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REMARKS ON NAIMARK'S DUALITY

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Abstract. We present an extension of a version of Naimark's dilation theorem which states that complete systems in a Hilbert space are projections of ω-linearly independent systems of elements of an ambient Hilbert space. This result is presented in the context of other known extensions of Naimark's theorem.

1. Introduction

The following result is known as the *Naimark's dilation theorem*; see, e.g., [17], [18], [1], [11].

Theorem 1.1 (Naimark's dilation theorem)**.** *Let $E(\lambda)$, $\lambda \in \mathbb{R}$, be a one-parameter family of bounded and self-adjoint operators on a Hilbert space $\mathbb{H}$, which satisfies the following conditions:*

(1) *$E(\lambda_2) - E(\lambda_1)$, $\lambda_2 > \lambda_1$, is a bounded positive operator on $\mathbb{H}$,*
(2) *$E(\lambda^-) = E(\lambda)$,*
(3) *$\lim_{\lambda \to -\infty} E(\lambda) = 0$,*
(4) *$\lim_{\lambda \to -\infty} E(\lambda) = Id_{\mathbb{H}}$.*

Then, there exists a Hilbert space $\mathbb{H}'$, containing $\mathbb{H}$ as a subspace, and an orthogonal resolution of the identity $F(\lambda)$, $\lambda \in \mathbb{R}$, for the space $\mathbb{H}'$, such that for all $f \in \mathbb{H}$:

$$E(\lambda)(f) = PF(\lambda)(f),$$

where P is the operator of projection onto $\mathbb{H}$.

The next theorem can be considered as a special case of *Naimark's dilation theorem.*

Theorem 1.2. *Any tight frame in a Hilbert space $\mathbb{H}$ is an orthogonal projection of an orthonormal basis of an ambient Hilbert space $\mathbb{H}'$, $\mathbb{H} \subset \mathbb{H}'$.*

Theorem 1.2 was stated in [12], where it was generalized to hold for arbitrary frames and Riesz bases, and the proof was straightforward and did not rely on the approach of Naimark. This generalization was independently discovered by others; see, e.g., [14]. On the other hand, it has been observed and used before that orthogonal projections of orthonormal bases are tight frames; see, e.g., [2].

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The results of [12] were extended by Casazza, Han, and Larson [8] to hold for an appropriate notion of Banach frames. Moreover, Theorem 7.6 in [8] shows that the property of a frame being a projection of a Riesz basis is directly connected to the notions of excesses and deficits of frames; see [3] and [4] for more on this subject. For remarks on the history of Theorem 1.2 we refer the reader to [9], [16] and [15], and to references therein.

In his recent work, Terekhin [20] showed a further extension of Theorem 1.2, which holds for representation systems and for Schauder bases. Terekhin's work is also more general in the sense that his results are stated for Banach spaces rather than Hilbert spaces.

In the main result of this paper, Theorem 2.1, we present the above results combined with our own extension. This way we obtain yet another *duality principle* in the theory of representation systems. This principle can be compared to a duality theory which holds for Gabor systems, i.e., systems generated by time-frequency shifts of a single function; see, e.g., [19]. We also refer the interested reader to a recent survey of analogous results in frame theory [10].

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2. Definitions and main result

Let $\mathbb{H}$ be a separable, infinite-dimensional Hilbert space over $\mathbb{C}$. We say that a collection $\{f_k : k = 1, \ldots\} \subset \mathbb{H}$ of vectors is a *frame* for $\mathbb{H}$, with the *frame bounds* A and B, if

$$\forall\, f \in \mathbb{H}, \quad A\|f\|_{\mathbb{H}}^2 \leq \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 \leq B\|f\|_{\mathbb{H}}^2.$$

We say that A and B are the lower and the upper frame bounds, respectively. A frame is *tight* if $A = B$. A system that satisfies only the upper inequality in the above estimate is called a *Bessel system*. We say that a frame is *exact* if it is no longer a frame after removal of any of its elements. Exact frames are also called *Riesz bases*. An equivalent characterization of Riesz bases states that a collection $\{f_k : k = 1, \ldots\} \subset \mathbb{H}$ of vectors is a Riesz basis if it is complete (i.e., $\overline{\operatorname{span}\{f_k : k = 1, \ldots\}} = \mathbb{H}$) and if there exist constants $0 < A \leq B < \infty$ such that

$$\forall\, c \in l^2(\mathbb{N}), \quad A\|c\|_2^2 \leq \Big\| \sum_{k\in\mathbb{N}} c(k) f_k \Big\|_2^2 \leq B\|c\|_2^2. \tag{1}$$

When only the second inequality in the above holds, we say that the system $\{f_k : k = 1, \ldots\} \subset \mathbb{H}$ satisfies the *upper Riesz inequality*. Such collections are in particular Bessel systems.

