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EMBEDDINGS OF LOCALLY FINITE METRIC SPACES INTO BANACH SPACES

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Abstract. We show that if X is a Banach space without cotype, then every locally finite metric space embeds metrically into X.

1. Introduction

Let X and Y be two Banach spaces. If X and Y are linearly isomorphic, the *Banach-Mazur distance* between X and Y, denoted by $d_{BM}(X,Y)$, is the infimum of $\|T\|\,\|T^{-1}\|$, over all linear isomorphisms T from X onto Y.

For $p \in [1,\infty]$ and $n \in \mathbb{N}$, ℓ_p^n denotes the space $\mathbb{R}^n$ equipped with the ℓ_p norm. We say that a Banach space X uniformly contains the ℓ_p^n's if there is a constant $C \geq 1$ such that for every integer n, X admits an n-dimensional subspace Y so that $d_{BM}(\ell_p^n, Y) \leq C$.

A metric space M is *locally finite* if any ball of M with finite radius is finite. If moreover, there is a function $C : (0,+\infty) \to \mathbb{N}$ such that any ball of radius r contains at most $C(r)$ points, we say that M has a *bounded geometry*.

Let (M,d) and (N,δ) be two metric spaces and $f : M \to N$ be a map. For $t > 0$ define

$$\rho_f(t) = \inf\{\delta(f(x), f(y)),\quad d(x,y) \geq t\}$$

and

$$\omega_f(t) = \sup\{\delta(f(x), f(y)),\quad d(x,y) \leq t\}.$$

We say that f is a *coarse embedding* if $\omega_f(t)$ is finite for all $t > 0$ and $\lim_{t\to\infty} \rho_f(t) = \infty$.

Suppose now that f is injective. We say that f is a *uniform embedding* if f and f^{-1} are uniformly continuous. Following [7], we also define the *distortion* of f to be

$$\operatorname{dist}(f) := \|f\|_{Lip}\|f^{-1}\|_{Lip} = \sup_{x\neq y\in M} \frac{\delta(f(x), f(y))}{d(x,y)} . \sup_{x\neq y\in M} \frac{d(x,y)}{\delta(f(x), f(y))}.$$

If the distortion of f is finite, we say that f is a *metric embedding* and that M metrically embeds into N and we denote $M \hookrightarrow N$. If $\operatorname{dist}(f) \leq C$, we use the notation $M \overset{C}{\hookrightarrow} N$.

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A celebrated result of I. Aharoni [1] asserts that every separable metric space metrically embeds into c_0 (the space of all real sequences converging to 0, equipped with the supremum norm). It is an open problem to know what are the Banach spaces that share this property with c_0. In other words, if c_0 metrically embeds into a separable Banach space X, do we have that X contains a closed subspace which is linearly isomorphic to c_0? Very recently, N.J. Kalton [3] made an important step in this direction by showing that one of the iterated duals of such a Banach space has to be nonseparable (in particular, X cannot be reflexive). In fact, he even proved it for X such that c_0 coarsely or uniformly embeds into X.

In another direction, N. Brown and E. Guentner proved in [2] that any metric space with bounded geometry coarsely embeds into a reflexive Banach space. This was also improved by Kalton in [3], who obtained that any locally finite metric space uniformly and coarsely embeds into a reflexive space. These results are connected with the coarse Novikov conjecture. Indeed, Kasparov and Yu have shown in [4] that if a metric space with bounded geometry coarsely embeds into a super-reflexive Banach space, then it satisfies this important conjecture. In this paper, we improve the results of Brown, Guentner and Kalton, by showing that any Banach space uniformly containing the ℓ_∞^n's is metrically universal for all locally finite metric spaces. It should also be noted that Brown and Guentner used a specific space, $(\sum \ell_{p_n})_2$, with p_n tending to infinity and that this space fulfils our hypothesis.

