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MEAN VALUE OF MIXED EXPONENTIAL SUMS

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ABSTRACT. For integers q, m, n, k with $q,k \geq 1$, and Dirichlet character χ mod q, we define a mixed exponential sum

$$C(m,n,k,\chi;q) := \sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+na}{q}\right),$$

where $\mathrm{e}(y)=\mathrm{e}^{2\pi iy}$, and $\sum_a'$ denotes the summation over all a with $(a,q)=1$. The main purpose of this paper is to study the mean value of

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}{}' |C(m,n,k,\chi;q)|^4,$$

and to give a related identity on the mean value of the general Kloosterman sum

$$K(m,n,\chi;q) := \sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma+n\overline{a}}{q}\right),$$

where $a\overline{a} \equiv 1 \bmod q$.

1. INTRODUCTION

For integers q, m, n, k with $q,k \geq 1$, and Dirichlet character χ mod q, we define a mixed exponential sum

$$C(m,n,k,\chi;q) := \sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+na}{q}\right),$$

where $\mathrm{e}(y)=\mathrm{e}^{2\pi iy}$, and $\sum_a'$ denotes the summation over all a with $(a,q)=1$.

If $q=p$ is a prime, it follows from A. Weil's work [12] that for all χ mod p, and $p \nmid m$,

$$|C(m,n,k,\chi;p)| \leq kp^{\frac{1}{2}}.$$

Let $q=p^\alpha$. Using the method established in [3] and [4], T. Cochrane and Z. Zheng [5] obtained the following bound for $C(m,n,k,\chi;p^\alpha)$.

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Proposition 1.1. *Let $k \geq 2$ and $\alpha \geq 1$, and let m, n be any integers with $p \nmid m$. If $p > 2$, then for all $\chi \bmod p^\alpha$, we have $|C(m,n,k,\chi;p^\alpha)| \leq kp^{\frac{2}{3}\alpha}(n,p^\alpha)^{\frac{1}{3}}$. If $p = 2$, then for all $\chi \bmod 2^\alpha$, we have $|C(m,n,k,\chi;2^\alpha)| \leq 2k2^{\frac{2}{3}\alpha}(n,2^\alpha)^{\frac{1}{3}}$.*

Moreover, for $\alpha \geq 2$ and $k \geq 2$, T. Cochrane and Z. Zheng [6] proved various upper bounds for $C(m,n,k,\chi;p^\alpha)$. Specifically, it was shown that if χ is primitive or $(p,m,n) = 1$, then one has $|C(m,n,k,\chi;p^\alpha)| \leq 2kp^{\frac{2}{3}\alpha}$. If χ has conductor p and $(p,m,n) = 1$, then the stronger bound $|C(m,n,k,\chi;p^\alpha)| \leq kp^{\frac{2}{3}\alpha}$ was given.

Furthermore, define

$$S := \sideset{}{'}\sum_{a=1}^{p^\alpha} \chi(g(a))\,\mathrm{e}\left(\frac{f(a)}{p^\alpha}\right); \qquad S(\chi,h) := \sum_{a=1}^{p-1} \chi(a)\mathrm{e}\left(\frac{h(a)}{p}\right);$$
$$S(\Psi,f;\chi,g) := \sum_{a\in\mathbb{F}_q\setminus\mathcal{S}} \chi(g(a))\,\Psi(f(a)),$$

where $\alpha \geq 2$, f and g are rational functions with integer coefficients, h is a Laurent polynomial, and Ψ is a non-trivial additive character on a finite field $\mathbb{F}_q$ of characteristic p. T. Cochrane (partly with other coauthors) obtained upper bounds on the above mixed exponential sums step by step in [7]–[10].

It may be interesting to study the mean value of mixed exponential sums. The main purpose of this paper is to study the mean value of

$$\sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |C(m,n,k,\chi;q)|^4,$$

and give the following identity.

Theorem 1.1. *Let q, k be positive integers with $(k,q) = 1$. Then for any integer n with $(n,q) = 1$, we have*

$$\begin{aligned}&\sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |C(m,n,k,\chi;q)|^4 \\ &= q^2\phi^2(q) \prod_{\substack{p^\alpha \| q \\ \alpha\geq 2}} \left(\alpha + 1 - \frac{\alpha+1+2(k,p-1)}{p}\right) \\ &\quad \times \prod_{p\|q} \left(2 - \frac{2\left((k,p-1)+1\right)}{p} - \frac{1}{p^2} + \frac{(k,p-1)^2}{p(p-1)}\right),\end{aligned}$$

where $\phi(q)$ is the Euler function, and $\prod_{p^\alpha\|q}$ denotes the product over all prime divisors p of q with $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

We immediately get the following two corollaries.