A *representation system* $\{f_k : k = 1, \ldots\} \subset \mathbb{H}$ is a system such that for any $f \in \mathbb{H}$ there exists a sequence $c = \{c(k) : k = 1, \ldots\} \subset \mathbb{C}$ such that the series

$$\sum_{k=1}^{\infty} c(k) f_k \tag{2}$$

converges to f in the norm of $\mathbb{H}$. The space of sequences for which the above series converges is called the *coefficient space* of the representation system

$\{f_k : k = 1, \ldots\}$. The coefficient space is a Banach space with norm

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} c(k) f_k \right\|_{\mathbb{H}}.$$

A representation system with the property that for each $f \in \mathbb{H}$ there exists a unique number sequence $c = \{c(k) : k = 1, \ldots\}$ such that (2) converges to f in $\mathbb{H}$ is called a *Schauder basis*. We define the *coefficient space of zero-series* to be the closed subspace N of the coefficient space, consisting of those sequences for which the expansion (2) converges to $0 \in \mathbb{H}$. Finally, a collection $\{f_k : k = 1, \ldots\} \subset \mathbb{H}$ is *ω-linearly independent for $l^2(\mathbb{N})$* if the fact that there exists $c \in l^2(\mathbb{N})$ such that $\sum_{k\in\mathbb{N}} c_k f_k = 0$ implies that $c = 0$.

In Theorem 2.1 we use the following notation: given two Hilbert spaces $\mathbb{H} \subset \mathbb{H}'$, let P denote the orthogonal projection of $\mathbb{H}'$ onto $\mathbb{H}$.

Theorem 2.1. *Let $\mathbb{H}$ be a separable Hilbert space.*

a. *The collection $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is a tight frame for $\mathbb{H}$ if and only if there exists a Hilbert space $\mathbb{H}'$ containing $\mathbb{H}$ and an orthogonal basis $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ for $\mathbb{H}'$ such that*

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

b. *The collection $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is a frame for $\mathbb{H}$ with frame constants $A \leq B$ if and only if there exists a Hilbert space $\mathbb{H}'$ containing $\mathbb{H}$ and a Riesz basis $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ for $\mathbb{H}'$ with the same constants $A \leq B$ such that*

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

c. *Assume that the space of zero-series for $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is complemented in the coefficient space of $\{f_j : j \in \mathbb{N}\}$. Then, the collection $\{f_j : j \in \mathbb{N}\}$ is a representation system in $\mathbb{H}$ if and only if there exists a Hilbert space $\mathbb{H}'$ containing $\mathbb{H}$ and a Schauder basis $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ for $\mathbb{H}'$ such that*

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

d. *Assume that the collection $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is a Bessel system for $\mathbb{H}$. Then, $\{f_j : j \in \mathbb{N}\}$ is complete in $\mathbb{H}$ if and only if there exists a Hilbert space $\mathbb{H}'$ containing $\mathbb{H}$ and a complete, ω-linearly independent for $l^2(\mathbb{N})$ system $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ for $\mathbb{H}'$ such that*

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

e. *The collection $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is a Bessel system for $\mathbb{H}$ with constant B if and only if there exists a Hilbert space $\mathbb{H}'$ containing $\mathbb{H}$ and a collection $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ satisfying the upper Riesz inequality with the same constant and such that*

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

Proof. ***a.*** This is, e.g., Proposition 1.1 in [12].

b. This statement follows, e.g., from Proposition 1.6 in [12]; see also Theorem 7.6 in [8] and Theorem 1 in [14].

c. This is the result of Terekhin, [20], stated in the context of Hilbert spaces.

d. ($\Longleftarrow$) Let $\mathbb{H} \subset \mathbb{H}'$, let $\{e_j : j \in \mathbb{N}\} \subset \mathbb{H}'$ be a complete system for $\mathbb{H}'$, and let $f_j = P(e_j)$, $j \in \mathbb{N}$. Then, the collection $\{f_j : j \in \mathbb{N}\} \subset \mathbb{H}$ is complete in $\mathbb{H}$.

