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A NOTE ON EQUILIBRIUM POINTS OF GREEN'S FUNCTION

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Abstract. We answer a question raised by Ahmet Sebbar and Thérèse Falliero (2007) by showing that for every finitely connected planar domain Ω there exists a compact subset $K \subset \Omega$, independent of w, containing all critical points of Green's function $G(z,w)$ of Ω with pole at $w \in \Omega$.

Let Ω be a domain on $\mathbb{C}$ bordered by $n \geq 2$ Jordan analytic curves and let $G(z,w)$ be Green's function of Ω with pole at $w \in \Omega$. For $w \in \Omega$, let $\mathbb{Z}_\Omega(w) = \{z \in \Omega : \frac{\partial}{\partial z} G(z,w) = 0\}$ be the set of critical (= equilibrium) points of $G(z,w)$. It is well known that for every $w \in \Omega$, $\#(\mathbb{Z}_\Omega(w)) = n-1$ counting multiplicity.

Theorem 1. *Let $\overline{\mathbb{Z}}_\Omega$ denote the closure of the set $\mathbb{Z}_\Omega = \bigcup_{w\in\Omega} \mathbb{Z}_\Omega(w)$. Then $\overline{\mathbb{Z}}_\Omega$ is a compact subset of Ω having at most $n-1$ connected components.*

Thus this theorem answers affirmatively a question raised by Ahmet Sebbar and Thérèse Falliero in [2, p. 314]. For the case when Ω is doubly-connected, this theorem was proved in [2] by direct computation involving an explicit expression of Green's function of a circular annulus.

To prove Theorem 1, we use the well-known Schiffer's formula linking Green's function with the Bergman kernel. The Bergman kernel and the adjoint Bergman kernel of Ω are defined as $K(z,w) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial \bar{w}} G(z,w)$ and $L(z,w) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial w} G(z,w)$, respectively. Necessary properties of the Bergman kernel can be found, for example, in [1]. It is convenient to use subscripts to denote differentiation so that $G_z(z,w) = \frac{\partial}{\partial z} G(z,w)$, $G_{z\bar{w}}(z,w) = \frac{\partial^2}{\partial z \partial \bar{w}} G(z,w)$, etc. To emphasize dependence on Ω if necessary, we will write $G_\Omega(z,w)$, $K_\Omega(z,w)$, etc.

Lemma 1. *Let Ω be a Dirichlet domain in the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$ having an open interval $I = (-1,1)$ on $\partial\Omega$ and such that $\widehat{\Omega} = \Omega \cup \Omega^* \cup I$ is a domain. Here $\Omega^* = \{z : \bar{z} \in \Omega\}$. Then*

$$K_\Omega(z,w) = K_{\widehat{\Omega}}(z,w) - L_{\widehat{\Omega}}(z,\bar{w}), \qquad z \in \Omega,\ w \in \Omega \cup I,\ z \neq w, \tag{1}$$

and

$$\lim_{\mathbb{H}\ni w \to 0} \frac{\frac{\partial}{\partial z} G_\Omega(z,w)}{\Im w} = \pi i K_\Omega(z,0) \tag{2}$$

uniformly on compact subsets of $\Omega \cup I \setminus \{0\}$.

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Proof. Since

$$G_\Omega(z,w) = G_{\widehat{\Omega}}(z,w) - G_{\widehat{\Omega}}(z,\bar{w}) \tag{3}$$

for all $z, w \in \Omega$ such that $z \neq w$, (1) follows from (3) after differentiation.

Let $\widehat{G}(z,w) = G_{\widehat{\Omega}}(z,w)$. Since $\widehat{G}(z,w)$ is harmonic in each variable if $z \neq w$, the function $\widehat{G}_z(z,w)$ has the following Taylor expansion at $w = 0$ if $z \neq 0$:

$$\widehat{G}_z(z,w) = G_z(z,0) + G_{zw}(z,0)w + G_{z\bar{w}}(z,0)\bar{w} + \text{higher powers of } w \text{ and } \bar{w}. \tag{4}$$

Using (3), (4), and (1), we find that

$$\begin{aligned} \lim_{\mathbb{H}\ni w\to 0} \frac{\frac{\partial}{\partial z} G_\Omega(z,w)}{\Im w} &= \lim_{w\to 0} \frac{\widehat{G}_z(z,w) - \widehat{G}_z(z,\bar{w})}{\Im w} = -2i(\widehat{G}_{z\bar{w}}(z,0) - \widehat{G}_{zw}(z,0)) \\ &= \pi i(K_{\widehat{\Omega}}(z,0) - L_{\widehat{\Omega}}(z,0)) = \pi i K_\Omega(z,0) \end{aligned} \tag{5}$$

and the limit is uniform on compact subsets of $\widehat{\Omega} \setminus \{0\}$. □

Proof of Theorem 1. Using Hurwitz's theorem one can easily prove that the set $\mathbb{Z}_\Omega(w)$ depends continuously on $w \in \Omega$. Since Ω is open and connected and for every $w \in \Omega$, $\sharp(\mathbb{Z}_\Omega(w)) = n-1$ counting multiplicity the latter implies that $\mathbb{Z}_\Omega$ has at most $n-1$ connected components.