Corollary 1.1. *Let q be a square-full number and k a positive integer with $(k,q) = 1$. Then for any integer n with $(n,q) = 1$, we have*

$$\sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |C(m,n,k,\chi;q)|^4 = q^2\phi^2(q) \prod_{p^\alpha\|q} \left(\alpha + 1 - \frac{\alpha+1+2(k,p-1)}{p}\right).$$

Corollary 1.2. *Let q be a square-free number and k a positive integer with $(k,q)=1$. Then for any integer n with $(n,q)=1$, we have*

$$\sum_{\chi \bmod q}\sideset{}{'}\sum_{m=1}^{q}|C(m,n,k,\chi;q)|^4 = q^2\phi^2(q)\prod_{p|q}\left(2-\frac{2\left((k,p-1)+1\right)}{p}-\frac{1}{p^2}+\frac{(k,p-1)^2}{p(p-1)}\right).$$

Moreover, from Theorem 1.1 we can give some identities on the mean value of general Kloosterman sums. For arbitrary integers m and n, the general Kloosterman sum $K(m,n,\chi;q)$ is defined by

$$K(m,n,\chi;q) := \sideset{}{'}\sum_{a=1}^{q}\chi(a)\mathrm{e}\left(\frac{ma+n\overline{a}}{q}\right),$$

where $a\overline{a}\equiv 1 \bmod q$. For $q=p$ a prime, S. Chowla [2] and A. V. Malyshev [11] proved an upper bound for $K(m,n,\chi;p)$. For general integer $q>2$, and fixed integer n with $(n,q)=1$, W. Zhang [13] showed the identity

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}|K(m,n,\chi;q)|^4 = \phi^2(q)q^2d(q)\prod_{p^\alpha\|q}\left(1-\frac{2}{\alpha+1}\cdot\frac{p^{\alpha-1}-1}{p^\alpha(p-1)}+\frac{\alpha-4p^{\alpha-1}}{(\alpha+1)p^\alpha}\right),$$

where $d(q)$ is the divisor function.

Now taking $k\equiv -1(\bmod \phi(q))$ in Theorem 1.1, we have the following:

Corollary 1.3. *Let q be a positive integer and n an integer with $(n,q)=1$. Then we have*

$$\sum_{\chi \bmod q}\sideset{}{'}\sum_{m=1}^{q}|K(m,n,\chi;q)|^4 = q^2\phi^2(q)\prod_{\substack{p^\alpha\|q\\ \alpha\geq 2}}\left(\alpha+1-\frac{\alpha+3}{p}\right)\prod_{p\|q}\left(2-\frac{4}{p}-\frac{1}{p^2}+\frac{1}{p(p-1)}\right).$$

2. Some lemmas

To prove Theorem 1.1 and Corollary 1.3, we need the following lemmas.

Lemma 2.1. *Let k, q_1, q_2 be positive integers with $(q_1,q_2)=1$. Then*

$$|C(m,n,k,\chi;q_1q_2)| = \left|C(mq_2^{k-1},n,k,\chi_1;q_1)\right|\cdot\left|C(mq_1^{k-1},n,k,\chi_2;q_2)\right|,$$

where $\chi=\chi_1\chi_2 \bmod q_1q_2$ with $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$.

Proof. From the properties of reduced residue systems we have

$$C(m,n,k,\chi;q_1q_2) = \sum_{a=1}^{q_1q_2}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+na}{q_1q_2}\right)$$
$$= \sum_{b=1}^{q_1}{}'\sum_{c=1}^{q_2}{}' \chi_1\chi_2\left(bq_2+cq_1\right)\mathrm{e}\left(\frac{m(bq_2+cq_1)^k+n(bq_2+cq_1)}{q_1q_2}\right)$$
$$= \chi_1(q_2)\chi_2(q_1)\sum_{b=1}^{q_1}{}'\sum_{c=1}^{q_2}{}' \chi_1(b)\chi_2(c)\mathrm{e}\left(\frac{m\left((bq_2)^k+(cq_1)^k\right)+n(bq_2+cq_1)}{q_1q_2}\right)$$
$$= \chi_1(q_2)\chi_2(q_1)\sum_{b=1}^{q_1}{}' \chi_1(b)\mathrm{e}\left(\frac{mq_2^{k-1}b^k+nb}{q_1}\right)\sum_{c=1}^{q_2}{}' \chi_2(c)\mathrm{e}\left(\frac{mq_1^{k-1}c^k+nc}{q_2}\right)$$
$$= \chi_1(q_2)\chi_2(q_1)C(mq_2^{k-1},n,k,\chi_1;q_1)C(mq_1^{k-1},n,k,\chi_2;q_2).$$

