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CLASSIFYING SERRE SUBCATEGORIES OF FINITELY PRESENTED MODULES

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Abstract. Given a commutative coherent ring R, a bijective correspondence between the thick subcategories of perfect complexes $\mathcal{D}_{\mathrm{per}}(R)$ and the Serre subcategories of finitely presented modules is established. To construct this correspondence, properties of the Ziegler and Zariski topologies on the set of isomorphism classes of indecomposable injective modules are used in an essential way.

Introduction

Throughout this paper we fix a commutative ring R with unit. Recall that a complex of R-modules is perfect if it is isomorphic in the derived category $\mathcal{D}(R)$ of R to a bounded complex of finitely generated projective modules. The (skeletally small) full subcategory of perfect complexes is denoted by $\mathcal{D}_{\mathrm{per}}(R)$. If R is noetherian the classification theorem of Hopkins [5] and Neeman [8] establishes a bijective correspondence between the thick subcategories of perfect complexes $\mathcal{D}_{\mathrm{per}}(R)$ and arbitrary unions of closed sets of $\mathsf{Spec}\, R$. Later Thomason [12] generalized the result to arbitrary commutative rings (and to quasi-compact, quasi-separated schemes).

For a regular coherent ring R, Hovey [6] shows that there is a 1-1 correspondence between the thick subcategories of perfect complexes $\mathcal{D}_{\mathrm{per}}(R)$ and the Serre subcategories of finitely presented modules (=wide subcategories in this case; see [6, 3.7]). The main result of the paper establishes the same correspondence for arbitrary coherent rings. One should remark that our approach is completely different from that of Hovey and is based on properties of the Ziegler and Zariski topologies on the set of isomorphism classes of indecomposable injective modules.

By $\mathrm{Mod}\, R$ we denote the category of all R-modules. The full subcategory of finitely presented modules is denoted by $\mathrm{mod}\, R$. The *Gabriel spectrum* of R is, by definition, the set $\mathsf{Inj}\, R$ of isomorphism classes of indecomposable injective R-modules. The collection of subsets

$$[M] = \{E \in \mathsf{Inj}\, R \mid \mathrm{Hom}_R(M, E) = 0\}$$

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with $M \in \operatorname{mod} R$ forms a basis of open subsets for the *Zariski topology* on $\mathsf{Inj}\, R$. This topological space will be denoted by $\mathsf{Inj}_{\mathrm{zar}}\, R$.

There is another topology on $\mathsf{Inj}\, R$. Whenever R is coherent the collection of subsets

$$(M) = \mathsf{Inj}_{\mathrm{zar}}\, R \setminus [M] = \{E \in \mathsf{Inj}\, R \mid \mathrm{Hom}_R(M, E) \neq 0\}$$

with $M \in \operatorname{mod} R$ forms a basis of quasi-compact open subsets for the *Ziegler topology* on $\mathsf{Inj}\, R$. This topological space will be denoted by $\mathsf{Inj}_{\mathrm{zg}}\, R$. It arises from Ziegler's work on the model theory of modules [13]. The points of the Ziegler spectrum of R are the isomorphism classes of indecomposable pure-injective R-modules, and the closed subsets correspond to complete theories of modules. If R is coherent, then the indecomposable injective R-modules form a closed subset in the Ziegler spectrum and the sets (M) above form a basis of open sets for the restriction of the Ziegler topology to this set.

The Zariski and Ziegler topologies on $\mathsf{Inj}\, R$ play a crucial role in our analysis. In the general commutative case there is an embedding

$$\alpha : \mathsf{Spec}\, R \to \mathsf{Inj}\, R, \quad P \mapsto E(R/P).$$

Here $E(R/P)$ stands for the injective hull of R/P. It need not be surjective in general. We shall identify $\mathsf{Spec}\, R$ with its image in $\mathsf{Inj}\, R$.

We first demonstrate the following:

Theorem A. *Let a ring R be commutative coherent. The space $\mathsf{Spec}\, R$ is dense and a retract in $\mathsf{Inj}_{\mathrm{zar}}\, R$. A left inverse to the embedding $\mathsf{Spec}\, R \hookrightarrow \mathsf{Inj}_{\mathrm{zar}}\, R$ takes an indecomposable injective module E to the prime ideal $P(E)$ which is the sum of annihilator ideals of non-zero elements of E. Moreover, $\mathsf{Inj}_{\mathrm{zar}}\, R$ is quasi-compact, the basic open subsets $[M], M \in \operatorname{mod} R$, are quasi-compact, the intersection of two quasi-compact open subsets is quasi-compact, and every non-empty irreducible closed subset has a generic point.*

Notice that neither $\mathsf{Inj}_{\mathrm{zar}}\, R$ nor $\mathsf{Inj}_{\mathrm{zg}}\, R$ is a spectral space in general, for these are not necessarily T_0.

