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ALL-TIME MORSE DECOMPOSITIONS OF LINEAR NONAUTONOMOUS DYNAMICAL SYSTEMS

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ABSTRACT. Morse decompositions provide inside information about the global asymptotic behavior of dynamical systems on compact metric spaces. Recently, the existence of Morse decompositions for *nonautonomous* dynamical systems was proved by restricting attention to the past or the future of the system, but in general, such a construction is not realizable for the entire time. In this article, it is shown that all-time Morse decompositions can be defined for linear systems on the projective space. Moreover, the dynamical properties are discussed and an analogue to the Theorem of Selgrade is proved.

1. INTRODUCTION

In his famous article *Isolated Invariant Sets and the Morse Index* [3], Charles C. Conley introduced Morse decompositions in order to describe the asymptotic behavior of dynamical systems acting on a compact phase space. This notion has far reaching implications and inspired many authors for future research. Recently, the existence of Morse decompositions for nonautonomous dynamical systems was proved in [7] by restricting attention to the past of the system. Via time reversal, an analogous construction is possible for the future. An example in [7], however, showed that, in general, it is not possible to construct Morse decompositions for the whole time. Nevertheless, it is shown in this article that all-time Morse decompositions can be defined for linear systems on the projective space.

The importance of Morse decompositions is due to the fact that they provide important information about the long-term behavior of dynamical systems. In fact, each state of the system converges in forward as well as in backward time to some Morse set, which is a component of the Morse decomposition. Thus, a Morse set has both attractive and repulsive properties, and suitable notions of attractor and repeller are needed for the definition of Morse decompositions. In Section 2 of this paper, new definitions of attractivity and repulsivity are introduced for nonautonomous dynamical systems which capture the local dynamical behavior of nonautonomous sets for the whole time. In Section 3, the construction of attractor-repeller pairs is explained, and Section 4 is devoted to the introduction

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of all-time Morse decompositions and the discussion of their basic dynamical properties. Finally, in Section 5, an analogue to the Theorem of Selgrade concerning the existence of a finest Morse decomposition is proved.

We close this introduction by pointing out the relationship of all-time Morse decompositions to the past and future Morse decompositions discussed in [7]. First, observe that the notions of all-time attractivity and repulsivity are stronger than the concepts for the past and the future of the system. This means that an all-time Morse decomposition is both a past and future Morse decomposition. We will use this fact repeatedly in this paper. In contrast to the notions of all-time attractivity and repulsivity, however, there is a lack of symmetry between attractivity and repulsivity in the past and future case; for instance, a past attractor is locally unique, but not a past repeller, and there is a formalism for the construction of a past attractor from a past repeller, but not vice versa.

2. Preliminaries

We denote by $\mathbb{R}$ the set containing all reals and by $\mathbb{R}^{N\times N}$ the set of all real $N\times N$ matrices. Given a metric space (X,d), we write $U_\varepsilon(x_0) = \big\{x \in X : d(x,x_0) < \varepsilon\big\}$ for the ε-neighborhood of a point $x_0 \in X$. For arbitrary nonempty sets $A, B \subset X$ and $x \in X$, let $d(x,A) := \inf\big\{d(x,y) : y \in A\big\}$ be the *distance* of x to A and let $d(A|B) := \sup\big\{d(x,B) : x \in A\big\}$ be the *Hausdorff semi-distance* of A and B.

The Euclidian space $\mathbb{R}^N$ is equipped with the Euclidian norm $\|\cdot\|$, which is induced by the scalar product $\langle\cdot,\cdot\rangle$, defined by $\langle x,y\rangle := \sum_{i=1}^N x_i, y_i$. To introduce the real projective space $\mathbb{P}^{N-1}$ of the $\mathbb{R}^N$, we say two nonzero elements $x,y \in \mathbb{R}^N$ are equivalent if there exists a $c \in \mathbb{R}$ such that $x = cy$. The equivalence class of $x \in \mathbb{R}^N$ is denoted by $\mathbb{P}x$, and we call the set of all equivalent classes the *projective space* $\mathbb{P}^{N-1}$. Equipped with the metric $d_{\mathbb{P}} : \mathbb{P}^{N-1}\times\mathbb{P}^{N-1} \to \big[0,\sqrt{2}\big]$, given by

$$d_{\mathbb{P}}(\mathbb{P}v,\mathbb{P}w) = \min\left\{\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\|, \left\|\frac{v}{\|v\|} + \frac{w}{\|w\|}\right\|\right\} \quad \text{for all } v,w \in \mathbb{R}^N,$$

the projective space is a compact metric space. We define $\mathbb{P}^{-1}v := \big\{x \in \mathbb{R}^N : \mathbb{P}x = v\big\} \cup \{0\}$ for any $v \in \mathbb{P}^{N-1}$. The $(N-1)$-*sphere* of the $\mathbb{R}^N$ is defined by $\mathbb{S}^{N-1} := \big\{x \in \mathbb{R}^N : \|x\| = 1\big\}$. We make repeated use of the following fundamental lemma, which follows from [2, Lemma B.1.17, p. 538].

