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CORE OF IDEALS OF NOETHERIAN LOCAL RINGS

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Abstract. The core of an ideal is the intersection of all its reductions. In 2005, Polini and Ulrich explicitly described the core as a colon ideal of a power of a single reduction and a power of I for a broader class of ideals, where I is an ideal in a local Cohen-Macaulay ring. In this paper, we show that if I is an ideal of analytic spread 1 in a Noetherian local ring with infinite residue field, then with some mild conditions on I, we have $\operatorname{core}(I) \supseteq J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I$ for any minimal reduction J of I and for $n \gg 0$.

1. Introduction

Let I be an ideal of a Noetherian ring. An ideal $J \subseteq I$ is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. The *core* of I, denoted by $\operatorname{core}(I)$, is defined to be the intersection of all reductions of I. This object was introduced by D. Rees and J. Sally [7] and later studied by C. Huneke and I. Swanson [3], who determined the core of integrally closed ideals in two dimensional regular local rings. Recently, Corso, Polini and Ulrich [1], [2] gave explicit descriptions for the core of certain ideals of Cohen-Macaulay local rings.

The purpose of this paper is to study the following conjecture raised by Corso, Polini and Ulrich in [2]:

Conjecture. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring with infinite residue field. Let $I \subseteq R$ be an ideal of analytic spread $l \geq 1$ that satisfies G_l and is weakly $(l-1)$-residually S_2. Let J be a minimal reduction of I and let r denote the reduction number of I with respect to J. Then

$$\operatorname{core}(I) = J(J^r : I^r) = I(J^r : I^r) = J^{r+1} : I^r.$$

This conjecture has been addressed and solved by several authors: In [5], it is shown by Hyry and Smith that the core of I is equal to $J^{n+1} : I^n$ for $n \gg 0$ provided that I is equi-multiple, R contains the rational numbers, and $R[It]$ is Cohen-Macaulay. Subsequently (at the same time as [6]) in [4], Huneke and Trung verified the conjecture for equi-multiple ideals in Cohen-Macaulay local rings of equi-characteristic zero without assuming the Cohen-Macaulayness of the Rees algebra. Independently, Polini and Ulrich in [6] established the formula $\operatorname{core}(I) = J^{r+1} : I^r$ for a broader class of ideals, which includes equi-multiple ideals as a

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special case. Here, we solve the case when the ideal has height at most one. Indeed we are able to show the following:

Theorem 1.1. (i) *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field. Let $I \subseteq R$ be an ideal of analytic spread 1. Suppose that for every minimal reduction J of I and for every positive integer n, $(0) : J = (0) : I^n$ and $((0) : I^n) \cap I = (0)$. Then*

$$\operatorname{core}(I) \supseteq J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I$$

for any minimal reduction J of I and for $n \geq r_J(I)$.

(ii) *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with infinite residue field of characteristic 0. Let $I \subseteq R$ be an ideal of analytic spread 1. Suppose that I satisfies G_1. Then*

$$\operatorname{core}(I) = J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I \subseteq J^{n+1} : I^n$$

for any minimal reduction J of I and for $n \geq r_J(I)$.

Notice that if the ideal I has height one, our result is covered by [4] and [6]. On the other hand, part (i) of the theorem above is not covered in any previously mentioned literature as the ideal we considered might have height zero.

2. Core of ideals of analytic spread one

Let R be a Noetherian ring and I be an ideal of R of height g. Recall that I satisfies the condition G_s if for each prime ideal P containing I with $\dim R_P \leq s-1$, the minimal number of generators $\mu(I_P)$ is at most $\dim R_P$. Recall also that if R is local with infinite residue field, then the minimal number of generators of every minimal reduction of I is the analytic spread of I. Given a reduction J of I, we write $r_J(I)$ for the least integer $n \geq 0$ such that $I^{n+1} = JI^n$.