($\Longrightarrow$) As it was observed in [14], it is enough to assume, without loss of generality, that $\mathbb{H} = l^2(\mathbb{N})$. Consider an infinite-dimensional matrix for which its jth column

is defined to be the sequence of coefficients of $f_j \in \mathbb{H}$, $j \in \mathbb{N}$. Let v_i, $i \in \mathbb{N}$, denote the rows of this matrix, i.e.,

$$\forall\, i, j \in \mathbb{N}, \quad f_j(i) = v_i(j).$$

With this definition, the assumption that $\{f_j : j \in \mathbb{N}\}$ forms a Bessel system for $\mathbb{H}$ implies that

$$\forall\, i \in \mathbb{N}, \quad v_i \in l^2(\mathbb{N}),$$

and, moreover, the system $\{v_i : i \in \mathbb{N}\}$ satisfies

$$\forall\, c \in l^2(\mathbb{N}), \quad \Big\| \sum_{i \in \mathbb{N}} c(i) v_i \Big\|_2^2 \le B \|c\|_2^2.$$

We first observe that $\{v_i : i \in \mathbb{N}\} \subset l^2(\mathbb{N})$ is ω-linearly independent for $l^2(\mathbb{N})$-sequences. Indeed, the fact that $\sum_{i \in \mathbb{N}} c(i) v_i = 0$ for some $c \in l^2(\mathbb{N})$ is equivalent to

$$\forall\, j \in \mathbb{N}, \quad \langle c, f_j \rangle = 0.$$

Thus, the completeness of $\{f_j : j \in \mathbb{N}\}$ implies that $c = 0$. (This fact can also be derived from Proposition 4.2 in [10], once we observe that $\{v_i : i \in \mathbb{N}\}$ is the R-dual sequence, which corresponds to $\{f_j : j \in \mathbb{N}\}$.)

Let V denote the closure in $l^2(\mathbb{N})$ of the linear span of $\{v_i : i \in \mathbb{N}\}$. Let $V^\perp$ be the orthogonal complement of V in $l^2(\mathbb{N})$ and let $\{w_k : k \in K\}$ be an orthonormal basis of $V^\perp$, where K denotes some set of indices (countable, finite, or empty, depending on the dimension of $V^\perp$), $K \cap \mathbb{N} = \emptyset$. It is not difficult to see that the collection $\{v_i : i \in \mathbb{N}\} \cup \{w_k : k \in K\} \subset l^2(\mathbb{N})$ is ω-linearly independent for $l^2(\mathbb{N})$-sequences.

Consider now a matrix whose rows are the sequences of coefficients of vectors $\{v_i : i \in \mathbb{N}\}$ and $\{w_k : k \in K\}$, where the index set $L = \mathbb{N} \cup K$ of rows is endowed with some fixed order. Moreover, choose $\mathbb{H}'$ to be the space $l^2(L)$.

Let $e_j \in l^2(L) = \mathbb{H}'$ be the jth column of this matrix, $j \in \mathbb{N}$. Clearly,

$$\forall\, j \in \mathbb{N}, \quad P(e_j) = f_j.$$

Since the row vectors $\{v_i : i \in \mathbb{N}\} \cup \{w_k : k \in K\} \subset l^2(\mathbb{N})$ are ω-linearly independent for $l^2(\mathbb{N})$-sequences, it follows that the collection $\{e_j : j \in \mathbb{N}\}$ is complete in $l^2(L) = \mathbb{H}'$.

Finally we need to show that $\{e_j : j \in \mathbb{N}\}$ is ω-linearly independent for $l^2(\mathbb{N})$. In order to do so, let $c \in l^2(\mathbb{N})$ be such that

$$\sum_{j \in \mathbb{N}} c(j) e_j = 0.$$

This implies that

$$\forall\, i \in \mathbb{N}, \quad \langle c, v_i \rangle = 0 \tag{3}$$

and

$$\forall\, k \in K, \quad \langle c, w_k \rangle = 0. \tag{4}$$

Write $c = c_V + c_{V^\perp}$, where $c_V \in V$ and $c_{V^\perp} \in V^\perp$. Then, since $\{w_k : k \in K\}$ is an orthonormal basis in $V^\perp$, (4) implies that $c_{V^\perp} = 0$. Thus, (3) may be rewritten as:

$$\forall\, i \in \mathbb{N}, \quad \langle c_V, v_i \rangle = 0.$$

By the definition of the space V, $\{v_i : i \in \mathbb{N}\}$ is complete in V, and so $c_V = 0$, as well.

e. This part follows trivially from the fact that a system $\{f_j : j \in \mathbb{N}\}$ satisfies the upper Riesz inequality if and only if it is a Bessel system; see the works of N. Bari [5], [6], cf., [7]. Compare this result with Theorem 9.8 in [13]. □

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