Lemma 2.1. *For all $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that for all nonzero $v,w \in \mathbb{R}^N$ with $\langle v,w\rangle^2/(\|v\|^2\|w\|^2) \ge 1-\delta$, we have $d_{\mathbb{P}}(\mathbb{P}v,\mathbb{P}w) \le \varepsilon$.*

The notion of a nonautonomous dynamical system has emerged in the late 1990s as an abstraction of both random dynamical systems (see, e.g., Arnold [1]) and continuous skew product flows (see, e.g., Sell [11, 12]). The definition is given as follows.

Definition 2.2. A *nonautonomous dynamical system* (NDS for short) with a base set P, a locally compact metric space (X,d) and a time $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ consists of the following two ingredients:

(i) A model of the nonautonomy, given by a dynamical system $\theta : \mathbb{T}\times P \to P$, the so-called *base flow*, i.e., for all $t,s \in \mathbb{T}$ and $p \in P$, we have

$$\theta(0,p) = p \quad \text{and} \quad \theta(t+s,p) = \theta(t,\theta(s,p)).$$

(ii) A model of the system under nonautonomous influence, given by a *cocycle* $\varphi : \mathbb{T} \times P \times X \to X$ over θ, i.e., for all $t, s \in \mathbb{T}$, $p \in P$ and $\xi \in X$, we have

$$\varphi(0, p, \xi) = \xi \quad \text{and} \quad \varphi(t + s, p, \xi) = \varphi(t, \theta(s, p), \varphi(s, p, \xi))\,,$$

and the mapping $\varphi(\cdot, p, \cdot) : \mathbb{T} \times X \to X$ is continuous for all $p \in P$.

For simplicity in notation, we write $\theta_t p$ instead of $\theta(t, p)$ and $\varphi(t, p)\xi$ instead of $\varphi(t, p, \xi)$.

Standard examples of nonautonomous dynamical systems are provided by nonautonomous differential equations $\dot{x} = f(t, x)$, $\mathbb{T} = \mathbb{R}$, and nonautonomous difference equations $x_{n+1} = f(n, x_n)$, $\mathbb{T} = \mathbb{Z}$, fulfilling conditions of global existence and uniqueness of solutions. The base set P can simply be chosen to be $\mathbb{T}$ with the base flow $(t, s) \mapsto t + s$; $\varphi(t, s, \xi)$ is the value at time $t + s$ of the solution fulfilling the initial condition $x(s) = \xi$. However, P is then noncompact, which may cause difficulties. This can be avoided for a special class of right hand sides f by considering the Bebutov flow on the hull of f (see, e.g., Sell [12]). Please note that we do not assume compactness of P in this article.

A subset M of the extended phase space $P \times X$ is called a *nonautonomous set*; we use the term *p-fiber of* M for the set $M(p) := \{x \in X : (p, x) \in M\}$, $p \in P$, and we call M *compact* if all p-fibers of M are compact. Finally, a nonautonomous set M is called *invariant* if $\varphi(t, p, M(p)) = M(\theta_t p)$ for all $t \in \mathbb{T}$ and $p \in P$.

We now state the definitions of all-time attractivity and repulsivity, which have been introduced in [6].

Definition 2.3 (All-time attractivity and repulsivity)**.** Let A and R be invariant and compact nonautonomous sets.

(i) A is called an *all-time attractor* if there exists an $\eta > 0$ such that

$$\lim_{t\to\infty} \sup_{p\in P} d\big(\varphi(t, p)U_\eta(A(p))\big|A(\theta_t p)\big) = 0\,.$$

(ii) R is called an *all-time repeller* if there exists an $\eta > 0$ such that

$$\lim_{t\to\infty} \sup_{p\in P} d\big(\varphi(-t, p)U_\eta(R(p))\big|R(\theta_{-t} p)\big) = 0\,.$$

Remark 2.4. The notions of all-time attractivity and repulsivity are stronger than the concepts of past and future attractivity and repulsivity considered in [7]. This can be directly seen from the following definitions: An invariant and compact nonautonomous set A is called a *past attractor* if there exists an $\eta > 0$ such that for all $p \in P$, there exists a $\hat{\tau} > 0$ with

$$\lim_{t\to\infty} d\big(\varphi(t, \theta_{-\tau-t} p)U_\eta(A(\theta_{-\tau-t} p))\big|A(\theta_{-\tau} p)\big) = 0 \quad \text{for all } \tau \geq \hat{\tau}\,,$$

and an invariant and compact nonautonomous set R is called *past repeller* if there exists an $\eta > 0$ such that for all $p \in P$, there exists a $\hat{\tau} > 0$ with

$$\lim_{t\to\infty} d\big(\varphi(-t, \theta_{-\tau} p)U_\eta(R(\theta_{-\tau} p))\big|R(\theta_{-\tau-t} p)\big) = 0 \quad \text{for all } \tau \geq \hat{\tau}\,.$$

The notions of future attractivity and repulsivity are obtained via time reversal.

