CODES OVER RINGS OF SIZE FOUR, HERMITIAN LATTICES, AND CORRESPONDING THETA FUNCTIONS

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Abstract. Let $K = \mathbb{Q} (\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$, and let $C = [n, k]$ be a linear code defined over $\mathcal{O}_K / 2\mathcal{O}_K$. The level $\ell$ theta function $\Theta_{\Lambda_\ell}(C)$ of $C$ is defined on the lattice $\Lambda_\ell(C) := \{ x \in \mathcal{O}_K^n : \rho_\ell(x) \in C \}$, where $\rho_\ell : \mathcal{O}_K \rightarrow \mathcal{O}_K / 2\mathcal{O}_K$ is the natural projection. In this paper, we prove that:

i) for any $\ell, \ell'$ such that $\ell \leq \ell'$, $\Theta_{\Lambda_\ell}(q)$ and $\Theta_{\Lambda_{\ell'}}(q)$ have the same coefficients up to $q^{\ell+1}/4$,

ii) for $\ell \geq 2(n+1)(n+2)/n - 1$, $\Theta_{\Lambda_\ell}(C)$ determines the code $C$ uniquely,

iii) for $\ell < 2(n+1)(n+2)/n - 1$, there is a positive dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$.

1. Introduction

Let $K = \mathbb{Q} (\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$. Then the image $\mathcal{O}_K / 2\mathcal{O}_K$ of the projection $\rho_\ell : \mathcal{O}_K \rightarrow \mathcal{O}_K / 2\mathcal{O}_K$ is $\mathbb{F}_4$ (resp., $\mathbb{F}_2 \times \mathbb{F}_2$) if $\ell \equiv 3 \mod 8$ (resp., $\ell \equiv 7 \mod 8$).

Let $\mathcal{R}$ be a ring isomorphic to $\mathbb{F}_4$ or $\mathbb{F}_2 \times \mathbb{F}_2$ and $C = [n, k]$ be a linear code over $\mathcal{R}$ of length $n$ and dimension $k$. An admissible level $\ell$ is an $\ell$ such that $\ell \equiv 3 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_4$ or $\ell \equiv 7 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Fix an admissible $\ell$ and define $\Lambda_\ell(C) := \{ x \in \mathcal{O}_K^n : \rho_\ell(x) \in C \}$.

Then, the level $\ell$ theta function $\Theta_{\Lambda_\ell(C)}(\tau)$ of the lattice $\Lambda_\ell(C)$ is given as the symmetric weight enumerator $swe_C$ of $C$, evaluated on the theta functions defined on cosets of $\mathcal{O}_K / 2\mathcal{O}_K$. In this paper we study the following two questions:

i) How do the theta functions $\Theta_{\Lambda_\ell(C)}(\tau)$ of the same code $C$ differ for different levels $\ell$?

ii) Can nonequivalent codes give the same theta functions for all levels $\ell$?

In an attempt to study the second question, Chua in [1] gives an example of two nonequivalent codes that give the same theta function for level $\ell = 7$ but not for higher level thetas. We will show in this paper how such an example is not a coincidence. Our main results are as follows:
Theorem 1. Let $C$ be a code defined over $R$. For all admissible $\ell, \ell'$ such that $\ell > \ell'$, the following holds:
\[
\Theta_{\Lambda_{\ell}}(C) = \Theta_{\Lambda_{\ell'}}(C) + \mathcal{O}(q^{\ell+1}).
\]

Theorem 2. Let $C$ be a code of size $n$ defined over $R$ and $\Theta_{\Lambda_{\ell}}(C)$ be its corresponding theta function for level $\ell$. Then the following hold:

i) For $\ell < \frac{2(n+1)(n+2)}{n} - 1$, there is a $\delta$-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_{\ell}}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{n} - \frac{n(\ell+1)}{4} - 1$.

ii) For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$, there is a unique symmetrized weight enumerator polynomial which corresponds to $\Theta_{\Lambda_{\ell}}(C)$.