If $M \in \operatorname{Mod} R$ we write

$$\mathsf{supp}(M) = \{P \in \mathsf{Spec}\, R \mid M_P \neq 0\}.$$

The next theorem was proved by Gabriel [2] for noetherian rings and by Hovey [6, 3.6] for regular coherent rings and [6, Sec. 4] for quotients of such rings by finitely generated ideals.

Theorem B. *Let a ring R be commutative coherent. The assignments*

$$\operatorname{mod} R \supseteq \mathcal{S} \mapsto \bigcup_{M \in \mathcal{S}} \mathsf{supp}(M) \text{ and } \mathsf{Spec}\, R \supseteq Y \mapsto \{M \in \operatorname{mod} R \mid \mathsf{supp}(M) \subseteq Y\}$$

induce bijections between

- ⋄ *the set of all Serre subcategories of* $\operatorname{mod} R$, *and*
- ⋄ *the set of all subsets $Y \subseteq \mathsf{Spec}\, R$ of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $\mathsf{Spec}\, R \setminus Y_i$ for all $i \in \Omega$.*

Given a coherent ring R and a family $\mathcal{X}$ of objects in $\operatorname{Mod} R$, by $\sqrt{\mathcal{X}}$ denote the least Serre subcategory in $\operatorname{mod} R$ containing $\mathcal{X}$.

Theorem C. *Let a ring R be commutative coherent. There are bijections between*

$\diamond$ *the set of all subsets* $Y \subseteq \mathsf{Inj}_{\mathrm{zar}} R$ *of the form* $Y = \bigcup_{i\in\Omega} Y_i$ *with each* Y_i *having quasi-compact open complement* $\mathsf{Inj}_{\mathrm{zar}} R \setminus Y_i$*, that is, the set of all open subsets of* $\mathsf{Inj}_{\mathrm{zg}}$,
$\diamond$ *the set of all Serre subcategories of* $\operatorname{mod} R$,
$\diamond$ *the set of all thick subcategories of* $\mathcal{D}_{\mathrm{per}}(R)$.

These bijections are defined as follows:

$$Y \mapsto \begin{cases} \mathcal{S} = \{M \mid (M) \subseteq Y\}, \\ \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid (H_n(X)) \subseteq Y \text{ for all } n \in \mathbf{Z}\}, \end{cases}$$

$$\mathcal{S} \mapsto \begin{cases} Y = \bigcup_{M\in\mathcal{S}} (M), \\ \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid H_n(X) \in \mathcal{S} \text{ for all } n \in \mathbf{Z}\}, \end{cases}$$

$$\mathcal{T} \mapsto \begin{cases} Y = \bigcup_{X\in\mathcal{T}, n\in\mathbf{Z}} (H_n(X)), \\ \mathcal{S} = \sqrt{\{H_n(X) \mid X \in \mathcal{T}, n \in \mathbf{Z}\}}. \end{cases}$$

1. Proof of Theorem A

Recall that a ring R is coherent if and only if the (small) category of finitely presented modules $\operatorname{mod} R$ is abelian (see [10, Sec. 2]). For a coherent ring R we are going to show that the Zariski topology on $\mathsf{Inj}\, R$ can be described in terms of finitely generated ideals.

Given any commutative ring R and any ideal I of R, let us set $D^{\mathrm{m}}(I) = \{E \in \mathsf{Inj}\, R \mid (R/I, E) = 0\}$ (“m” for “morphism”). Since $D^{\mathrm{m}}(I) \cap D^{\mathrm{m}}(J) = D^{\mathrm{m}}(I \cap J)$ (for the non-immediate inclusion, note that any morphism from $R/(I\cap J)$ to E extends, by injectivity of E, to one from $R/I \oplus R/J$) these form a basis for topology on $\mathsf{Inj}\, R$. Note, however, that if $I = \sum_\lambda I_\lambda$, then clearly $D^{\mathrm{m}}(I) \supseteq \bigcup_\lambda D^{\mathrm{m}}(I_\lambda)$ but, as it is shown in [9, after 9.3], the inclusion may be proper.