We begin this section with some useful lemmas.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring. If $I \subseteq R$ is an ideal satisfying G_1, then $((0) : I^n) \cap I = (0)$ and $(0) : J = (0) : I^n$ for every minimal reduction J of I and for every positive integer n.*

Proof. Let

$$q_1 \cap \cdots \cap q_n$$

be the minimal primary decomposition of (0) and $\sqrt{q_i} = P_i$; then P_i is minimal for every i. We may assume that $I \subseteq P_i$ for $i \leq t$ and $I \nsubseteq P_i$ for $i > t$. Since I satisfies G_1, $I_{P_i} = 0$ for $i \leq t$, so that $I \subseteq q_i$ for $i \leq t$, it follows that $(0) : I^n = \bigcap_{i>t} q_i$ for every positive integer n. Therefore, $((0) : I^n) \cap I \subseteq q_1 \cap \cdots \cap q_n = (0)$. Moreover, if $J = (x)$ is a minimal reduction of I, then $x \in q_i$ for $i \leq t$ and $x \notin P_i$ for $i > t$ so that $(0) : J = \bigcap_{i>t} q_i = (0) : I^n$ for every positive integer n. □

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with infinite residue field. Let $I \subseteq R$ be an ideal of analytic spread 1. Suppose that I satisfies G_1. Then $J^n : I^m$ is an $\mathfrak{m}$ primary ideal for any minimal reduction J of I and for any positive integers m, n.*

Proof. Let P be a minimal prime of R and J be a minimal reduction of I. Assume that $J^n : I^m \subseteq P$. It is clear that $V(J^n : I^m) \subseteq V(I)$ as J is a reduction of I. Therefore, $I \subseteq P$. Moreover, as I satisfies G_1, it clearly follows that $ht(J^n : I^m) \geq 1$, a contradiction. Thus, we conclude that $J^n : I^m$ is an $\mathfrak{m}$ primary ideal. □

From now on, let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field. Let I be an ideal of R of analytic spread 1 such that $((0) : I) \cap I = (0)$ and $(0) : J = (0) : I$ for every minimal reduction J of I.

Lemma 2.3. *Let $J = (x)$ be a minimal reduction of I. Then $(x^{n+1}) : I^n = x((x^n) : I^n) + (0) : I$ for $n \geq 1$.*

Proof. Since $(0) : I = (0) : I^n$ for every positive integer n and $x((x^n) : I^n) + (0) : I^n \subseteq (x^{n+1}) : I^n$, it is enough to show that $(x^{n+1}) : I^n \subseteq x((x^n) : I^n) + (0) : I$. For this, let $a \in (x^{n+1}) : I^n$; then $a \in (x^{n+1}) : (x^n)$, so that $(a - bx)x^n = 0$ for some $b \in R$. It follows that $(a - bx)x \in ((0) : I^{n-1}) \cap I = (0)$, which implies that $a \equiv bx \ (mod\ (0) : I)$ as $(0) : x = (0) : I$. Now, it remains to show that $b \in (x^n) : I^n$. For this, let $u \in I^n$; then $bux = au \in (x^{n+1})$, and it follows that $bu \in (x^n)$ as $((0) : I) \cap I = (0)$. □

Lemma 2.4. *Let $J_1 = (x)$ and $J_2 = (y)$ be any two minimal reductions of I. Then*

$$(x^n) : I^n = (y^n) : I^n$$

and

$$(x^{n+1}) : I^n = (y^{n+1}) : I^n$$

for every $n \geq \max\{r_{J_1}(I), r_{J_2}(I)\}$.