This paper is organized as follows. In the second section, we give a basic introduction of lattices and theta functions. We define a lattice $\Lambda$ over a number field $K$ in general, the theta series of a lattice, and the one-dimensional theta series and its shadow. Then we discuss the lattices over imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-\ell})$ with a ring of integers $\mathcal{O}_K$, where $\ell$ is a square free integer such that $\ell \equiv 3 \mod 4$. The ring $\mathcal{O}_K/(2\mathcal{O}_K)$ is equivalent to either the field of order 4 or a ring of order 4 depending on whether $\ell \equiv 3 \mod 8$ or $\ell \equiv 7 \mod 8$. We define bi-dimensional theta functions for the four cosets of $\mathcal{O}_K/(2\mathcal{O}_K)$.

In the third section, we define codes over $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$, the weight enumerators of a code, and recall the main result of [1]. We simplify the expressions for bi-dimensional theta series and prove Theorem 1.

In the fourth section, we study families of codes corresponding to the same theta function. We call an acceptable theta series $\Theta(q)$ a series for which there exists a code $C$ such that $\Theta(q) = \Theta_{\Lambda_{\ell}}(C)(q)$. For any given $\ell$ and an acceptable theta series $\Theta(q)$ we can determine a family of symmetrized weight enumerators that correspond to $\Theta(q)$. For small $\ell$ this is a positive dimensional family, where the dimension is given by Theorem 2i). Hence, the example given in [1] is no surprise. For large $\ell$ (see Theorem 2ii)) this is a 0-dimensional family of symmetrized weight enumerators that correspond to $\Theta(q)$. Therefore, the example that Chua provides cannot occur for larger $\ell$.

2. Introduction to lattices and theta functions

Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. A lattice $\Lambda$ over $K$ is an $\mathcal{O}_K$-submodule of $K^n$ of full rank. The Hermitian dual is defined by

\[
\Lambda^* = \{ x \in K^n \mid x \cdot \bar{y} \in \mathcal{O}_K, \text{ for all } y \in \Lambda \},
\]

where $x \cdot y := \sum_{i=1}^n x_i y_i$. In the case that $\Lambda$ is a free $\mathcal{O}_K$-module, for every $\mathcal{O}_K$ basis $\{v_1, v_2, ..., v_n\}$ we can associate a Gram matrix $G(\Lambda)$ given by $G(\Lambda) = (v_i \cdot v_j)_{i,j=1}^n$ and the determinant $\det \Lambda := \det(G)$ defined up to squares of units in $\mathcal{O}_K$. If $\Lambda = \Lambda^*$, then $\Lambda$ is Hermitian self-dual (or unimodular) and integral if and only if $\Lambda \subset \Lambda^*$. An integral lattice has the property $\Lambda \subset \Lambda^* \subset \frac{1}{\det \Lambda} \Lambda$. An integral lattice is called even if $x \cdot x \equiv 0 \mod 2$ for all $x \in \Lambda$, and otherwise it is odd. An odd unimodular lattice is called a Type 1 lattice, and an even unimodular lattice is called a Type 2 lattice.

The theta series of a lattice $\Lambda$ in $K^n$ is given by $\Theta_\Lambda(\tau) = \sum_{z \in \Lambda} e^{\pi i \tau z \bar{z}}$, where $\tau \in H = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. Usually we let $q = e^{\pi i \tau}$. Then, $\Theta_\Lambda(q) = \sum_{z \in \Lambda} q^{z \bar{z}}$. 

The 1-dimensional theta series and its shadow are given by

\[ \theta_3(q) := \sum_{m \in \mathbb{Z}} q^{m^2}, \quad \theta_2(q) := \sum_{m \in \mathbb{Z} + 1/2} q^{m^2}. \]

Let \( \ell > 0 \) be a square free integer and \( K = \mathbb{Q}(\sqrt{-\ell}) \) be the imaginary quadratic field with discriminant \( d_K \). Recall that \( d_K = -\ell \) if \( \ell \equiv 3 \mod 4 \) and \( d_K = -4\ell \) otherwise.