For a coherent ring R the sets $D^{\mathrm{m}}(I)$ with I running over finitely generated ideals form a basis for a topology on $\mathsf{Inj}\, R$ that we call the *fg-ideals topology* (use the fact that the intersection of two finitely generated ideals is finitely generated in a coherent ring [11, I.13.3]). By definition, $D^{\mathrm{m}}(I) = [R/I]$ for I a finitely generated ideal. Let M be a finitely presented R-module. It is finitely generated by $b_1, ..., b_n$, say. Set $M_k = \sum_{j\leqslant k} b_j R$, $M_0 = 0$. Each factor $C_j = M_j/M_{j-1}$ is cyclic and, we claim $[M] = [C_1] \cap ... \cap [C_n]$. For, if there is a non-zero morphism from C_j to E, then, by injectivity of E, this extends to a morphism from M/M_{j-1} to E and hence there is induced a non-zero morphism from M to E. Conversely, if $f : M \to E$ is non-zero, let j be minimal such that the restriction of f to M_j is non-zero. Then f induces a non-zero morphism from C_j to E. Since each C_j is cyclic and finitely presented there are finitely generated ideals I_j, $1 \leqslant j \leqslant n$, such that $C_j \cong R/I_j$. It follows that each $[C_j]$ coincides with $D^{\mathrm{m}}(I_j)$, and hence $[M] = D^{\mathrm{m}}(I)$ with $I = \bigcap_{1\leqslant j\leqslant n} I_j$ finitely generated. Thus we have shown the following:

Proposition 1.1. *Given a coherent ring* R*, the Zariski topology on* $\mathsf{Inj}\, R$ *coincides with the fg-ideals topology.*

If $\mathcal{A}$ is an abelian category, then a Serre subcategory is a full subcategory $\mathcal{S}$ such that if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$, then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$. Given a subcategory $\mathcal{X}$ in $\operatorname{mod} R$ with R coherent, we may consider the smallest Serre subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$. This Serre subcategory we denote, following Herzog [3], by

$$\sqrt{\mathcal{X}} = \bigcap \{\mathcal{S} \subseteq \operatorname{mod} R \mid \mathcal{S} \supseteq \mathcal{X} \text{ is Serre}\}.$$

There is an explicit description of $\sqrt{\mathcal{X}}$.

Proposition 1.2 ([3, 3.1])**.** *Let R be commutative coherent and let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$*. A finitely presented module M is in $\sqrt{\mathcal{X}}$ if and only if there is a finite filtration of M by finitely presented submodules*

$$M = M_0 \geqslant M_1 \geqslant \cdots \geqslant M_n = 0$$

and, for each $i < n$, there is $N_i \in \mathcal{X}$ and there are finitely presented submodules

$$N_i \geqslant N_{i_1} \geqslant N_{i_2}$$

such that $M_i/M_{i+1} \cong N_{i_1}/N_{i_2}$.

Given a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, denote by

$$[\mathcal{X}] = \{E \in \mathsf{Inj}\, R \mid \operatorname{Hom}_R(M, E) = 0 \text{ for all } M \in \mathcal{X}\}.$$

We shall also write $(\mathcal{X})$ to denote $\mathsf{Inj}\, R \setminus [\mathcal{X}]$.

Corollary 1.3 ([3, 3.3])**.** *Given a coherent ring R and $\mathcal{X} \subseteq \operatorname{mod} R$, we have $[\mathcal{X}] = [\sqrt{\mathcal{X}}]$ and $(\mathcal{X}) = (\sqrt{\mathcal{X}})$.*

Proof. This immediately follows from Proposition 1.2 and the fact that the functor $\operatorname{Hom}_R(-, E)$ with E injective preserves exact sequences. □

Let E be any indecomposable injective R-module. Set $P = P(E)$ to be the sum of annihilator ideals of non-zero elements, equivalently non-zero submodules, of E. Since E is uniform, the set of annihilator ideals of non-zero elements of E is closed under finite sum. It is easy to check ([9, 9.2]) that $P(E)$ is a prime ideal.

Recall that for any ideal I of a ring, R, and $r \in R$ we have an isomorphism $R/(I : r) \cong (rR + I)/I$, where $(I : r) = \{s \in R \mid rs \in I\}$, induced by sending $1 + (I : r)$ to $r + I$.

The next result, from [9], is crucial in our analysis. We give a proof here for the convenience of the reader. We use the notation E_P to denote $E(R/P)$.

Theorem 1.4 (Prest [9, 9.6])**.** *Let R be commutative coherent, let E be an indecomposable injective module and let $P(E)$ be the prime ideal defined before. Then E and $E_{P(E)}$ are topologically indistinguishable in $\mathsf{Inj}_{\mathrm{zg}}\, R$ and hence also in $\mathsf{Inj}_{\mathrm{zar}}\, R$.*

Proof. Let I be such that $E = E(R/I)$. For each $r \in R \setminus I$ we have, by the remark just above, that the annihilator of $r + I \in E$ is $(I : r)$ and so, by definition of $P(E)$, we have $(I : r) \leqslant P(E)$. The natural projection $(rR+I)/I \cong R/(I : r) \longrightarrow R/P(E)$ extends to a morphism from E to $E_{P(E)}$, which is non-zero on $r + I$. Forming the product of these morphisms as r varies over $R \setminus I$, we obtain a morphism from E to a product of copies of $E_{P(E)}$ which is monic on R/I and hence is monic. Therefore E is a direct summand of a product of copies of $E_{P(E)}$, and so $E \in (M)$ implies $E_{P(E)} \in (M)$, where $M \in \operatorname{mod} R$.