Example 2.5. The nonautonomous differential equation

$$\dot{x} = a(t)x + b(t)x^3 =: f(t, x)$$

with continuous functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to (\gamma, \infty)$ for some $\gamma > 0$ generates a nonautonomous dynamical system with $P = \mathbb{R}$ (see above). For $t \in \mathbb{R}$ with $a(t) \geq 0$,

$f(t,x)$ has the same sign as $x \in \mathbb{R}$. In case $a(t) < 0$, we have $f\big(t, \pm\sqrt{-a(t)/b(t)}\big) = 0$, and therefore, $f(t,x)$ has opposite sign as x in a vicinity of 0. Thus, $\mathbb{R} \times \{0\}$ is an all-time attractor if $\inf_{t\in\mathbb{R}} -a(t)/b(t) > 0$, and an all-time repeller if $\inf_{t\in\mathbb{R}} a(t) \geq 0$. These conditions are only sufficient for stability of the trivial solution but not necessary.

We henceforth suppose that $(\theta : \mathbb{T} \times P \to P, \varphi : \mathbb{T} \times P \times \mathbb{R}^N \to \mathbb{R}^N)$ is a linear nonautonomous dynamical system, i.e., given $\alpha, \beta \in \mathbb{R}$, we have

$$\varphi(t,p,\alpha x + \beta y) = \alpha\varphi(t,p,x) + \beta\varphi(t,p,y) \quad \text{for all } t \in \mathbb{T}, p \in P \text{ and } x, y \in \mathbb{R}^N .$$

Thus, there exists a matrix-valued function $\Phi : \mathbb{T} \times P \to \mathbb{R}^{N\times N}$ with $\Phi(t,p)x = \varphi(t,p,x)$ for all $t \in \mathbb{T}$, $p \in P$ and $x \in \mathbb{R}^N$. The NDS (θ,φ) canonically induces a nonautonomous dynamical system $(\theta, \mathbb{P}\Phi)$ on $\mathbb{P}^{N-1}$ by defining

$$\mathbb{P}\Phi(t,p)\mathbb{P}x := \mathbb{P}(\Phi(t,p)x) \quad \text{for all } \tau, t \in \mathbb{T}, p \in P \text{ and } x \in \mathbb{R}^N$$

(see Colonius & Kliemann [2, Lemma 5.2.1, p. 149]).

A finite sum $M_1 + \cdots + M_n$ of invariant nonautonomous sets such that $M_i(p)$ is a linear subspace of the $\mathbb{R}^N$, $i \in \{1,\dots,n\}$ and $p \in P$, is called a *Whitney sum* $M_1 \oplus \cdots \oplus M_n$ if the relation $M_i \cap M_j = \mathbb{T} \times \{0\}$ is satisfied for $i \neq j$.

3. All-time attractor-repeller pairs

In case of past attractivity and repulsivity, the construction of attractor-repeller pairs is possible only in one direction, i.e., a past repeller implies a past attractor (see [7, Theorem 4.3]), but [7, Example 4.4] showed that, in general, a past repeller cannot be constructed from a past attractor. The past attractor from [7, Example 4.4] is also an all-time attractor, and no corresponding all-time repeller exists, since this would be also a past repeller. Similarly, one can show that, in general, there is no method to construct an all-time attractor from an all-time repeller.

In this section, we show that in our linear setting, it is possible to obtain an all-time repeller from an all-time attractor and vice versa. A step towards this result is the following proposition, which says, among other things, that all-time attractors and repellers in $\mathbb{P}^{N-1}$ give rise to linear subspaces in $\mathbb{R}^N$.

Proposition 3.1. *The following statements hold:*

(i) *Let $A \notin \{\emptyset, P \times X\}$ be an all-time attractor of $(\theta, \mathbb{P}\Phi)$. Then, for all $\beta \in (0,\sqrt{2})$, we have*

$$\lim_{t\to\infty} \sup_{p\in P} \frac{\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}A(p)} \|\Phi(-t,p)v\|}{\inf_{w\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}(\mathbb{P}^{N-1}\setminus U_\beta(A(p)))} \|\Phi(-t,p)w\|} = 0 . \tag{3.1}$$

Moreover, for all $p \in P$, the set $\mathbb{P}^{-1}A(p)$ is a linear subspace of the $\mathbb{R}^N$.

(ii) *Let $R \notin \{\emptyset, P \times X\}$ be an all-time repeller of $(\theta, \mathbb{P}\Phi)$. Then, for all $\beta \in (0,\sqrt{2})$, we have*

$$\lim_{t\to\infty} \sup_{p\in P} \frac{\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}R(p)} \|\Phi(t,p)v\|}{\inf_{w\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}(\mathbb{P}^{N-1}\setminus U_\beta(R(p)))} \|\Phi(t,p)w\|} = 0 .$$

Moreover, for all $p \in P$, the set $\mathbb{P}^{-1}R(p)$ is a linear subspace of the $\mathbb{R}^N$.