Let \( \mathcal{O}_K \) be the ring of integers of \( K \). The Hermitian lattice \( \Lambda \) over \( K \) is an \( \mathcal{O}_K \)-submodule of \( K^n \) of full rank. Let \( \ell \equiv 3 \mod 4 \) and let \( d \) be a positive number such that \( \ell = 4d - 1 \). Then, \( -\ell \equiv 1 \mod 4 \). This implies that the ring of integers \( \mathcal{O}_K = \mathbb{Z}[\omega] \), where \( \omega = \frac{-1 + \sqrt{-\ell}}{2} \) and \( \omega^2 + \omega + d = 0 \). The principal norm form of \( K \) is given by \( Q_d(x, y) = |x - y\omega|^2 = x^2 + xy + dy^2 \). Since \( \ell \equiv 3 \mod 4 \), we can consider two cases:

(1) If \( \ell \equiv 3 \mod 8 \), then \( -\ell \equiv 5 \mod 8 \). Thus, the prime ideal \( (2) \subset \mathbb{Z} \) lifts to a prime \( 2\mathcal{O}_K \subset \mathcal{O}_K \). Since the ring of integers \( \mathcal{O}_K \) is a Dedekind domain, \( 2\mathcal{O}_K \) is a maximal ideal. Therefore \( \mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_4 \).

(2) If \( \ell \equiv 7 \mod 8 \), then \( -\ell \equiv 1 \mod 8 \). Then the prime ideal \( (2) \subset \mathbb{Z} \) splits in \( K \). Therefore \( 2\mathcal{O}_K \) splits in \( \mathcal{O}_K \). Hence, \( \mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_2 \times \mathbb{F}_2 \). In either case, a complete set of coset representatives is \( \{0, 1, \omega, 1 + \omega\} \).

Let the following be the bi-dimensional theta series for the four cosets:

\[
\begin{align*}
A_d(q) &:= \Theta_{2\mathcal{O}_K}(\tau) = \sum_{m, n \in \mathbb{Z}} q^{4Q_d(m, n)}, \\
C_d(q) &:= \Theta_{1+2\mathcal{O}_K}(\tau) = \sum_{m, n \in \mathbb{Z}} q^{4Q_d(m + \frac{1}{2}, n)}, \\
G_d(q) &:= \Theta_{\omega+2\mathcal{O}_K}(\tau) = \sum_{m, n \in \mathbb{Z}} q^{4Q_d(m, n + \frac{1}{2})}, \\
H_d(q) &:= \Theta_{1+\omega+2\mathcal{O}_K}(\tau) = \sum_{m, n \in \mathbb{Z}} q^{4Q_d(m + \frac{1}{2}, n + \frac{1}{2})}.
\end{align*}
\]

Then we have the following lemma.

**Lemma 1.** Bi-dimensional theta series can be further expressed in terms of the standard one-dimensional theta series and its shadow:

\[
\begin{align*}
A_d(q) &= \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell}), \\
C_d(q) &= \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}), \\
G_d(q) &= H_d(q) = \frac{\theta_2(q)\theta_2(q^{\ell})}{2}.
\end{align*}
\]

Moreover,

\[
2G_d(q) = A_d(q^{1/4}) - A_d(q) - C_d(q).
\]