For the converse, take a basic Ziegler-open neighbourhood of $E_{P(E)}$: by (the proof of) Proposition 1.1 this has the form (R/I) for a finitely generated ideal I of R. Now, $E_{P(E)} \in (R/I)$ means that there is a non-zero morphism $f : R/I \longrightarrow E_{P(E)}$. Since $R/P(E)$ is essential in $E_{P(E)}$, the image of f has non-zero intersection with $R/P(E)$ so there is an ideal J, without loss of generality finitely generated, with $I < J \leqslant R$ and such that the restriction, f', of f to J/I is non-zero (and the image is contained in $R/P(E)$). Since $R/P(E) = \varinjlim R/I_\lambda$, where I_λ ranges over the annihilators of non-zero elements of E, and J/I is finitely presented, f' factorises through one of the maps $R/I_\lambda \longrightarrow R/P(E)$. In particular, there is a non-zero morphism $J/I \longrightarrow E$ and hence, by injectivity of E, an extension to a morphism $R/I \longrightarrow E$, showing that $E \in (R/I)$, as required. □

Remark. In fact, Prest [9] proves a slightly stronger result: the injective modules E and $E_{P(E)}$ are elementary equivalent in the first order language of modules.

Recall the definition of the *Zariski spectrum*, $\mathsf{Spec}\, R$, of a commutative ring R. The points are the prime ideals of R and a basis of open sets for the topology is given by the sets $D(r) = \{P \in \mathsf{Spec}\, R \mid r \notin P\}$ for $r \in R$. The open sets are, therefore, those of the form $D(I) = \{P \in \mathsf{Spec}\, R \mid I \nsubseteq P\}$ for I an ideal of R.

We also consider the sets $D^{\mathrm{m}}(r) = \{E \in \mathsf{Inj}\, R \mid \mathrm{Hom}_R(R/rR, E) = 0\} = [R/rR]$ for $r \in R$. By [9, 9.4], if I is a finitely generated ideal of R and $I = \sum_1^n I_i$, then $D^{\mathrm{m}}(I) = \bigcup_1^n D^{\mathrm{m}}(I_i)$. It follows that $D^{\mathrm{m}}(I) = \bigcup_1^n D^{\mathrm{m}}(r_i)$ with $r_1, \ldots, r_n$ generators of I. Therefore, by Proposition 1.1, the sets $D^{\mathrm{m}}(r)$, $r \in R$, form a basis for the space $\mathsf{Inj}_{\mathrm{zar}}\, R$.

We are going to examine the relation between the spaces $\mathsf{Spec}\, R$ and $\mathsf{Inj}_{\mathrm{zar}}\, R$. We identify $\mathsf{Spec}\, R$ with a subset of $\mathsf{Inj}_{\mathrm{zar}}\, R$ via the map α defined in the introduction. The following example, pointed out by T. Kucera, shows that the injective map $\mathsf{Spec}\, R \to \mathsf{Inj}\, R$, $P \mapsto E_P$, need not be surjective.

Example. Let $R = k[X_n (n \in \omega)]$ be a polynomial ring over a field k in infinitely many commuting indeterminates. Then R is obviously coherent. Let $I = (X_n^{n+1} : n \in \omega)$. Then I is not prime (since $X_1 \notin I$ but $X_1^2 \in I$) but $E = E(R/I)$ is an indecomposable injective [9]. On the other hand, E does not have the form $E(R/P)$ for any prime P [9, 9.1].

Lemma 1.5 ([9, 9.2])**.** *Let the ring R be commutative. A module $E \in \mathsf{Inj}\, R$ has the form E_P for some prime ideal P if and only if the set of annihilator ideals of non-zero elements of E has a maximal member, namely $P(E)$, in which case $E = E_{P(E)}$.*

We are now in a position to prove Theorem A.

Proof of Theorem A. For any ideal I of R we have

$$D^{\mathrm{m}}(I) \cap \mathsf{Spec}\, R = D(I) \tag{1.1}$$

(see [9, 9.5]). From this relation, Proposition 1.1, and Theorem 1.4 it follows that $\mathsf{Spec}\, R$ is dense in $\mathsf{Inj}_{\mathrm{zar}}\, R$ and that $\alpha : \mathsf{Spec}\, R \to \mathsf{Inj}_{\mathrm{zar}}\, R$ is a continuous map.

It follows from Lemma 1.5 that

$$\beta : \mathsf{Inj}_{\mathrm{zar}}\, R \to \mathsf{Spec}\, R, \quad E \mapsto P(E),$$

is left inverse to α. The relation (1.1) implies that β is continuous as well. Thus $\mathsf{Spec}\, R$ is a retract of $\mathsf{Inj}_{\mathrm{zar}}\, R$.