Proof. It is enough to show that $(x^n) : I^n \subseteq (y^n) : I^n$. Let $a \in (x^n) : I^n$ and $u = z^n \in I^n$ for some minimal reduction (z) of I; then $au = bx^n$ for some element $b \in R$. We claim that $b \in (u) : I^n$. To see the claim, let $v \in I^n$; then $bvx^n = avu = cx^n u$ for some $c \in R$ by the choice of a. Since x is a minimal reduction of I, $((0) : (x^n)) \cap I = (0)$, $bv \in (u)$. The claim follows. To complete the proof, let $w \in I^n$; then $awu = bx^n w \in bI^{2n} = by^n I^n$ as $n \geq r_{J_2}(I)$. Therefore there is an element $v \in I^n$ such that $awu = bvy^n$. Now by the claim, $bv = cu$ for some $c \in R$. Since $u = z^n$ and (z) is a minimal reduction of I, $((0) : (u)) \cap I = (0)$, $aw = cy^n \in (y^n)$. □

Corollary 2.5. (i) *Let J be a minimal reduction of I with reduction number r. Then $J^r : I^r = J^n : I^n$ and $J^{r+1} : I^r = J^{n+1} : I^n$ for all $n \geq r$.*

(ii) *Let $J_1 = (x)$ and $J_2 = (y)$ be any two minimal reductions of I. Then*

$$(x^r) : I^r = (y^t) : I^t$$

and

$$(x^{r+1}) : I^r = (y^{t+1}) : I^t,$$

where $r = r_{J_1}(I)$ and $t = r_{J_2}(I)$.

Proof. (i) Let $J = (x)$. The equality follows from the facts that $I^n = x^{n-r} I^r$ and $((0) : (x^{n-r})) \cap I = (0)$.

(ii) We may assume that $r > t$. By Lemma 2.4 and (i), $(x^r) : I^r = (y^r) : I^r = (y^t) : I^t$. □

Corollary 2.6. *Let J be a minimal reduction of I with reduction number r and let $K = J^r : I^r$. Then the following hold.*

(i) *Let $J_1 = (x)$ and $J_2 = (y)$ be any two minimal reductions of I; then $Kx^n = Ky^n$ for every positive integer n.*

(ii) *$(0) : I^n \subseteq K$ for every n.*

(iii)
$$J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I \subseteq \operatorname{core}(I)$$
for $n \geq r$.

Proof. (i) Let $n \geq \max\{r_{(x)}(I), r_{(y)}(I)\}$; then $K = (x^n y^{n+1}) : I^{2n+1} = (x^{n+1} y^n) : I^{2n+1}$, and it follows by Lemma 2.3 that

$$\begin{aligned} Kx + (0) : I &= x(x^n y^{n+1}) : I^{2n+1} + (0) : I \\ &= (x^{n+1} y^{n+1}) : I^{2n+1} \\ &= y(x^{n+1} y^n) : I^{2n+1} + (0) : I \\ &= Ky + (0) : I. \end{aligned}$$

Therefore $Kx = (Kx + (0) : I) \cap I = (Ky + (0) : I) \cap I = Ky$. Consequently, $Kx^n = Kx^{n-1}y = \cdots = Kxy^{n-1} = Ky^n$.

(ii) It is easy to see that $(0) : I^n = (0) : I \subseteq (x^r) : I^r = K$.

(iii) $J(J^n : I^n) \subseteq \operatorname{core}(I)$ follows from (i) and Corollary 2.5(i). $J(J^n : I^n) = I(J^n : I^n)$ follows from (i) and the fact that I can be generated by its minimal reductions. Finally, $J(J^n : I^n) = (J^{n+1} : I^n) \cap I$ follows from Lemma 2.3 and the fact that $((0) : I) \cap I = (0)$. □

The following lemma and its proof can also be seen in [2] and [6], which are inspired by [3].