**Proof.** See [3] for details. \( \square \)

3. Codes over \( \mathbb{F}_4 \) and \( \mathbb{F}_2 \times \mathbb{F}_2 \)

Let \( \mathbb{F}_4 = \{0, 1, \omega, \omega^2\} \), where \( \omega^2 + \omega + 1 = 0 \), be the finite field of four elements. The conjugation is given by \( \bar{x} = x^2 \), \( x \in \mathbb{F}_4 \). In particular \( \bar{\omega} = \omega^2 = \omega + 1 \). Let \( R_4 = \mathbb{F}_2 + \omega\mathbb{F}_2 \) where the new equation for \( \omega \) being \( \omega^2 + \omega = 0 \). Notice that \( R_4 \) has two maximal ideals, namely \( \langle \omega \rangle \) and \( \langle \omega + 1 \rangle \). Furthermore, one can show that
$R_4/(\omega)$ and $R_4/(\omega + 1)$ are both isomorphic to $\mathbb{F}_2$. The Chinese remainder theorem tells us that $R_4 = \langle \omega \rangle \oplus \langle \omega + 1 \rangle$. Therefore, $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$. The conjugate of $\omega$ is $\omega + 1$. Let $\mathcal{R}$ be the field $\mathbb{F}_4$ if $\ell \equiv 3 \mod 8$ or the ring $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$ when $\ell \equiv 7 \mod 8$. A linear code $C$ of length $n$ over $\mathcal{R}$ is an $\mathcal{R}$-submodule of $\mathcal{R}^n$. The dual is defined as $C^\perp = \{ u \in \mathcal{R} : u \cdot v = 0 \text{ for all } v \in C \}$. If $C = C^\perp$, then $C$ is self-dual.

We define $\Lambda_\ell(C) := \{ x \in \mathcal{O}_k^\ell : \rho_\ell(x) \in C \}$ where $\rho_\ell : \mathcal{O}_k \rightarrow \mathcal{O}_K/2\mathcal{O}_k \rightarrow \mathcal{R}$. In other words, $\Lambda_\ell(C)$ consists of all vectors in $\mathcal{O}_K^n$ which when taken mod $2\mathcal{O}_K$ componentwise are in $\rho_\ell^{-1}(C)$. The following is immediate.

**Lemma 2.**

1. $\Lambda_\ell(C)$ is an $\mathcal{O}_K$-lattice.
2. $\Lambda_\ell(C^\perp) = 2\Lambda_\ell(C)^*$. 
3. $C$ is self-dual if and only if $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ is self-dual.

Let $u = (u_1, u_2, \ldots, u_n) \in \mathcal{R}^n$ be a codeword and $\alpha \in \mathcal{R}$. Then the counting function $n_\alpha(u)$ is defined as the number of elements in the set $\{ j : u_j = \alpha \}$. For a code $C$ we define the complete weight enumerator (cwe), symmetrized weight enumerator (swe) and Hamming weight enumerator (W) to be

\[ \text{cwe}_C(X, Y, Z, W) := \sum_{u \in C} X^{n_0(u)} Y^{n_1(u)} Z^{n_2(u)} W^{n_3(u)}, \]

\[ \text{swe}_C(X, Y, Z) := \text{cwe}_C(X, Y, Z, Z), \]

\[ W_C(X, Y) := \text{swe}_C(X, Y, Y). \]

Then we have the following.

**Proposition 1.** Let $\ell \equiv 3 \mod 4$, $C$ be a linear code over $\mathcal{R}$, and $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ be a Hermitian lattice constructed via the construction $A$. Then

\[ \theta_{\Lambda_\ell(C)}(\tau) = \text{swe}_C(A_d(q), C_d(q), G_d(q)) \]

where $A_d(q), C_d(q)$, and $G_d(q)$ are given as in (2.4).

For a proof of the above statement the reader can see [1]. From the definition of a one-dimensional theta series we have

\[ \theta_2(q) = 2q^{1/4} \sum_{i \in S} q^i, \quad \theta_3(q^4) = 2q \sum_{i: \text{odd}} q^{i^2-1}, \quad \theta_4(q^4) = 1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}, \]

where $S = \{ j^2-4 : j \equiv 1 \mod 2 \}$. From (2.4) we can write

\[ G_d(q) = \frac{\theta_2(q)\theta_2(q^4)}{2} = q^{\frac{\ell+1}{2}} \alpha_1, \]

where $\alpha_1 = \sum_{i \in S} q^i \sum_{j \in S} q^{ij}$. Then,

\[ A_d(q) = \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell}) \]

\[ = (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)})(1 + 2q^{4\ell} \sum_{j \in \mathbb{Z}^+} q^{4\ell(j^2-1)}) \]