Let us show that each basic open set $[M]$, $M \in \mathrm{mod}\, R$, is quasi-compact (in particular $\mathsf{Inj}_{\mathrm{zar}}\, R = [0]$ is quasi-compact). By Proposition 1.1 $[M] = D^{\mathrm{m}}(I)$ for some finitely generated ideal I of R.

Let $D^{\mathrm{m}}(I) = \bigcup_{i \in \Omega} D^{\mathrm{m}}(I_i)$ with each I_i finitely generated. It follows from (1.1) that $D(I) = \bigcup_{i \in \Omega} D(I_i)$. Since I is finitely generated, $D(I)$ is quasi-compact in $\mathsf{Spec}\, R$. We see that $D(I) = \bigcup_{i \in \Omega_0} D(I_i)$ for some finite subset $\Omega_0 \subset \Omega$.

Assume $E \in D^{\mathrm{m}}(I) \setminus \bigcup_{i \in \Omega_0} D^{\mathrm{m}}(I_i)$. It follows from Theorem 1.4 that $E_{P(E)} \in D^{\mathrm{m}}(I) \setminus \bigcup_{i \in \Omega_0} D^{\mathrm{m}}(I_i)$. But $E_{P(E)} \in D^{\mathrm{m}}(I) \cap \mathsf{Spec}\, R = D(I) = \bigcup_{i \in \Omega_0} D(I_i)$, and hence it is in $D(I_{i_0}) = D^{\mathrm{m}}(I_{i_0}) \cap \mathsf{Spec}\, R$ for some $i_0 \in \Omega_0$ — a contradiction. So $[M]$ is quasi-compact.

It follows from the above that every quasi-compact open subset in $\mathsf{Inj}_{\mathrm{zar}}\, R$ is of the form $[M]$ with M finitely presented. Therefore, the intersection $[M] \cap [N] = [M \oplus N]$ of two quasi-compact open subsets is quasi-compact.

Finally, by [9, 9.14] a subset V of $\mathsf{Inj}_{\mathrm{zar}}\, R$ is Zariski-closed and irreducible if and only if there is a prime ideal Q of R such that $V = \{E \mid P(E) \geqslant Q\}$. Theorem 1.4, Lemma 1.5, and (1.1) obviously imply that the point $E_Q \in V$ is generic. Theorem A is proved. $\square$

Given a spectral topological space, X, Hochster [4] endows the underlying set with a new, "dual", topology by taking as open sets those of the form $Y = \bigcup_{i \in \Omega} Y_i$ where Y_i has quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$. The spaces, X, which we consider here are not in general spectral; nevertheless, we make the same definition and denote the space so obtained by X^*. We also write $\mathsf{Spec}^*\, R$ for $(\mathsf{Spec}\, R)^*$.

Corollary 1.6. *Let a ring R be commutative coherent. The following relations hold:*

$$\mathsf{Inj}_{\mathrm{zar}}\, R = (\mathsf{Inj}_{\mathrm{zg}}\, R)^* \text{ and } \mathsf{Inj}_{\mathrm{zg}}\, R = (\mathsf{Inj}_{\mathrm{zar}}\, R)^*.$$

Proof. This follows from Theorem A and the fact that a Ziegler-open subset $\mathcal{O}$ is quasi-compact if and only if it is one of the basic open subsets $(M), M \in \mathrm{mod}\, R$ (see [13, 4.9], [3, 3.9] or [7, 4.5]). $\square$

2. Proof of Theorem B

Lemma 2.1. *Let a ring R be commutative coherent. The maps*

$$\mathsf{Spec}^*\, R \supseteq \mathcal{O} \overset{\varphi}{\mapsto} \mathcal{Q}_{\mathcal{O}} = \{E \in \mathsf{Inj}_{\mathrm{zg}}\, R \mid P(E) \in \mathcal{O}\}$$

and

$$\mathsf{Inj}_{\mathrm{zg}}\, R \supseteq \mathcal{Q} \overset{\psi}{\mapsto} \mathcal{O}_{\mathcal{Q}} = \{P(E) \in \mathsf{Spec}^*\, R \mid E \in \mathcal{Q}\} = \mathcal{Q} \cap \mathsf{Spec}^*\, R$$

induce a 1-1 correspondence between the lattices of open sets of $\mathsf{Spec}^*\, R$ and those of $\mathsf{Inj}_{\mathrm{zg}}\, R$.