Lemma 2.7. *Let* (y) *and* (z) *be two minimal reductions of* I. *Let* $K = (y^n) : I^n$ *with* $n \gg 0$. *Let* $l > \dim_{R/\mathfrak{m}}(K : \mathfrak{m}/K)$ *and let* $\lambda_1, \ldots, \lambda_l$ *be units in* R *that are not all congruent modulo* $\mathfrak{m}$. *Then*

$$(K : \mathfrak{m})y \cap (K : \mathfrak{m})(y + \lambda_1 z) \cap \cdots \cap (K : \mathfrak{m})(y + \lambda_l z) \subset [(K : \mathfrak{m}) \cap ((y) : (z))]y.$$

Proof. Let α be an element of the intersection on the left hand side. Write

$$\alpha = ys = (y + \lambda_1 z)s_1 = \cdots = (y + \lambda_l z)s_l,$$

where s and all s_i belong to $K : \mathfrak{m}$. It is enough to show that $s \in (y) : (z)$. For this, observe that $s_i \in (y) : (z)$ for every i as λ_i is a unit. If $s_i \in K$ for some i, then

$$ys = (y + \lambda_i z)s_i = yK + zK = yK \subseteq y((y) : (z))$$

as $zK = yK \subseteq (y)$ implies $K \subseteq (y) : (z)$. It follows that $s - t \in (0) : (y)$ for some $t \in (y) : (z)$. However $(0) : (y) = (0) : I = (0) : (z) \subseteq (y) : (z)$, $s \in (y) : (z)$. So, we may assume that $s_i \notin K$ for every i.

Let '$-$' denote the image in $\bar{R} = R/K$. Now, $\bar{s}_1, \ldots, \bar{s}_l$, or equivalently $\bar{\lambda}_1\bar{s}_1, \ldots, \bar{\lambda}_l\bar{s}_l$ are l nonzero elements of the $R/\mathfrak{m}$-vector space $\overline{K : \mathfrak{m}}$. Let P_c be the property that there are units $\lambda_1, \ldots, \lambda_c$ that are not all congruent modulo $\mathfrak{m}$ and $\{t_1, \ldots, t_c\} \subseteq \{s_1, \ldots, s_l\}$ such that $\{\bar{\lambda}_1\bar{t}_1, \ldots, \bar{\lambda}_c\bar{t}_c\}$ are linearly dependent over $R/\mathfrak{m}$. Since R has P_l, we can find a smallest integer $c \geq 2$ so that R has P_c but R has no P_{c-1}. Without loss of generality, we may assume $c = l$. Therefore there are elements $\delta_1, \ldots, \delta_l$ in R not all in $\mathfrak{m}$ such that $\sum_{i=1}^l \bar{\delta}_i\bar{\lambda}_i\bar{s}_i = \bar{0}$. Notice that $\sum_{i=1}^l \bar{\delta}_i\bar{s}_i \neq \bar{0}$ by the minimality of l. Since $\sum_{i=1}^l \delta_i\lambda_i s_i \in K$ and $yK = zK$, $z \cdot (\sum_{i=1}^l \delta_i\lambda_i s_i) = yt$ for some $t \in K$. Setting $\delta = \sum_{i=1}^l \delta_i$ and multiplying both sides by α, we obtain

$$\delta sy = \delta\alpha = \sum_{i=1}^l \delta_i\alpha = \sum_{i=1}^l \delta_i(y + \lambda_i z)s_i = \sum_{i=1}^l \delta_i y s_i + yt \subseteq y((y) : (z))$$

as $K \subseteq (y) : (z)$. It is clear that δ is a unit, and therefore $s \in (y) : (z) + (0) : (y) = (y) : (z) + (0) : (z) = (y) : (z)$. $\square$

The following lemma as well as its proof are inspired by [6].

Lemma 2.8. *Let (y) and (z) be two minimal reductions of I. Assume that there are s, s' and $n \geq 1$ such that $s \in ((y) : (z)) \cap \cdots \cap ((y^{n-1}) : (z^{n-1}))$ and $sy = s'(z + \lambda y)$, where λ is a unit. Then $s' \in ((y) : (z)) \cap \cdots \cap ((y^n) : (z^n))$.*

Proof. We prove the lemma by induction on n. The case $n = 1$ is clear. Let $n \geq 2$. By assumption, there is an element $t \in R$ such that $sz^{n-1} = ty^{n-1}$. Therefore, $ty^n = sz^{n-1}y = s'z^{n-1}(z + \lambda y)$. By induction, $s' \in (y^{n-1}) : (z^{n-1})$; hence there is an element t' such that $s'z^{n-1} = t'y^{n-1}$. It follows that $ty = t'(z + \lambda y)$ and $t' \in (y) : (z)$. Therefore $s' \in (y^n) : (z^n)$. $\square$