Proof. We first note that β is supposed to be less than $\sqrt{2}$ only to guarantee that the infimum in the denominator of (3.1) is taken over a nonempty set. A is also a past attractor, and hence, [7, Proposition 8.2] implies that for all $p \in P$, the set

$\mathbb{P}^{-1}A(p)$ is a linear subspace. The definition of an all-time attractor implies the existence of an $\eta > 0$ such that

$$\lim_{t\to\infty} \sup_{p\in P} d\big(\mathbb{P}\Phi(t,p)U_{2\eta}(A(p))\big|A(\theta_t p)\big) = 0\,. \tag{3.2}$$

Due to Lemma 2.1, there exists a $\delta \in (0,1)$ such that for all nonzero $u_1, u_2 \in \mathbb{R}^N$ with $\langle u_1, u_2\rangle^2/(\|u_1\|^2\|u_2\|^2) \geq 1-\delta$, we have $d_{\mathbb{P}}(\mathbb{P}u_1, \mathbb{P}u_2) \leq \eta$. We assume to the contrary that there exist sequences $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, $\{p_n\}_{n\in\mathbb{N}}$ in P, $\{v_n\}_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}A(p_n)$ and $\{w_n\}_{n\in\mathbb{N}}$ in $\in \mathbb{S}^{N-1}\cap\mathbb{P}^{-1}\big(\mathbb{P}^{N-1}\setminus U_\beta(A(p))\big)$ such that $\lim_{n\to\infty} t_n = -\infty$ and the following property are fulfilled: There exists a $\gamma > 0$ with $\|\Phi(t_n,p_n)w_n\|/\|\Phi(t_n,p_n)v_n\| \leq \gamma$ for all $n \in \mathbb{N}$. We write $\Phi_n := \Phi(t_n,p_n)$. For nonzero $c \in \mathbb{R}$ with $|c|$ sufficiently small, this implies that for all $n\in\mathbb{N}$,

$$\begin{aligned}&\frac{\big\langle \Phi_n(cw_n+v_n), \Phi_n v_n\big\rangle^2}{\|\Phi_n(cw_n+v_n)\|^2\|\Phi_n v_n\|^2}\\ &= \frac{c^2\big\langle\Phi_n w_n, \Phi_n v_n\big\rangle^2 + 2c\|\Phi_n v_n\|^2\big\langle\Phi_n w_n,\Phi_n v_n\big\rangle + \|\Phi_n v_n\|^4}{c^2\|\Phi_n w_n\|^2\|\Phi_n v_n\|^2 + 2c\|\Phi_n v_n\|^2\big\langle\Phi_n w_n,\Phi_n v_n\big\rangle + \|\Phi_n v_n\|^4} \geq 1-\delta\end{aligned}$$

holds. We fix such a c and obtain

$$d_{\mathbb{P}}\big(\mathbb{P}\Phi(t_n,p_n)\mathbb{P}(cw_n+v_n), A(\theta_{t_n}p_n)\big) \leq \eta \quad \text{for all } \ n\in\mathbb{N}\,,$$

and this means that

$$\begin{aligned}&\lim_{n\to\infty} d_{\mathbb{P}}\big(\mathbb{P}(cw_n+v_n), A(p_n)\big)\\ &= \lim_{n\to\infty} d_{\mathbb{P}}\big(\mathbb{P}\Phi(-t_n,\theta_{t_n}p_n)\underbrace{\mathbb{P}\Phi(t_n,p_n)\mathbb{P}(cw_n+v_n)}_{\in U_{2\eta}(A(\theta_{t_n}p_n))}, A(p_n)\big) \overset{(3.2)}{=} 0\,.\end{aligned}$$

This is a contradiction, because $\mathbb{P}w_n \notin U_\beta(A(p_n))$ implies that there exists an $\alpha > 0$ with $\mathbb{P}(cw_n+v_n) \notin U_\alpha(A(p_n))$ for all $n\in\mathbb{N}$. ☐

Remark 3.2. Even in case the base set P is a topological space, the fiber $A(p)$ of an all-time attractor A and the dimension of this subspace $\dim A(p)$ do not depend continuously on $p\in P$. This follows from the definition of an all-time attractor, where all trajectories of the base flow $\{\theta_t p : t\in\mathbb{T}\}$, $p\in P$, can be considered as independent of each other.

An all-time attractor is also a future attractor, and hence, the formalism in [7] of constructing attractor-repeller pairs leads to a corresponding future repeller, which is given by (3.4) below. In our linear situation, the rate of attraction of the future attractor equals the rate of repulsion of the future repeller, and therefore, the future repeller is also an all-time repeller, since the future attractor is also an all-time attractor. This fact is the main idea for the proof of the following theorem.

Theorem 3.3 (All-time attractor-repeller pairs). *The following statements hold:*

(i) *Given an all-time attractor A of $(\theta, \mathbb{P}\Phi)$, i.e., there exists an $\eta > 0$ with*

$$\lim_{t\to\infty} \sup_{p\in P} d_{\mathbb{P}}\big(\mathbb{P}\Phi(t,p)U_\eta(A(p))\big|A(\theta_t p)\big) = 0\,. \tag{3.3}$$

Then the nonautonomous set A^*, defined by

$$A^*(p) := \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\mathbb{P}\Phi(-t,\theta_t p)\big(\mathbb{P}^{N-1}\setminus U_\eta(A(\theta_t p))\big)} \quad \textit{for all } \ p\in P\,, \tag{3.4}$$

is an all-time repeller such that $U_\eta(A(p)) \cap A^*(p) = \emptyset$ *for all* $p \in P$*, and we call* (A, A^*) *an* all-time attractor-repeller pair*. Moreover, we have the following decomposition in a Whitney sum:*

$$\mathbb{P}^{-1}A \oplus \mathbb{P}^{-1}A^* = P \times \mathbb{R}^N . \tag{3.5}$$