Proof. First note that $E_P \in \mathcal{Q}_{\mathcal{O}}$ for any $P \in \mathcal{O}$ (see Lemma 1.5). Let us check that $\mathcal{Q}_{\mathcal{O}}$ is an open set in $\mathsf{Inj}_{\mathrm{zg}}\, R$. Given an ideal I of R, denote $V(I) := \mathsf{Spec}\, R \setminus D(I)$ and $V^{\mathrm{m}}(I) := \mathsf{Inj}_{\mathrm{zg}}\, R \setminus D^{\mathrm{m}}(I)$. By definition, each $V(I)$ with I a finitely generated ideal of R is a basic open set in $\mathsf{Spec}^*\, R$. It follows from (1.1) that

$$V(I) = V^{\mathrm{m}}(I) \cap \mathsf{Spec}^*\, R. \tag{2.1}$$

Every closed subset of $\mathsf{Spec}\, R$ with quasi-compact complement has the form $V(I)$ for some finitely generated ideal, I, of R (see [1, Chpt. 1, Exc. 17(vii)]), so there are finitely generated ideals $I_\lambda \subseteq R$ such that $\mathcal{O} = \bigcup_\lambda V(I_\lambda)$. Since the points E and $E_{P(E)}$ are, by Theorem 1.4, indistinguishable in $\mathsf{Inj}_{\mathrm{zg}}\, R$, we see that $\mathcal{Q}_{\mathcal{O}} = \bigcup_\lambda V^{\mathrm{m}}(I_\lambda)$, hence this set is open in $\mathsf{Inj}_{\mathrm{zg}}\, R = (\mathsf{Inj}_{\mathrm{zar}}\, R)^*$.

The same arguments imply that $\mathcal{O}_{\mathcal{Q}}$ is open in $\mathsf{Spec}^*\, R$. It is now easy to see that $\mathcal{O}_{\mathcal{Q}_{\mathcal{O}}} = \mathcal{O}$ and $\mathcal{Q}_{\mathcal{O}_{\mathcal{Q}}} = \mathcal{Q}$. $\square$

If P is a prime ideal of a commutative ring R, its complement as a subset of R is a multiplicatively closed set S. For a module M one denotes the module of fractions $M[S^{-1}]$ by M_P. There is a corresponding Gabriel topology

$$\mathfrak{F}_P = \{I \mid P \notin V(I)\}.$$

The $\mathfrak{F}_P$-torsion modules are characterized by the property that $M_P = 0$ (see [11, p. 151]). Given an injective R-module E, denote by $\mathfrak{F}_E$ the Gabriel topology cogenerated by E. By definition, this corresponds to the localizing subcategory $\mathcal{S}_E = \{M \in \operatorname{Mod} R \mid \operatorname{Hom}_R(M, E) = 0\}$ (for localization in locally coherent categories see, for example, [3, Sec. 2]; there the term hereditary torsion subcategory, rather than localizing subcategory, is used). It is easy to see that

$$\mathfrak{F}_E = \{I \mid \operatorname{Hom}_R(R/I, E) = 0\}.$$

Lemma 2.2. *Let R be commutative and $P \in \mathsf{Spec}\, R$. Then $\mathfrak{F}_P = \mathfrak{F}_{E_P}$.*

Proof. Let $I \in \mathfrak{F}_{E_P}$ be such that $I \notin \mathfrak{F}_P$. It follows that $I \subseteq P$, and hence there is a non-zero map $R/I \to E_P = E(R/P)$ — a contradiction. Now suppose $I \in \mathfrak{F}_P$. If there existed a non-zero map $f : R/I \to E_P$, it would follow that I is contained in the annihilator ideal of $0 \neq f(1) \in E_P$. Lemma 1.5 would imply that $I \subseteq P = P(E_P)$ — a contradiction. □

Corollary 2.3. *Let R be commutative coherent, $M \in \operatorname{mod} R$, and $E \in \mathsf{Inj}\, R$. Then $E \in (M)$ if and only if $M_{P(E)} \neq 0$ (or equivalently $P(E) \in \mathsf{supp}(M)$).*

Proof. By Theorem 1.4, $E \in (M)$ if and only if $E_{P(E)} \in (M)$. The assertion now follows from the preceding lemma. □

It follows from the preceding corollary that

$$\mathsf{supp}(M) = (M) \cap \mathsf{Spec}^* R. \tag{2.2}$$

Hence $\mathsf{supp}(M)$ is an open set of $\mathsf{Spec}^* R$ by Lemma 2.1. More generally, we have for any $\mathcal{X} \subseteq \operatorname{mod} R$:

$$\mathsf{supp}(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \mathsf{supp}(M) = (\mathcal{X}) \cap \mathsf{Spec}^* R.$$

Since $(\mathcal{X}) = (\sqrt{\mathcal{X}})$ by Corollary 1.3, it follows that

$$\mathsf{supp}(\mathcal{X}) = \mathsf{supp}(\sqrt{\mathcal{X}}). \tag{2.3}$$

We are now in a position to prove Theorem B.