\[ + 4q^{\ell+1} \sum_{i: \text{odd}} q^{i^2-1} \sum_{j: \text{odd}} q^{j^2-1}\ell \]

\[ = \alpha_2 + q^{\ell+1} \alpha_3 + q^{4\ell} \alpha_4, \]
where $\alpha_2, \alpha_3$ and $\alpha_4$ have the following forms:

\[
\alpha_2 = 1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)},
\]

\[
\alpha_3 = 4 \sum_{i: \text{odd}} q^{i^2-1} \sum_{j: \text{odd}} q^{j^2-1} \ell,
\]

\[
\alpha_4 = 2 \sum_{j \in \mathbb{Z}^+} q^{4(i^2-1)} (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}).
\]

Furthermore,

\[
C_d(q) = \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}) = 2q \sum_{i: \text{odd}} q^{i^2-1} (1 + 2q^{4\ell} \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)})
\]

\[
+ (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}) q^{\ell} \sum_{i: \text{odd}} q^{i^2-1} \ell
\]

\[
= \alpha_5 + q^{\ell+1} \alpha_6 + q^{4\ell+1} \alpha_7,
\]

where $\alpha_5, \alpha_6$ and $\alpha_7$ have the following forms:

\[
\alpha_5 = 2 \sum_{i: \text{odd}} q^{i^2-1},
\]

\[
\alpha_6 = 2 \sum_{j: \text{odd}} q^{(j^2-1)\ell} (1 + 2q^4 \sum_{i \in \mathbb{Z}^+} q^{4(i^2-1)}),
\]

\[
\alpha_7 = 4 \sum_{i: \text{odd}} q^{i^2-1} \sum_{j \in \mathbb{Z}^+} q^{4\ell(j^2-1)}.
\]

The next result shows that for large enough admissible $\ell$ and $\ell'$ the theta functions $\Theta_{\Lambda_{\ell}}(C)$ and $\Theta_{\Lambda_{\ell'}}(C)$ are virtually the same.

**Theorem 3.** Let $C$ be a code defined over $R$. For all admissible $\ell, \ell'$ such that $\ell > \ell'$, the following holds:

\[
(3.3) \quad \Theta_{\Lambda_{\ell}}(C) = \Theta_{\Lambda_{\ell'}}(C) + O(q^{\ell' + 1}).
\]

**Proof.** Let

\[
s\text{we}_C(X, Y, Z) = \sum_{i+j+k=n} a_{i,j,k} \cdot X^i Y^j Z^k
\]

be a degree $n$ polynomial. Write this as a polynomial in $Z$. Then

\[
s\text{we}_C(Z) = \sum_{k=0}^{n} L_k Z^k = L_0 + Z(\sum_{k=1}^{n} L_k Z^{k-1}).
\]

Terms in $L_0$ are of the form of $a_{i,j} X^i Y^j$, where $i + j = n$. From the above we have

\[
A_d(q)^i \cdot C_d(q)^j = (\alpha_2 + q^{\ell+1} \alpha_3 + q^{4\ell} \alpha_4)^i \cdot (\alpha_5 + q^{\ell} \alpha_6 + q^{4\ell+1} \alpha_7)^j
\]

\[
= \text{(terms independent from $\ell$)} + q^j(\cdots).
\]

Also we have seen that $G_d(q) = q^{(\ell+1)/4} \alpha_1$. This gives

\[
\Theta_{\Lambda_{\ell}}(C) = s\text{we}_C(A_d(q), C_d(q), G_d(q))
\]

\[
= \text{(terms independent from $\ell$)} + O(q^{\ell'/4}).
\]
Then the result follows. □

**Example 1.** Let $C$ be a code defined over $R_4$ that has symmetrized weight enumerator

$$sw_{\mathcal{C}}(X, Y, Z) = X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3.$$ 

Then we have the following:

$$\Theta_{\Lambda_63}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 12q^{16} + 6q^{18} + 48q^{20} + 30q^{22} + \cdots,$$

(3.4) \quad $$\Theta_{\Lambda_{79}}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 6q^{16} + 30q^{20} + 6q^{22} + 48q^{24} + \cdots,$$

$$\Theta_{\Lambda_{79}}(C) = \theta_{\Lambda_{63}}(C) + \mathcal{O}(q^{16}).$$

4. **A FAMILY OF CODES CORRESPONDING TO THE SAME THETA FUNCTION**

If we are given the code over $\mathcal{R}$ and its symmetrized weight enumerator polynomial, then by (3.2) we can find the theta function of the lattice constructed from the code by using the construction $A$. Now, we would like to give a way to construct families of codes corresponding to the same theta function.