(ii) *Given an all-time repeller* R *of* $(\theta, \mathbb{P}\Phi)$*, i.e., there exists an* $\eta > 0$ *with*

$$\lim_{t\to\infty} \sup_{p\in P} d_{\mathbb{P}}\big(\mathbb{P}\Phi(-t,p)U_\eta(R(p))\big|R(\theta_{-t}p)\big) = 0 .$$

Then the nonautonomous set R^**, defined by*

$$R^*(p) := \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \mathbb{P}\Phi(t,\theta_{-t}p)\big(\mathbb{P}^{N-1} \setminus U_\eta(R(\theta_{-t}p))\big)} \quad \textit{for all } p \in P ,$$

is an all-time attractor such that $R^*(p) \cap U_\eta(R(p)) = \emptyset$ *for all* $p \in P$*, and we call* (R^*, R) *an* all-time attractor-repeller pair*. Moreover, we have the following decomposition in a Whitney sum:* $\mathbb{P}^{-1}R^* \oplus \mathbb{P}^{-1}R = P \times \mathbb{R}^N$*.*

(iii) *We have* $(A^*)^* = A$ *and* $(R^*)^* = R$ *for all-time attractors* A *and all-time repellers* R*.*

Proof. (i) We first note that $A^*(p) \cap U_\eta(A(p)) = \emptyset$ for all $p \in P$. Otherwise, since A and A^* are compact nonautonomous sets, (3.3) leads to $A^*(p) \cap A(p) \neq \emptyset$ for some $p \in P$, but this is a contradiction due to [7, Theorem 4.5 (i)] (it is proved there that past and future attractor-repeller pairs are disjoint, and (3.4) is the formula for the corresponding future repeller of a given future attractor A). Hence, due to Proposition 3.1, for fixed $\gamma \in (0, \eta)$, we have

$$\lim_{t\to\infty} \sup_{p\in P} \frac{\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}A(p)} \|\Phi(-t,p)v\|}{\inf_{w\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))} \|\Phi(-t,p)w\|} = 0 . \tag{3.6}$$

The relation (3.5) is a consequence of [7, Proposition 8.3]. The remaining proof is divided into two steps.

Step 1. We have

$$\begin{aligned} 1 &= \lim_{t\to\infty} \inf_{p\in P} \inf_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))} \frac{\|\Phi(-t,p)v_r^p\|}{\|\Phi(-t,p)v\|} \\ &= \lim_{t\to\infty} \sup_{p\in P} \sup_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))} \frac{\|\Phi(-t,p)v_r^p\|}{\|\Phi(-t,p)v\|} , \end{aligned}$$

where $v = v_a^p + v_r^p$ *with* $v_a^p \in \mathbb{P}^{-1}A(p)$ *and* $v_r^p \in \mathbb{P}^{-1}A^*(p)$*.*

The first assertion follows from

$$\begin{aligned} &\lim_{t\to\infty} \inf_{p\in P} \inf_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))} \frac{\|\Phi(-t,p)v_r^p\|}{\|\Phi(-t,p)v\|} \\ &\geq \left(\lim_{t\to\infty} \sup_{p\in P} \sup_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))} \frac{\|\Phi(-t,p)v_a^p\|}{\|\Phi(-t,p)v_r^p\|} + 1\right)^{-1} \\ &= \left(\lim_{t\to\infty} \sup_{p\in P} \sup_{v\in\mathbb{P}^{-1}U_\gamma(A^*(p)), v_a^p\neq 0} \frac{\|v_a^p\| \left\|\Phi(-t,p)\frac{v_a^p}{\|v_a^p\|}\right\|}{\|v_r^p\| \left\|\Phi(-t,p)\frac{v_r^p}{\|v_r^p\|}\right\|} + 1\right)^{-1} \overset{(3.6)}{=} 1 \end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to\infty}\inf_{p\in P}\inf_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))}\frac{\|\Phi(-t,p)v_r^p\|}{\|\Phi(-t,p)v\|}\\
&\le\left(\lim_{t\to\infty}\sup_{p\in P}\sup_{0\neq v\in\mathbb{P}^{-1}U_\gamma(A^*(p))}\left|1-\frac{\|\Phi(-t,p)v_a^p\|}{\|\Phi(-t,p)v_r^p\|}\right|\right)^{-1}\\
&=\left(\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{P}^{-1}U_\gamma(A^*(p)),v_a^p\neq 0}\left|1-\frac{\|v_a^p\|\left\|\Phi(-t,p)\frac{v_a^p}{\|v_a^p\|}\right\|}{\|v_r^p\|\left\|\Phi(-t,p)\frac{v_r^p}{\|v_r^p\|}\right\|}\right|\right)^{-1}\stackrel{(3.6)}{=}1\,.
\end{aligned}$$

In both relations, the last equality holds, because the set $V_a:=\{v_a^p:p\in P, v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))\}$ is bounded and the set $V_r:=\{v_r^p:p\in P,v\in\mathbb{S}^{-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))\}$ is bounded away from zero. The second assertion concerning the supremum instead of the infimum follows analogously.

