))_8$, where for a CW-complex X by X_r we denote the r-skeleton of X. By [3] $(\Omega Sp(2))_8$ is stably homotopy equivalent to $Z \vee S^8$, where Z is a stably indecomposable CW-complex. Therefore if $\Omega Sp(n)$ is stably split as $\Omega Sp(n) \simeq X(1) \vee X(2)$, where $H_2(X(1)) \cong \mathbb{F}_2$; then $X(2)$ is 7-connected. By Lemma 2.4 this implies that $X(2)$ must be trivial and completes the proof of the theorem. $\square$

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