The proof of $\overline{\mathbb{Z}}_\Omega \subset \Omega$ is by contradiction. Suppose there is a sequence $w_k \in \Omega$ with $w_k \to w_0$ as $k \to \infty$ such that there exists a sequence $z_k \in \Omega$ with $z_k \to z_0 \in \partial\Omega$ such that $G_z(z_k, w_k) = 0$. Now we consider two cases.

(1) If $w_0 \in \Omega$, then let $z_1^0, \ldots, z_{n-1}^0$ be zeros of $G_z(z, w_0)$ counting multiplicity. Let $\varepsilon > 0$ be sufficiently small. By Hurwitz's theorem, there is a positive integer N such that for all $k \geq N$ the set $\bigcup_{j=1}^{n-1}\{z : |z - z_j^0| < \varepsilon\}$ contains exactly $n-1$ zeros of $G_z(z, w_k)$ counting multiplicity. Since $\#(\mathbb{Z}_\Omega(w_k)) = n-1$ and $z_j^0 \in \Omega$ for all $j = 1, \ldots, n-1$, the latter contradicts the assumption that $G_z(z, w_k)$ has a zero, say z_k^1, such that $z_k^1 \to z_0 \in \partial\Omega$ as $k \to \infty$.

(2) Suppose now that $w_0 \in \partial\Omega$. Since Green's function and the Bergman kernel are conformally invariant, we may use Koebe's theorem on the conformal mapping onto a circular domain to reduce the problem to the case of domain Ω in $\mathbb{H}$ bounded by the real axis $\mathbb{R}$ and $n-1$ disjoint circles in $\mathbb{H}$. In addition, we may assume that $w_0 = 0$. Let $f_k(z) = G_z(z, w_k)/(\Im w_k)$. By equation (2) of Lemma 1, $f_k(z) \to K_\Omega(z, 0)$ uniformly on compact subsets of Ω.

By a theorem of N. Suita and A. Yamada in [3], if $0 \in \partial\Omega$ the Bergman kernel $K_\Omega(z,0)$ has exactly $n-1$ zeros, say $z_1^0, \ldots, z_{n-1}^0$, in Ω counting multiplicity. Applying Hurwitz's theorem to the sequence $f_k(z)$ as in **(1)**, we conclude that for a given sufficiently small $\varepsilon > 0$ there is a positive integer N such that for all $k \geq N$ the set $\bigcup_{j=1}^{n-1}\{z : |z - z_j^0| < \varepsilon\}$ contains exactly $n-1$ zeros of $f_k(z)$ (which coincide with zeros of $G_z(z, w_k)$) counting multiplicity. Since $\#(\mathbb{Z}_\Omega(w_k)) = n-1$ and $z_j^0 \in \Omega$ for all $j = 1, \ldots, n-1$, the latter contradicts the assumption that $G_z(z, w_k)$ has a zero, say z_k^1, such that $z_k^1 \to z_0 \in \partial\Omega$ as $k \to \infty$. The theorem is proved. □

Remarks. **(1)** If Ω is infinitely connected, then $\mathbb{Z}_\Omega(w) = \{z_k(w)\}_{k=1}^\infty$ is infinite for every $w \in \Omega$. Since $G_z(z,w)$ is not constant, the uniqueness theorem for analytic functions implies that $z_k(w) \to \partial\Omega$ as $k \to \infty$. Thus, Theorem 1 fails for every infinitely connected domain.

(2) We finish this note with a remark on the boundary of $\mathbb{Z}_\Omega$. Let $z_0 \in \partial\mathbb{Z}_\Omega$. If, in addition, $z_0 \in \mathbb{Z}_\Omega$, then $G_z(z_0, w_0) = 0$ for some $w_0 \in \Omega$, $z_0 \neq w_0$. Let $\bigtriangledown_w G_z(z, w)$ denote the determinant of the Jacobian matrix of $G_z(z, w)$ in the variable w. Then

$$\bigtriangledown_w G_z(z, w) = \frac{\pi^2}{4}\left(|L_\Omega(z, w)|^2 - |K_\Omega(z, w)|^2\right).$$

By the implicit function theorem, if $|K_\Omega(z_0, w_0)| \neq |L_\Omega(z_0, w_0)|$, then z_0 is an interior point of $\mathbb{Z}_\Omega$. Therefore for $z_0 \in \partial\mathbb{Z}_\Omega$, there are two possibilities: 1) either $K_\Omega(z_0, w_0) = 0$ for some $w_0 \in \partial\Omega$ or 2) $|K_\Omega(z_0, w_1)| = |L_\Omega(z_0, w_1)|$ for every $w_1 \in \Omega$ such that $G_z(z_0, w_1) = 0$. The latter possibility seems unlikely, but we could not exclude this case.

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