Therefore,

$$|C(m,n,k,\chi;q_1q_2)| = \left|C(mq_2^{k-1},n,k,\chi_1;q_1)\right|\cdot\left|C(mq_1^{k-1},n,k,\chi_2;q_2)\right|. \qquad \square$$

Lemma 2.2. *Let q, k be positive integers and n_0 an integer with $(n_0,q)=1$. Then*

$$\sum_{m=1}^{q}{}' |C(m,n_0,k,\chi;q)|^4 = \frac{1}{\phi(q)}\sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}' |C(m,n,k,\chi;q)|^4.$$

Proof. From the properties of reduced residue systems we have

$$\phi(q)\sum_{m=1}^{q}{}' |C(m,n_0,k,\chi;q)|^4 = \phi(q)\sum_{m=1}^{q}{}'\left|\sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+n_0a}{q}\right)\right|^4$$
$$= \sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}'\left|\sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+n_0a}{q}\right)\right|^4$$
$$= \sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}'\left|\sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{m\overline{n}^ka^k+n_0a}{q}\right)\right|^4$$
$$= \sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}'\left|\chi(n)\sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+n_0na}{q}\right)\right|^4$$
$$= \sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}'\left|\sum_{a=1}^{q}{}' \chi(a)\mathrm{e}\left(\frac{ma^k+na}{q}\right)\right|^4$$
$$= \sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}' |C(m,n,k,\chi;q)|^4. \qquad \square$$

Lemma 2.3. *Let p be a prime, and let k and α be positive integers. Define*

$$\Psi_1(k,p^\alpha) := \sum_{\substack{a=1 \\ p^\alpha|(a^k-1)(b^k-1) \\ p^\alpha|(a-1)(b-1)}}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' .$$

Then we have

$$\Psi_1(k,p^\alpha)=(\alpha+1)\phi(p^\alpha)-p^{\alpha-1}.$$

Proof. Since $p^\alpha \mid (a-1)(b-1)$, then $p^\alpha \mid (a^k-1)(b^k-1)$. Therefore,

$$\begin{aligned}
\Psi_1(k,p^\alpha) &= \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a-1)(b-1)}}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} \\
&=\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha}\sideset{}{'}\sum_{\substack{a=1\\(a-1,p^\alpha)=p^\beta}}^{p^\alpha}\sideset{}{'}\sum_{\substack{b=1\\(b-1,p^\alpha)=p^\gamma\\ p^\alpha|(a-1)(b-1)}}^{p^\alpha} \\
&=\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha}\sideset{}{'}\sum_{\substack{u=1\\(up^\beta+1,p)=1}}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{\substack{v=1\\(vp^\gamma+1,p)=1\\ p^\alpha|uvp^{\beta+\gamma}}}^{p^{\alpha-\gamma}} \\
&=\sum_{\substack{\beta=0\\ \beta+\gamma\geq\alpha}}^{\alpha}\sum_{\gamma=0}^{\alpha}\sideset{}{'}\sum_{\substack{u=1\\(up^\beta+1,p)=1}}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{\substack{v=1\\(vp^\gamma+1,p)=1}}^{p^{\alpha-\gamma}} \\
&=\sum_{\substack{\beta=1\\ \beta+\gamma\geq\alpha}}^{\alpha-1}\sum_{\gamma=1}^{\alpha-1}\sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}}+2\sum_{\beta=0}^{\alpha-1}\sideset{}{'}\sum_{\substack{u=1\\(up^\beta+1,p)=1}}^{p^{\alpha-\beta}}+1 \\
&=\sum_{\beta=1}^{\alpha-1}\phi(p^{\alpha-\beta})\sum_{\gamma=\alpha-\beta}^{\alpha-1}\phi(p^{\alpha-\gamma})+2\left(\sideset{}{'}\sum_{\substack{u=1\\(u+1,p)=1}}^{p^\alpha}+\sum_{\beta=1}^{\alpha-1}\sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}}\right)+1 \\
&=\big((\alpha-1)\phi(p^\alpha)-p^{\alpha-1}+1\big)+2p^{\alpha-1}(p-2)+2(p^{\alpha-1}-1)+1 \\
&=(\alpha+1)\phi(p^\alpha)-p^{\alpha-1}.
\end{aligned}$$