Lemma 2.9. *Let (x) and (y) be two minimal reductions of I. Let $K = (x^n) : I^n$ for $n \gg 0$. If $ax^n = by^n$ for some elements $a, b \in R$, then $a \in K : \mathfrak{m}$ if and only if $b \in K : \mathfrak{m}$.*

Proof. Let $a \in K : \mathfrak{m}$; then $by^n\mathfrak{m} \subseteq Kx^n = Ky^n$ by Corollary 2.6(i), so that $b\mathfrak{m} \subseteq K + (0) : (y^n) = K + (0) : I^n = K$. It follows that $b \in K : \mathfrak{m}$. $\square$

Lemma 2.10. *Let (x) be a minimal reduction of I with reduction number r. Let $z_1, \ldots, z_k \in I$ such that z_i generates a minimal reduction of I for every i and $I = (z_1, \ldots, z_k)$. Then*

$$((x^r) : (z_1^r))x \cap \cdots \cap ((x^r) : (z_k^r))x = ((x^r) : I^r)x.$$

Proof. It is enough to show that $((x^r) : (z_1^r))x \cap ((x^r) : (z_2^r))x \subseteq [(x^r) : (z_1^r, z_2^r)]x$. For this, let $\alpha \in ((x^r) : (z_1^r))x \cap ((x^r) : (z_2^r))x$. Write $\alpha = xu = xu'$, where $u \in (x^r) : (z_1^r)$ and $u' \in (x^r) : (z_2^r)$. We need to show that $u \in (x^r) : (z_2^r)$. Notice that $u - u' \in (0) : (x) = (0) : I$. Hence $uz_2^r - u'z_2^r \in ((0) : I) \cap I = (0)$. Therefore $u \in (x^r) : (z_2^r)$ and $ux \in [(x^r) : (z_1^r, z_2^r)]x$. $\square$

As we mentioned in the introduction, there are some known results in [3], [4], [5] and [6] that are related to the conjecture. However, our main result is different from those as we solve the conjecture when the ideal has height at most one, which is not covered in the previous literature. In fact, a modification of [3] gives the proof of the following.

Theorem 2.11. (i) *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field. Let $I \subseteq R$ be an ideal of analytic spread 1. Suppose that for every minimal reduction J of I and for every positive integer n, $(0) : J = (0) : I^n$ and $((0) : I^n) \cap I = (0)$. Then*

$$\operatorname{core}(I) \supseteq J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I$$

for any minimal reduction J of I and for $n \geq r_J(I)$.

(ii) *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring with infinite residue field of characteristic 0. Let $I \subseteq R$ be an ideal of analytic spread 1. Suppose that I satisfies G_1. Then*

$$\operatorname{core}(I) = J(J^n : I^n) = I(J^n : I^n) = (J^{n+1} : I^n) \cap I \subseteq J^{n+1} : I^n$$

for any minimal reduction J of I and for $n \geq r_J(I)$.

Proof. (i) follows from Corollary 2.6(iii).

(ii) Let (x) be a minimal reduction of I and $r = r_{(x)}(I) = r(I)$. Let $K = (x^r) : I^r$ and $K' = (x^{r+1}) : I^r$. To show the theorem, it suffices to show that $(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq K'$ by Lemma 2.2 and [3, Lemma 3.8]. We claim that for any two minimal reductions (y) and (z) of I,

$$(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq [(K : m) \cap (\bigcap_{i=1}^{j}((y^i) : (z^i)))]y \subseteq ((y^j) : (z^j))y$$

for every $j \geq 0$. Set $y = x$ and $j = r$ in the claim. Since I^r can be generated by the set $\{z^r \mid z \in I\}$, we see that $(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq Kx \subseteq K'$ by Lemma 2.10.