Let $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be an acceptable theta series for level $\ell$ and

$$f(x, y, z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree $n$ generic ternary homogeneous polynomial. We want to find out how many polynomials $f(x, y, z)$ correspond to $\Theta(q)$ for a fixed $\ell$.

We have the following lemma.

**Lemma 3.** Let $C$ be a code of size $n$ defined over $\mathcal{R}$ and $\Theta(q)$ be its theta function for level $\ell$. Then, $\Theta(q)$ is uniquely determined by its first $\frac{n(\ell+1)}{4}$ coefficients.

**Proof.** Let $C$ be a code over $\mathcal{R}$, $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be its theta series, $s = \frac{n(\ell+1)}{4}$ and

$$f(x, y, z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree $n$ generic ternary homogeneous polynomial. Find $A_d(q), C_d(q), G_d(q)$ for the given $\ell$ and substitute it in $f(x, y, z)$. Hence $f(x, y, z)$ is now written as a series in $q$. Recall that a generic degree $n$ ternary polynomial has $r = \frac{(n+1)(n+2)}{2}$ coefficients. So, the corresponding coefficients of the two sides of the equation are equal:

$$f(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i.$$

Consider the term

$$c_{i,j,k} (\alpha_2 + q^{\ell+1} \alpha_3 + q^{4\ell} \alpha_4)^i (\alpha_5 + q^{\ell} \alpha_6 + q^{4\ell+1} \alpha_7)^j (q^{\frac{\ell(\ell+1)}{4}} \alpha_1)^k.$$

Then $c_{i,j,k}$ appears first as a coefficient of $q^{j + \frac{k(\ell+1)}{4}}$. For all such $j, k$ we have $j + \frac{k(\ell+1)}{4} \leq \frac{n(\ell+1)}{4}$. Consider the equations where $c_{i,j,k}$ appears first. This is a system of equations with $\leq \frac{(n+1)(n+2)}{2}$ equations. Let us denote this system of equations as $\Xi$. Solve this system for $c_{i,j,k}$. Hence, $c_{i,j,k}$ is a function of $\lambda_0, \ldots, \lambda_s$. For each $\mu > s$, $\lambda_\mu$ is a function of $c_{i,j,k}$ for $i, j, k = 0, \ldots, n$, and therefore a rational function on $\lambda_0, \ldots, \lambda_s$. This completes the proof. □
Next we have the following theorem:

**Theorem 2.** Let $C$ be a code of size $n$ defined over $R$ and $\Theta_{\Lambda_\ell}(C)$ be its corresponding theta function for level $\ell$. Then the following hold:

i) For $\ell < \frac{2(n+1)(n+2)}{n} - 1$, there is a $\delta$-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{2} - \frac{n((\ell+1))}{4} - 1$.

ii) For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$, there is a unique symmetrized weight enumerator polynomial that corresponds to $\Theta_{\Lambda_\ell}(C)$.

**Proof.** We want to find out how many polynomials $f(x, y, z)$ correspond to $\Theta_{\Lambda_\ell}(C)$ for a fixed $\ell$. $\Theta_{\Lambda_\ell}(C)$ and $f(x, y, z)$ are defined as above. Consider the system of equations $\Xi$.

If $\frac{n(\ell+1)}{4} < r$, then our system has more variables than equations. Since the system is linear, the solution space is a family of positive dimension.