Step 2. We have

$$\lim_{t\to\infty}\sup_{p\in P}d_{\mathbb{P}}\big(\mathbb{P}\Phi(-t,p)U_\gamma(A^*(p))\big|A^*(\theta_{-t}p)\big)=0\,,\tag{3.7}$$

i.e., A^* is an all-time repeller.

With v_r^p and v_a^p defined as in Step 1, for all $t\ge 0$, $p\in P$ and $v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))$, the relation

$$\begin{aligned}
&\frac{\langle\Phi(-t,p)v,\Phi(-t,p)v_r^p\rangle^2}{\|\Phi(-t,p)v\|^2\|\Phi(-t,p)v_r^p\|^2}\\
&=\frac{\langle\Phi(-t,p)v_a^p,\Phi(-t,p)v_r^p\rangle^2}{\|\Phi(-t,p)v\|^2\|\Phi(-t,p)v_r^p\|^2}+\frac{\|\Phi(-t,p)v_r^p\|^2}{\|\Phi(-t,p)v\|^2}+\frac{2\langle\Phi(-t,p)v_a^p,\Phi(-t,p)v_r^p\rangle}{\|\Phi(-t,p)v\|^2}
\end{aligned}$$

holds. Using the Cauchy-Schwartz inequality, we obtain the following relations:

$$\begin{aligned}
0&\le\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))}\frac{\langle\Phi(-t,p)v_a^p,\Phi(-t,p)v_r^p\rangle^2}{\|\Phi(-t,p)v\|^2\|\Phi(-t,p)v_r^p\|^2}\\
&\le\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))}\frac{\|\Phi(-t,p)v_a^p\|^2}{\|\Phi(-t,p)v\|^2}\stackrel{(3.6)}{=}0
\end{aligned}$$

and

$$\begin{aligned}
0&\le\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))}\frac{2\big|\langle\Phi(-t,p)v_a^p,\Phi(-t,p)v_r^p\rangle\big|}{\|\Phi(-t,p)v\|^2}\\
&\le\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))}2\,\frac{\|\Phi(-t,p)v_a^p\|}{\|\Phi(-t,p)v\|}\,\frac{\|\Phi(-t,p)v_r^p\|}{\|\Phi(-t,p)v\|}\\
&\stackrel{\text{Step 1}}{=}\lim_{t\to\infty}\sup_{p\in P}\sup_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))}\frac{2\|\Phi(-t,p)v_a^p\|}{\|\Phi(-t,p)v\|}\stackrel{(3.6)}{=}0
\end{aligned}$$

(we also use the fact from Step 1 that V_a is bounded). Hence, due to Step 1, we have

$$\begin{aligned}
&\lim_{t\to\infty} \inf_{p\in P} \inf_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))} \frac{\big\langle\Phi(-t,p)v, \Phi(-t,p)v_r^p\big\rangle^2}{\|\Phi(-t,p)v\|^2\|\Phi(-t,p)v_r^p\|^2} \\
&= \lim_{t\to\infty} \inf_{p\in P} \inf_{v\in\mathbb{S}^{N-1}\cap\mathbb{P}^{-1}U_\gamma(A^*(p))} \Bigg(\frac{\big\langle\Phi(-t,p)v_a^p, \Phi(-t,p)v_r^p\big\rangle^2}{\|\Phi(-t,p)v\|^2\|\Phi(-t,p)v_r^p\|^2} + \frac{\|\Phi(-t,p)v_r^p\|^2}{\|\Phi(-t,p)v\|^2} \\
&\qquad + \frac{2\big\langle\Phi(-t,p)v_a^p, \Phi(-t,p)v_r^p\big\rangle}{\|\Phi(-t,p)v\|^2}\Bigg) = 1\,.
\end{aligned}$$

Using Lemma 2.1, this implies (3.7).

(ii) can be proved analogously to (i).

(iii) Let A be an all-time attractor. Then A is also a future attractor, and due to [7, Theorem 4.3], A^* given by (3.4) is the maximal future repeller outside of A (with respect to set inclusion). Reapplication of [7, Theorem 4.3] shows that $(A^*)^*$ is the maximal past attractor outside of A^*. This shows $A \subset (A^*)^*$. Assume to the contrary that there exist $p \in P$ and $x \in (A^*)^*(p) \setminus A(p)$. Then there exists a $\tau \geq 0$ such that $\mathbb{P}\Phi(-t,p)x \notin U_\eta(A(\theta_{-t}p))$ for all $t \geq \tau$, because otherwise, (3.3) would lead to $x \in A(p)$ (note that A is a compact nonautonomous set). Hence, (3.4) implies that $x \in A^*(p)$, and this is a contradiction, since A^* and $(A^*)^*$ are disjoint due to (i). □

Remark 3.4. In [5], so-called *generalized attractor-repeller pairs* are introduced, which consist of two invariant and linear nonautonomous sets A and R of Φ fulfilling the following three conditions:

(a) $A(p) \oplus R(p) = \mathbb{R}^N$ for all $p \in P$,

(b) given $p \in P$, $0 \neq \xi \in A(p)$ and $0 \neq \eta \in R(p)$, we have

$$\frac{\|\Phi(t,p)\eta\|}{\|\Phi(t,p)\xi\|} \to 0 \text{ as } t \to \infty \quad \text{and} \quad \frac{\|\Phi(t,p)\xi\|}{\|\Phi(t,p)\eta\|} \to 0 \text{ as } t \to -\infty\,,$$

(c) the angle between $A(p)$ and $R(p)$ is bounded below by a positive number.