Proof of Theorem B. The map

$$\tau : \operatorname{mod} R \supseteq \mathcal{S} \mapsto \bigcup_{M \in \mathcal{S}} \mathsf{supp}(M)$$

factors as

$$\operatorname{mod} R \supseteq \mathcal{S} \overset{\delta}{\mapsto} \mathcal{Q} = \bigcup_{M \in \mathcal{S}} (M) \overset{\psi}{\mapsto} \bigcup_{M \in \mathcal{S}} \mathsf{supp}(M),$$

where ψ is the map of Lemma 2.1 (we have used here relation (2.2)). By [3, 3.8] and [7, 4.2] the map δ induces a 1-1 correspondence between the Serre subcategories of $\operatorname{Mod} R$ and the open sets $\mathcal{Q}$ of $\mathsf{Inj}_{\mathsf{zg}}\, R$. By Lemma 2.1, the map ψ induces a 1-1 correspondence between lattices of open sets of $\mathsf{Inj}_{\mathsf{zg}}\, R$ and those of $\mathsf{Spec}^* R$. Therefore, the map τ induces the desired 1-1 correspondence between the Serre subcategories of $\operatorname{mod} R$ and the open sets of $\mathsf{Spec}^* R$. The inverse map to this correspondence is induced by the composite

$$\mathsf{Spec}^* R \supseteq \mathcal{O} \overset{\varphi}{\mapsto} \mathcal{Q}_{\mathcal{O}} \overset{\zeta}{\mapsto} \mathcal{S}(\mathcal{Q}_{\mathcal{O}}) := \{M \in \operatorname{mod} R \mid (M) \subseteq \mathcal{Q}_{\mathcal{O}}\}$$

where ζ yields the inverse to the correspondence induced by τ (see [3, 3.8] and [7, 4.2]) and ψ yields the inverse to the correspondence induced by φ (see Lemma 2.1). Theorem B is proved. □

Recall that a Serre subcategory $\mathcal{L}$ of $\operatorname{Mod} R$ is *localizing* if it is closed under direct limits. It is *of finite type* if the canonical functor from the quotient category $\operatorname{Mod} R/\mathcal{L} \to \operatorname{Mod} R$ respects direct limits. If the ring R is noetherian, then every localizing subcategory in $\operatorname{Mod} R$ is of finite type, but this is not true for general rings.

To conclude the section, we give a classification of the localizing subcategories of finite type in $\operatorname{Mod} R$ with R commutative coherent in terms of open sets of $\mathsf{Spec}^* R$ (cf. [11, p. 151]). For commutative noetherian rings, the next result is due to Hovey [6, 5.2].

Corollary 2.4. *Let the ring R be commutative coherent. The assignments*

$$\operatorname{Mod} R \supseteq \mathcal{L} \mapsto \bigcup_{M \in \mathcal{L}} \mathsf{supp}(M)$$

and

$$\mathsf{Spec}^* R \supseteq \mathcal{O} \mapsto \{\varinjlim_{\lambda} M_\lambda \mid M_\lambda \in \operatorname{mod} R, \mathsf{supp}(M_\lambda) \subseteq \mathcal{O}\}$$

induce bijections between

- ◇ *the set of all localizing subcategories of finite type in* $\operatorname{Mod} R$*,*
- ◇ *the set of all open subsets* $\mathcal{O} \subseteq \mathsf{Spec}^* R$*.*

Proof. By [3, 2.8] and [7, 2.10] there is a 1-1 correspondence between the Serre subcategories of $\operatorname{mod} R$ and the localizing subcategories of finite type in $\operatorname{Mod} R$. This correspondence is given by

$$\mathcal{S} \mapsto \vec{\mathcal{S}} := \{\varinjlim_{\lambda} M_\lambda \mid M_\lambda \in \mathcal{S}\} \quad \text{and} \quad \mathcal{L} \mapsto \mathcal{L} \cap \operatorname{mod} R.$$

Since the functor of P-localization with $P \in \mathsf{Spec}\, R$ commutes with direct limits, we see that

$$\bigcup_{M \in \mathcal{L}} \mathsf{supp}(M) = \bigcup_{M \in \mathcal{L} \cap \operatorname{mod} R} \mathsf{supp}(M).$$

Now our assertion follows from Theorem B. □

3. Proof of Theorem C

We shall write $\mathcal{L}(\mathsf{Spec}^* R)$, $\mathcal{L}(\mathsf{Inj}_{\mathrm{zg}} R)$, $\mathcal{L}_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R))$, $\mathcal{L}_{\mathrm{Serre}}(\operatorname{mod} R)$ to denote

- ◇ the lattice of all open subsets of $\mathsf{Spec}^* R$,
- ◇ the lattice of all open subsets of $\mathsf{Inj}_{\mathrm{zg}} R$ with R coherent,
- ◇ the lattice of all thick subcategories of $\mathcal{D}_{\mathrm{per}}(R)$,
- ◇ the lattice of all Serre subcategories of $\operatorname{mod} R$ with R coherent.