If $\frac{n(\ell+1)}{4} \geq r$, then for each equation in $\Xi$ (see the proof of the previous lemma) we have only one $c_{i,j,k}$ appearing for the first time. Otherwise suppose $c_{i,j,k}$ and $c_{i',j',k'}$ appear for the first time in an equation of $\Xi$. Then $j + \frac{k(\ell+1)}{4} = j' + \frac{k'(\ell+1)}{4}$. This implies

\[ 4(j - j') = (k' - k)(\ell + 1). \]

Without loss of generality, assume $k' \geq k$. We can consider three cases.

Case 1: If $k' - k \geq 2$, then from (4.1) we have $4n(j - j') = n(k' - k)(\ell + 1) \geq 4r(k' - k)$. Then we have $n(j - j') \geq (n + 1)(n + 2)$. Since $n \geq (j - j')$, we have a contradiction.

Case 2: If $k' - k = 1$, then by (4.1), $j - j' = \frac{\ell+1}{4}$. Since $j - j' \leq n$ and $\frac{\ell+1}{4} > n$, we get a contradiction.

Case 3: If $k' - k = 0$, then by (4.1) we have $j = j'$. Hence $i = i'$.

Notice that $c_{n,0,0}$ is uniquely determined by the equation corresponding to the equation of the coefficient of $q^0$. Solve the system $\Xi$ in the order of the equation that corresponds to the power of $q$. We have a unique solution for $c_{i,j,k}$. \hfill \Box

### 4.1 Families of codes of length 3

In this section we discuss the codes of length 3 for different levels $\ell$. Our main goal is to investigate the example provided in [1] and provide some computational evidence for the above two cases. We assume that the symmetrized weight enumerator polynomial is a generic homogenous polynomial of degree three.

Let $P(x, y, z)$ be a generic ternary cubic homogeneous polynomial given as below:

\[ P(x, y, z) = c_1 x^3 + c_2 y^3 + c_3 z^3 + c_4 x^2 y + c_5 x^2 z + c_6 y^2 x + c_7 y^2 z + c_8 z^2 x + c_9 z^2 y + c_{10} xyz. \]

Assume that there is a code $C$, of length 3, defined over $R$ such that $swe_C(x, y, z) = P(x, y, z)$. First we have to fix the level $\ell$. When we fix the level, we can find $A_d(q), C_d(q), G_d(q)$. By equating both sides of

\[ p(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i, \]
we can get a system of equations. When $\ell = 7$, we are in the first case of the previous theorem. The system of equations is given by the following:

$$
\begin{align*}
&c_1 - \lambda_0 = 0, \\
&2c_4 - \lambda_1 = 0, \\
&4c_6 + 2c_5 - \lambda_2 = 0, \\
&8c_2 + 4c_{10} - \lambda_3 = 0
\end{align*}
$$

The solution for the above system is given by $c_1 = \lambda_0$, $c_4 = \frac{1}{2}\lambda_1$, and

$$
\begin{align*}
c_2 &= \frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_3 - \frac{1}{8}\lambda_5 + c_9, \\
c_3 &= \frac{3}{2}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + \frac{1}{8}\lambda_6 + c_7, \\
c_5 &= -3\lambda_0 + \frac{1}{2}\lambda_4 - 4c_7 - 2c_8, \\
c_6 &= \frac{3}{2}\lambda_0 + \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + 2c_7 + c_8, \\
c_{10} &= -\lambda_1 + \frac{1}{4}\lambda_5 - 2c_9
\end{align*}
$$

where $c_7, c_8, c_9$ are free variables. By giving different triples $(c_7, c_8, c_9)$, we can construct different polynomials $P(x, y, z)$ for the same $\sum_{i=0}^{\infty} \lambda_i q^i$.

Consider the following theta function. From [1] there are two nonisomorphic codes that give this theta function for level $\ell = 7$:

$$
\theta_{\sqrt{2}\mathcal{K}_3^3} = 1 + 6q^2 + 24q^4 + 56q^6 + 114q^8 + 168q^{10} + 280q^{12} + 294q^{14} + \cdots
$$

For this particular theta function, we can rewrite the solution (Eq. (4.4)) as follows: $c_1 = 1, c_2 = c_9, c_3 = 1 + c_7, c_4 = 0, c_5 = 9 - 4c_7 - 2c_8, c_6 = -3 - 2c_7 + c_8, c_{10} = -2c_9$.