It is easy to see that all-time attractor-repeller pairs are also generalized attractor-repeller pairs, but in general, the reversal is not true, since the limit relation in (b) is not so strong as relation (3.1).

Example 3.5. Given a nonautonomous differential equation

$$\dot{x} = B(t)x \tag{3.8}$$

with a continuous function $B : \mathbb{R} \to \mathbb{R}^{N\times N}$. We denote the corresponding linear nonautonomous dynamical system by Φ and suppose that (3.8) admits an (nonhyperbolic) exponential dichotomy (see, e.g., [4]). Thus, there exist constants $\alpha, K > 0$, a growth rate $\beta \in \mathbb{R}$ and invariant and linear nonautonomous sets A and R of Φ such that for all $p \in P$, $\xi \in R(p)$ and $\eta \in A(p)$, we have

$$\|\Phi(t,p)\xi\| \leq Ke^{(\beta-\alpha)t}\|\xi\| \quad \text{and} \quad \|\Phi(-t,p)\eta\| \leq Ke^{-(\beta+\alpha)t}\|\eta\| \quad \text{for all } t \geq 0\,.$$

Then, $(\mathbb{P}A, \mathbb{P}R)$ is an all-time attractor-repeller pair of $\mathbb{P}\Phi$ (for a proof, see [6, 8]).

4. All-time Morse decompositions

In this section, the notion of an all-time attractor-repeller pair is generalized by considering Morse decompositions.

Definition 4.1 (All-time Morse decomposition)**.** A family $\{M_1, M_2, \dots, M_n\}$ of nonautonomous sets, the so-called *Morse sets*, is called an *all-time Morse decomposition* if the representation $M_i = A_i \cap A_{i-1}^*$ for all $i \in \{1, \dots, n\}$ holds with all-time attractors $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = P \times \mathbb{P}^{N-1}$.

Remark 4.2.

- Let (A, R) be an all-time attractor-repeller pair such that $\emptyset \subsetneq A \subsetneq P \times X$. Then $\{A, R\}$ is an all-time Morse decomposition.
- An all-time Morse decomposition is both a past and future Morse decomposition.

Proposition 4.3 (Basic properties of all-time Morse decompositions)**.** *The Morse sets of an all-time Morse decomposition $\{M_1, \dots, M_n\}$ are nonempty, invariant, pairwise disjoint and* isolated*, i.e., there exists a $\beta > 0$ such that for all $1 \le i < j \le n$, we have $U_\beta(M_i(p)) \cap U_\beta(M_j(p)) = \emptyset$ for all $p \in P$. Moreover, the sets $\mathbb{P}^{-1}M_i(p)$, $i \in \{1, \dots, n\}$, are linear subspaces of the $\mathbb{R}^N$ for all $p \in P$, and we have the following decomposition in a Whitney sum: $\mathbb{P}^{-1}M_1 \oplus \cdots \oplus \mathbb{P}^{-1}M_n = \mathbb{T} \times \mathbb{R}^N$.*

Proof. Due to the fact that every all-time Morse decomposition is a past Morse decomposition, [7, Proposition 5.3] yields that the Morse sets are nonempty, invariant and pairwise disjoint. The Morse sets are isolated, since this is an obvious consequence of the fact that attractor-repeller pairs are isolated (cf. Theorem 3.3). To show the decomposition in a Whitney sum, we first note that for $1 \le i < j \le n$, we have $\mathbb{P}^{-1}M_i \cap \mathbb{P}^{-1}M_j = \mathbb{T} \times \{0\}$. Furthermore, (3.5) implies

$$\begin{aligned}\mathbb{T} \times \mathbb{R}^N &= \mathbb{P}^{-1}A_1 + \mathbb{P}^{-1}A_1^* = \mathbb{P}^{-1}M_1 + \big(\mathbb{P}^{-1}A_1^* \cap (\mathbb{P}^{-1}A_2 + \mathbb{P}^{-1}A_2^*)\big)\\ &= \mathbb{P}^{-1}M_1 + \big(\mathbb{P}^{-1}A_1^* \cap \mathbb{P}^{-1}A_2\big) + \mathbb{P}^{-1}A_2^* = \mathbb{P}^{-1}M_1 + \mathbb{P}^{-1}M_2 + \mathbb{P}^{-1}A_2^*\,.\end{aligned}$$

Here, we used the fact that linear subspaces $E, F, G \subset \mathbb{R}^N$ with $E \supset G$ fulfill $E \cap (F + G) = (E \cap F) + G$. It follows inductively that

$$\mathbb{T} \times \mathbb{R}^N = \mathbb{P}^{-1}M_1 + \cdots + \mathbb{P}^{-1}M_n + \mathbb{P}^{-1}R_n = \mathbb{P}^{-1}M_1 + \cdots + \mathbb{P}^{-1}M_n\,.$$

This finishes the proof of this theorem. □

The following theorem shows that the Morse sets are crucial for the asymptotic behavior of the system.