(A thick subcategory is a triangulated subcategory closed under direct summands.)

Given a perfect complex $X \in \mathcal{D}_{\mathrm{per}}(R)$ denote by $\mathsf{supp}(X) = \{P \in \mathsf{Spec}\, R \mid X \otimes_R^L R_P \neq 0\}$. It is easy to see that

$$\mathsf{supp}(X) = \bigcup_{n \in \mathbf{Z}} \mathsf{supp}(H_n(X)),$$

where $H_n(X)$ is the nth homology group of X.

Theorem 3.1 (Thomason [12])**.** *Let R be a commutative ring. The assignments*

$$\mathcal{T} \in \mathcal{L}_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R)) \overset{\mu}{\mapsto} \bigcup_{X \in \mathcal{T}} \mathsf{supp}(X)$$

and

$$\mathcal{O} \in \mathcal{L}(\mathsf{Spec}^* R) \overset{\nu}{\mapsto} \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid \mathsf{supp}(X) \subseteq \mathcal{O}\}$$

are mutually inverse lattice isomorphisms.

We are now in a position to prove Theorem C.

Proof of Theorem C. Since $\mathsf{Inj}_{\mathrm{zar}}\, R = (\mathsf{Inj}_{\mathrm{zg}}\, R)^*$ and $\mathsf{Inj}_{\mathrm{zg}}\, R = (\mathsf{Inj}_{\mathrm{zar}}\, R)^*$ by Lemma 1.6, it is enough to establish bijective correspondences between $\mathcal{L}(\mathsf{Inj}_{\mathrm{zg}}\, R)$, $\mathcal{L}_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R))$, and $\mathcal{L}_{\mathrm{Serre}}(\mathrm{mod}\, R)$. Consider the following diagram:

$$\begin{tikzcd} \mathcal{L}(\mathsf{Spec}^* R) \arrow[r, shift left, "\nu"] \arrow[d, shift left, "\varphi"] & \mathcal{L}_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R)) \arrow[l, shift left, "\mu"] \arrow[d, shift left, "\rho"] \\ \mathcal{L}(\mathsf{Inj}_{\mathrm{zg}}\, R) \arrow[u, shift left, "\psi"] \arrow[r, shift left, "\zeta"] & \mathcal{L}_{\mathrm{Serre}}(\mathrm{mod}\, R), \arrow[l, shift left, "\delta"] \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

where φ, ψ are as in Lemma 2.1, ζ, δ are defined in the proof of Theorem B, and μ, ν are as in Theorem 3.1. The map σ takes a Serre subcategory $\mathcal{S}$ of $\mathrm{mod}\, R$ to the thick subcategory $\sigma(\mathcal{S})$ of perfect complexes whose homology groups are in $\mathcal{S}$. The map ρ takes a thick subcategory $\mathcal{T}$ of $\mathcal{D}_{\mathrm{per}}(R)$ to the Serre subcategory $\rho(\mathcal{T}) := \sqrt{\{H_n(X) \mid X \in \mathcal{T}, n \in \mathbf{Z}\}}$. Moreover, $\nu = \mu^{-1}$ by Theorem 3.1, $\varphi = \psi^{-1}$ by Lemma 2.1, and $\zeta = \delta^{-1}$ by [3, 3.8], [7, 4.2].

By construction,

$$\sigma\zeta\varphi(\mathcal{O}) = \{X \mid \bigcup_{n \in \mathbf{Z}} \mathsf{supp}(H_n(X)) \subseteq \mathcal{O}\} = \{X \mid \mathsf{supp}(X) \subseteq \mathcal{O}\}$$

for all $\mathcal{O} \in \mathcal{L}(\mathsf{Spec}^* R)$. Thus $\sigma\zeta\varphi = \nu$. Since ζ, φ, ν are bijections, then so is σ.

On the other hand,

$$\psi\delta\rho(\mathcal{T}) = \bigcup_{X \in \mathcal{T}, n \in \mathbf{Z}} \mathsf{supp}(H_n(X)) = \bigcup_{X \in \mathcal{T}} \mathsf{supp}(X)$$

for any $\mathcal{T} \in \mathcal{L}_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R))$. We have used here the relation (2.3). One sees that $\psi\delta\rho = \mu$. Since δ, ψ, μ are bijections, then so is ρ. Obviously, $\sigma = \rho^{-1}$ and the diagram above yields the desired bijective correspondences. Theorem C is proved. $\square$

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