Since $(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq (K' : \mathfrak{m}) \cap (y) = (Kx : y\mathfrak{m})y = (Ky : y\mathfrak{m})y \subseteq ((K + (0) : y) : \mathfrak{m})y = (K : \mathfrak{m})y$, the claim holds for $j = 0$.

Let $l > \dim(K : \mathfrak{m})/K$. Since the residue field of R is infinite, there are units $\lambda_1, \ldots, \lambda_l$ in R that are not all congruent modulo $\mathfrak{m}$ such that $z + \lambda_i y$ is a minimal reduction of I for every i. Notice that

$$(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq [\bigcap_i (K : \mathfrak{m})(z + \lambda_i y)] \cap (K : \mathfrak{m})y$$

from the above. By Lemma 2.7,

$$(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq [(K : \mathfrak{m}) \cap ((y) : (z))]y.$$

Suppose that we have shown that

$$(K' : \mathfrak{m}) \cap \operatorname{core}(I) \subseteq [(K : m) \cap (\bigcap_{i=1}^{j}((y^i) : (z^i)))]y$$

for some $j \geq 1$. Choose units $\lambda_1, \ldots, \lambda_l$ in R such that $z+\lambda_i y$ is a minimal reduction of I for every i and that λ_i are not all congruent modulo $\mathfrak{m}$. Let

$$sy \in [(K : m) \cap (\bigcap_{i=1}^{j}((y^i) : (z^i)))]y \cap [\bigcap_i (K : m)(z + \lambda_i y)]$$

for some $s \in (K : m) \cap (\bigcap_{i=1}^{j}((y^i) : (z^i)))$. Then there are $s_i \in K : \mathfrak{m}$ such that $sy = s_i(z + \lambda_i y)$ for every i. Note that $s \in ((y) : (z)) \cap \cdots \cap ((y^{j-1}) : (z^{j-1}))$. Therefore by Lemma 2.8, $s_i \in (y^j) : (z^j)$ for every i. Moreover, since $s \in (y^j) : (z^j)$, $sz^j = ty^j$ for some $t \in R$. Therefore $ty^{j+1} = syz^j = s_i z^j(z + \lambda_i y) = t_i y^j(z + \lambda_i y)$ for some $t_i \in R$, and it follows that $ty = t_i(z + \lambda_i y)$ for every i. Since $t, t_i \in K : \mathfrak{m}$ by Lemma 2.9, we obtain by Lemma 2.7 that $t \in (y) : (z)$; it follows that $s \in (y^{j+1}) : (z^{j+1})$. The proof is now complete. □

In general the formula $\operatorname{core}(I) = J(J^r : I^r)$ in the conjecture stated in the introduction does not hold even if $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring and I is an equi-multiple $\mathfrak{m}$ primary ideal of R. We provide an example in the following to show this fact. More examples can be found in [6].

Example 2.12. Let

$$R = \frac{k[|x, y, z, u, v, w|]}{(uz - xw, vz - yw, z^3, z^2w, (u, v, w)^2)},$$

where k is a field containing $\mathbb{Q}$. Then $\{x, y\}$ is a regular sequence of R and $(R, \mathfrak{m})$ is a 2-dimensional Cohen-Macaulay local ring, where $\mathfrak{m} = (x, y, z, u, v, w)R$. Moreover, $J = (x, y)R$ is a minimal reduction of $\mathfrak{m}$ with $\mathfrak{m}^3 = (x, y)\mathfrak{m}^2$. Now, by [4, Theorem 3.7], $\operatorname{core}(\mathfrak{m}) = J^3 : \mathfrak{m}^2$. However, it is easy to see that $J^3 : \mathfrak{m}^2 \neq J(J^2 : \mathfrak{m}^2)$ as $xv - yu \in J^3 : \mathfrak{m}^2$ but $xv - yu \notin J(J^2 : \mathfrak{m}^2)$.

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