Theorem 4.4 (Dynamical behavior of all-time Morse decompositions)**.** *For an all-time Morse decomposition $\{M_1, \dots, M_n\}$, the following statements are fulfilled:*

(i) Convergence in forward time. *For all $(p, x) \in P \times \mathbb{P}^{N-1}$, there exists an $i \in \{1, \dots, n\}$ with* $\lim_{t\to\infty} d_{\mathbb{P}}\big(\mathbb{P}\Phi(t, p)x, M_i(\theta_t p)\big) = 0$.

(ii) Convergence in backward time. *For all $(p, x) \in P \times \mathbb{P}^{N-1}$, there exists an $i \in \{1, \dots, n\}$ with* $\lim_{t\to\infty} d_{\mathbb{P}}\big(\mathbb{P}\Phi(-t, p)x, M_i(\theta_{-t} p)\big) = 0$.

Proof. This follows from [7, Theorem 8.5], since an all-time Morse decomposition is both a past and a future Morse decomposition. □

Example 4.5. Consider again (3.8) from Example 3.5, and denote by $\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$ the dichotomy spectrum of (3.8) (see [9, 13]). To each spectral interval $[a_i, b_i]$ corresponds a spectral manifold $\mathcal{W}_i$, which is obtained as intersections of sets of the form A and R from Example 3.5 (note that for each $\beta \in [b_i, a_{i+1}]$, the equation (3.8) admits an exponential dichotomy with growth rate β). From the observation in Example 3.5, it follows that $\{\mathbb{P}\mathcal{W}_1, \dots, \mathbb{P}\mathcal{W}_n\}$ is a Morse decomposition of $\mathbb{P}\Phi$ (see [8] for details).

5. Finest all-time Morse decomposition

Now, we restrict our attention to the special situation $P = \mathbb{T}$ and $\theta(t,s) = t+s$ for all $t, s \in \mathbb{T}$. As described in Section 2, this setting includes arbitrary nonautonomous differential or difference equations. We prove an analogue to the Theorem of Selgrade (see [10, Theorem 9.7] and [2, Theorem 5.2.5]).

Theorem 5.1 (Finest all-time Morse decomposition)**.** *We suppose $P = \mathbb{T}$ and $\theta(t,s) = t+s$ for all $t, s \in \mathbb{T}$. Then there exists a* finest all-time Morse decomposition *$\{M_1, \dots, M_n\}$, i.e., given another all-time Morse decomposition $\{\bar{M}_1, \dots, \bar{M}_m\}$, then for all $i \in \{1, \dots, m\}$, there exists $j \in \{1, \dots, n\}$ with $M_j \subset \bar{M}_i$. Moreover, we have $n \le N$, and the following decomposition in a Whitney sum is fulfilled:*

$$\mathbb{P}^{-1} M_1 \oplus \cdots \oplus \mathbb{P}^{-1} M_n = \mathbb{T} \times \mathbb{R}^N .$$

Proof. We first prove that any two all-time attractors A and $\bar{A}$ fulfill either the relation $A \subset \bar{A}$ or $A \supset \bar{A}$. Supposing the contrary, due to $P = \mathbb{T}$, there exist a $\tau \in \mathbb{T}$ and elements $x \in \mathbb{S}^{N-1} \cap \big(\mathbb{P}^{-1} A(\tau) \setminus \mathbb{P}^{-1} \bar{A}(\tau)\big)$ and $\bar{x} \in \mathbb{S}^{N-1} \cap \big(\mathbb{P}^{-1} \bar{A}(\tau) \setminus \mathbb{P}^{-1} A(\tau)\big)$. Because of Proposition 3.1, we obtain $\lim_{t\to\infty} \|\Phi(-t,\tau)x\| / \|\Phi(-t,\tau)\bar{x}\| = 0$ and $\lim_{t\to\infty} \|\Phi(-t,\tau)\bar{x}\| / \|\Phi(-t,\tau)x\| = 0$. This is a contradiction. Proposition 3.1 also implies that the fibers of all-time attractors correspond to linear subspaces. Thus, there are at most $N+1$ all-time attractors of $(\theta, \mathbb{P}\Phi)$, namely $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathbb{T} \times \mathbb{P}^{N-1}$ with $n \le N$. We denote by $\{M_1, \dots, M_n\}$ the corresponding all-time Morse decomposition. Let $\{\bar{M}_1, \dots, \bar{M}_m\}$ be another past Morse decomposition, obtained by the sequence $\emptyset = \bar{A}_0 \subsetneq \bar{A}_1 \subsetneq \cdots \subsetneq \bar{A}_m = \mathbb{T} \times \mathbb{P}^{N-1}$ of all-time attractors. Then, for each $i \in \{0, \dots, m\}$, there exists an $n_i \in \{0, \dots, n\}$ such that $\bar{A}_i = A_{n_i}$. This implies $\bar{A}^*_{i-1} \supset A^*_{n_i - 1}$, and hence, $\bar{M}_i = \bar{A}_i \cap \bar{A}^*_{i-1} \supset A_{n_i} \cap A^*_{n_i-1} = M_{n_i}$. This finishes the proof of this theorem. ☐

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