2. Results

Theorem 2.1. *There exists a universal constant $C > 1$ such that for every Banach space X uniformly containing the ℓ_∞^n's and every locally finite metric space (M, d), $M \overset{C}{\hookrightarrow} X$.*

Proof. Let X be a Banach space uniformly containing the ℓ_∞^n's (or equivalently without any nontrivial cotype). Our first lemma follows directly from the classical work of B. Maurey and G. Pisier [6].

Lemma 2.2. *For any finite codimensional subspace Y of X, any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists a subspace F of Y such that $d_{BM}(\ell_\infty^n, F) < 1 + \varepsilon$.*

We shall also need the following version of Mazur's Lemma (see for instance [5], page 4, for a proof).

Lemma 2.3. *Let X be an infinite dimensional Banach space, E be a finite dimensional subspace of X and $\varepsilon > 0$. Then there is a finite codimensional subspace Y of X such that:*

$$\forall x \in E \quad \forall y \in Y, \quad \|x\| \leq (1+\varepsilon)\|x+y\|.$$

Now consider a locally finite metric space (M, d). For $t \in M$ and $r \geq 0$, $B(t, r) = \{s \in M,\ d(s,t) \leq r\}$. We fix a point t_0 in M and denote $|t| = d(t, t_0)$, for $t \in M$. Since M is locally finite, we may assume, by multiplying d by a constant if necessary, that $B(t_0, 1) = \{t_0\}$. For any nonnegative integer n, we denote $B_n = B(t_0, 2^{n+1})$. Using Lemmas 2.2 and 2.3, together with the fact that each ball B_n is finite, we can build inductively finite dimensional subspaces $(F_n)_{n=0}^\infty$ of X and $(T_n)_{n=0}^\infty$ so that for every $n \geq 0$, T_n is a linear isomorphism from $\ell_\infty(B_n)$ onto F_n satisfying

$$\forall u \in \ell_\infty(B_n), \qquad \frac{1}{2}\|u\| \leq \|T_n u\| \leq \|u\|$$

and also such that $(F_n)_{n=0}^{\infty}$ is a Schauder finite dimensional decomposition of its closed linear span Z. More precisely, if P_n is the projection from Z onto $F_0 \oplus ... \oplus F_n$ with kernel $\overline{\mathrm{sp}}\,(\bigcup_{i=n+1}^{\infty} F_i)$, we will assume, as we may, that $\|P_n\| \leq 2$. We now denote $\Pi_0 = P_0$ and $\Pi_n = P_n - P_{n-1}$ for $n \geq 1$. We have that $\|\Pi_n\| \leq 4$.

We now consider $\varphi_n : B_n \to \ell_\infty(B_n)$ defined by

$$\forall t \in B_n, \quad \varphi_n(t) = \big(d(s,t) - d(s,t_0)\big)_{s \in B_n} = \big(d(s,t) - |s|\big)_{s \in B_n}.$$

The map φ_n is clearly an isometric embedding of B_n into $\ell_\infty(B_n)$. Then we set:

$$\forall t \in B_n, \quad f_n(t) = T_n(\varphi_n(t)) \in F_n.$$

Finally we construct a map $f : M \to X$ as follows:

(i) $f(t_0) = 0$.

(ii) For $n \geq 0$ and $2^n \leq |t| < 2^{n+1}$:

$$f(t) = \lambda f_n(t) + (1-\lambda) f_{n+1}(t), \quad \text{where} \quad \lambda = \frac{2^{n+1} - |t|}{2^n}.$$

We will show that $\mathrm{dist}(f) \leq 9 \times 24 = 216$.

Note first that for any t in M,

$$\frac{1}{16}|t| \leq \|f(t)\| \leq |t|.$$

We start by showing that f is Lipschitz. Let $t, t' \in M \setminus \{t_0\}$ and assume, as we may, that $1 \leq |t| \leq |t'|$ (we recall that $B(t_0, 1) = \{t_0\}$).

I) If $|t| \leq \frac{1}{2}|t'|$, then

$$\|f(t) - f(t')\| \leq |t| + |t'| \leq \frac{3}{2}|t'| \leq 3(|t'| - |t|) \leq 3\, d(t,t').$$

II) If $\frac{1}{2}|t'| < |t| \leq |t'|$, we have two different cases to consider.