For the triple $(1, 2, 0)$ (resp., $(0, 3, 0)$) we get the symmetrized weight enumerator polynomial for the code $C_{3,2}$ (resp. $C_{3,3}$). That is, $swe_{C_{3,2}}(X, Y, Z) = X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3$ (resp., $swe_{C_{3,3}}(X, Y, Z) = X^3 + 3X^2Z + 3XZ^2 + Z^3$), where $C_{3,2}$ and $C_{3,3}$ are given by

$$
\begin{align*}
C_{3,2} &= \omega([0,1,1]) + (\omega + 1)[[0,1,1]]^\perp, \\
C_{3,3} &= \omega([0,0,1]) + (\omega + 1)[[0,0,1]]^\perp.
\end{align*}
$$

When $\ell = 15$, we are in the second case of the above theorem. The system of equations is as follows:

$$
\begin{align*}
&c_1 - \lambda_0 = 0, \\
&2c_4 - \lambda_1 = 0, \\
&4c_6 - \lambda_2 = 0, \\
&8c_2 - \lambda_3 = 0, \\
&6c_1 + 2c_5 - \lambda_4 = 0, \\
&8c_4 + 4c_{10} - \lambda_5 = 0, \\
&2c_4 + 8c_7 + 8c_6 - \lambda_6 = 0, \\
&4c_8 + 8c_7 + 12c_1 + 8c_5 - \lambda_8 = 0, \\
&10c_4 + 8c_9 + 8c_{10} - \lambda_9 = 0, \\
&8c_7 + 8c_5 + 12c_8 + 8c_3 + 8c_1 - \lambda_{12} = 0.
\end{align*}
$$

Each $c_i$ appears first in exactly one equation. For example, consider the seventh equation. $c_7$ is the only variable that appears first in the seventh equation. Solve the system in the given order. The solution for the above system is given by:

$$
\begin{align*}
c_3 &= -\lambda_0 - \frac{1}{2}\lambda_2 + \frac{3}{4}\lambda_4 + \frac{1}{4}\lambda_6 - \frac{3}{8}\lambda_8 + \frac{1}{8}\lambda_{12}, \\
c_5 &= -3\lambda_0 + \frac{1}{2}\lambda_4, \\
c_7 &= \frac{3}{4}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{8}\lambda_4 + \frac{1}{8}\lambda_6, \\
c_8 &= \frac{3}{2}\lambda_0 + \frac{1}{2}\lambda_2 - \frac{3}{4}\lambda_4 - \frac{1}{4}\lambda_6 + \frac{1}{4}\lambda_8, \\
c_9 &= \frac{3}{8}\lambda_1 - \frac{1}{4}\lambda_5 + \frac{1}{8}\lambda_9, \\
c_{10} &= -\lambda_1 + \frac{1}{4}\lambda_5.
\end{align*}
$$
We have a unique solution. This implies that two nonequivalent codes cannot give the same theta function for $\ell = 15$ and $n = 3$.

5. Concluding remarks

The main goal of this paper was to find out how theta functions determine the codes over a ring of size 4. First we have shown how the theta functions of the same code $C$ differ for different levels $\ell$. The first $\frac{\ell + 1}{4}$ terms of the theta functions for levels $\ell$ and $\ell'$ are the same, where $\ell' \geq \ell$.

In [1], two nonisomorphic codes that give the same theta function for level $\ell = 7$ but not under higher level constructions are given. We justified the reason why we don’t have a similar situation for higher level constructions. In this note we have addressed a method that we can use for finding a family of polynomials that correspond to a given acceptable theta series for some fixed level $\ell$. We have studied two cases depending upon $\ell$ that give either a positive dimensional family of polynomials or a unique polynomial.

References


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