1) $2^n \leq |t| \leq |t'| < 2^{n+1}$, for some $n \geq 0$. Then, let

$$\lambda = \frac{2^{n+1} - |t|}{2^n} \quad \text{and} \quad \lambda' = \frac{2^{n+1} - |t'|}{2^n}.$$

We have that

$$|\lambda - \lambda'| = \frac{|t'| - |t|}{2^n} \leq \frac{d(t,t')}{2^n},$$

so

$$\begin{aligned}
\|f(t) - f(t')\| &= \|\lambda f_n(t) - \lambda' f_n(t') + (1-\lambda) f_{n+1}(t) - (1-\lambda') f_{n+1}(t')\| \\
&\leq \lambda \|f_n(t) - f_n(t')\| + (1-\lambda)\|f_{n+1}(t) - f_{n+1}(t')\| + 2|\lambda - \lambda'|\,|t'| \\
&\leq d(t,t') + 2^{n+2}|\lambda - \lambda'| \leq 5\, d(t,t').
\end{aligned}$$

2) $2^n \leq |t| < 2^{n+1} \leq |t'| < 2^{n+2}$, for some $n \geq 0$. Then, let

$$\lambda = \frac{2^{n+1} - |t|}{2^n} \quad \text{and} \quad \lambda' = \frac{2^{n+2} - |t'|}{2^{n+1}}.$$

We have that

$$\lambda \leq \frac{d(t,t')}{2^n}, \quad \text{so} \quad \lambda |t| \leq 2\, d(t,t').$$

Similarly

$$1-\lambda' = \frac{|t'|-2^{n+1}}{2^{n+1}} \leq \frac{d(t,t')}{2^{n+1}} \quad \text{and} \quad (1-\lambda')|t'| \leq 2\, d(t,t').$$

$$\begin{aligned}\|f(t)-f(t')\| &= \|\lambda f_n(t) + (1-\lambda) f_{n+1}(t) - \lambda' f_{n+1}(t') - (1-\lambda') f_{n+2}(t')\| \\ &\leq \lambda(\|f_n(t)\| + \|f_{n+1}(t)\|) + (1-\lambda')(\|f_{n+1}(t')\| + \|f_{n+2}(t')\|) \\ &\quad + \|f_{n+1}(t) - f_{n+1}(t')\| \\ &\leq d(t,t') + 2\lambda|t| + 2(1-\lambda')|t'| \leq 9\, d(t,t').\end{aligned}$$

We have shown that f is 9-Lipschitz.

We shall now prove that f^{-1} is Lipschitz. We consider $t, t' \in M\backslash\{t_0\}$ and assume again that $1 \leq |t| \leq |t'|$. We need to study three different cases. In our discussion, whenever $|t|$ (respectively $|t'|$) will belong to $[2^m, 2^{m+1})$, for some integer m, we shall denote

$$\lambda = \frac{2^{m+1}-|t|}{2^m} \quad (\text{respectively} \quad \lambda' = \frac{2^{m+1}-|t'|}{2^m}).$$

1) $2^n \leq |t| \leq |t'| < 2^{n+1}$, for some $n \geq 0$. Then

$$\Pi_n(f(t)-f(t')) = T_n(\lambda\varphi_n(t) - \lambda'\varphi_n(t')) \quad \text{and}$$
$$[\lambda\varphi_n(t) - \lambda'\varphi_n(t')]_{|s=t'} = \lambda d(t,t') + (\lambda'-\lambda)|t'|.$$

So

$$2\|\Pi_n(f(t)-f(t'))\| \geq \lambda d(t,t') + (\lambda'-\lambda)|t'|.$$

On the other hand

$$\Pi_{n+1}(f(t)-f(t')) = T_{n+1}((1-\lambda)\varphi_{n+1}(t) - (1-\lambda')\varphi_{n+1}(t')) \quad \text{and}$$
$$[(1-\lambda)\varphi_{n+1}(t) - (1-\lambda')\varphi_{n+1}(t')]_{|s=t'} = (1-\lambda)d(t,t') + (\lambda-\lambda')|t'|.$$

So

$$2\|\Pi_{n+1}(f(t)-f(t'))\| \geq (1-\lambda)d(t,t') + (\lambda-\lambda')|t'|.$$

Therefore

$$16\|f(t)-f(t')\| \geq d(t,t').$$

2) $2^n \leq |t| < 2^{n+1} \leq |t'| < 2^{n+2}$, for some $n \geq 0$.

$$2\|\Pi_n(f(t)-f(t'))\| = 2\lambda\|T_n(\varphi_n(t))\| \geq \lambda|t|,$$
$$2\|\Pi_{n+2}(f(t)-f(t'))\| = 2(1-\lambda')\|T_{n+2}(\varphi_{n+2}(t'))\| \geq (1-\lambda')|t'|,$$
$$\Pi_{n+1}(f(t)-f(t')) = T_{n+1}((1-\lambda)\varphi_{n+1}(t) - \lambda'\varphi_{n+1}(t')) \quad \text{and}$$
$$[\lambda'\varphi_{n+1}(t') - (1-\lambda)\varphi_{n+1}(t)]_{|s=t} = \lambda' d(t,t') - \lambda'|t| + (1-\lambda)|t|.$$

So

$$2\|\Pi_{n+1}(f(t)-f(t'))\| \geq \lambda' d(t,t') - \lambda'|t| + (1-\lambda)|t|.$$

Combining our three estimates, we obtain

$$24\|f(t)-f(t')\| \geq \lambda' d(t,t') + (1-\lambda')(|t|+|t'|) \geq d(t,t').$$

3) $2^n \leq |t| < 2^{n+1} < 2^p \leq |t'| < 2^{p+1}$ for some integers n and p.

$$2\|\Pi_p(f(t)-f(t'))\| = 2\lambda'\|T_p(\varphi_p(t'))\| \geq \lambda'|t'| \quad \text{and}$$
$$2\|\Pi_{p+1}(f(t)-f(t'))\| = 2(1-\lambda')\|T_{p+1}(\varphi_{p+1}(t'))\| \geq (1-\lambda')|t'|.$$

So

$$24\|f(t) - f(t')\| \geq \frac{3}{2}|t'| \geq |t'| + |t| \geq d(t, t').$$

All possible cases are settled, and we have shown that f^{-1} is 24-Lipschitz. □

We conclude by mentioning the following easy fact.

Proposition 2.4. *Let N be a metric space. The following assertions are equivalent:*
(i) For any locally finite metric space M, $M \hookrightarrow N$.
(ii) There exists $C \geq 1$ such that for any locally finite metric space $M \overset{C}{\hookrightarrow} N$.

This is an immediate consequence of the following lemma.

Lemma 2.5. *Let $(M_p, d_p)_{p=1}^{\infty}$ be a sequence of locally finite metric spaces. Then there exists a locally finite metric space (M, d) such that each M_p embeds isometrically into M.*

Proof. For any $p \in \mathbb{N}$, pick $x_0^p \in M_p$. Consider $M = \bigcup_{p=1}^{\infty} \{p\} \times M_p$. Let $x \in M_p$ and $y \in M_q$. We define $d((p, x), (q, y)) = d_p(x, y)$ if $p = q$ and $d((p, x), (q, y)) = \max\{p, q, d_p(x_0^p, x), d(x_0^q, y)\}$ if $p \neq q$. We leave it to the reader to check that (M, d) is a locally finite metric space. □

Remark 2.6. We do not know if the converse of Theorem 2.1 is true. So the question is: if every locally finite metric space metrically embeds in a given Banach space X, do we have that X uniformly contains the ℓ_∞^n's? However, it follows from the work of Mendel and Naor in [7] that such a Banach space cannot be K-convex, or equivalently, it must contain the ℓ_1^n's uniformly.

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