□

Lemma 2.4. *Let p be a prime, and let k and α be positive integers with $(k,p)=1$. Define*

$$\Psi_2(k,p^\alpha):=\sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|(a^k-1)(b^k-1)\\ p^{\alpha-1}|(a-1)(b-1)}}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}.$$

Then we have

$$\Psi_2(k,p^\alpha)=\begin{cases}(\alpha-1+2(k,p-1))\,\phi(p^\alpha)-p^{\alpha-1}, & \alpha\geq 2;\\ 2(p-1)(k,p-1)-(k,p-1)^2, & \alpha=1.\end{cases}$$

Proof. First we consider the case $\alpha \geq 2$. We have

$$\begin{aligned}
\Psi_2(k,p^\alpha) &= \sideset{}{'}\sum_{\substack{a=1 \\ p^\alpha|(a^k-1)(b^k-1) \\ p^{\alpha-1}|(a-1)(b-1)}}^{p^\alpha} \sideset{}{'}\sum_{b=1}^{p^\alpha} = \sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha} \sideset{}{'}\sum_{\substack{a=1 \\ (a-1,p^\alpha)=p^\beta \\ p^\alpha|(a^k-1)(b^k-1) \\ p^{\alpha-1}|(a-1)(b-1)}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1 \\ (b-1,p^\alpha)=p^\gamma}}^{p^\alpha} \\
&= \sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ (up^\beta+1,p)=1 \\ p^\alpha|((up^\beta+1)^k-1)((vp^\gamma+1)^k-1) \\ p^{\alpha-1}|uvp^{\beta+\gamma}}}^{p^{\alpha-\beta}} \sideset{}{'}\sum_{\substack{v=1 \\ (vp^\gamma+1,p)=1}}^{p^{\alpha-\gamma}} \\
&= \sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ (up^\beta+1,p)=1 \\ p^\alpha|\sum_{i=1}^k \binom{k}{i}u^i \sum_{j=1}^k \binom{k}{j} v^j p^{\beta i+\gamma j} \\ p^{\alpha-1}|p^{\beta+\gamma}}}^{p^{\alpha-\beta}} \sideset{}{'}\sum_{\substack{v=1 \\ (vp^\gamma+1,p)=1}}^{p^{\alpha-\gamma}} .
\end{aligned}$$

Since $p^{\alpha-1} \mid p^{\beta+\gamma}$, then $p^\alpha \mid p^{\beta i+\gamma j}$ for $\beta,\gamma \geq 1$ and $i+j>2$. Therefore,

$$\Psi_2(k,p^\alpha) = \sum_{\beta=1}^{\alpha}\sum_{\gamma=1}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ p^\alpha|uvp^{\beta+\gamma} \\ p^{\alpha-1}|p^{\beta+\gamma}}}^{p^{\alpha-\beta}} \sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}} + 2\sum_{\gamma=1}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1 \\ p^\alpha|\sum_{i=1}^k \binom{k}{i}u^i \sum_{j=1}^k \binom{k}{j}v^j p^{\gamma j} \\ p^{\alpha-1}|p^\gamma}}^{p^\alpha} \sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}} := \Gamma_1 + \Gamma_2. \tag{2.1}$$

It is not hard to show that

$$\begin{aligned}
\Gamma_1 &= \sum_{\substack{\beta=1 \\ \beta+\gamma\geq\alpha}}^{\alpha}\sum_{\gamma=1}^{\alpha} \sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}} \sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}} = \sum_{\beta=1}^{\alpha-1} \phi(p^{\alpha-\beta}) \sum_{\gamma=\alpha-\beta}^{\alpha-1} \phi(p^{\alpha-\gamma}) + 2\sum_{\beta=1}^{\alpha-1} \sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}} +1 \\
&= (\alpha-1)\,\phi(p^\alpha) + p^{\alpha-1}. \qquad (2.2)
\end{aligned}$$

On the other hand, if $p^{\alpha-1} \mid p^\gamma$, then $p^\alpha \mid p^{\gamma j}$ for $j \geq 2$. Therefore,

$$\begin{aligned}
\Gamma_2 &= 2\sum_{\gamma=1}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1 \\ p^\alpha|((u+1)^k-1)\sum_{j=1}^k \binom{k}{j}v^j p^{\gamma j} \\ p^{\alpha-1}|p^\gamma}}^{p^\alpha} \sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}} = 2\sum_{\gamma=1}^{\alpha} \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1 \\ p^\alpha|((u+1)^k-1)kvp^\gamma \\ p^{\alpha-1}|p^\gamma}}^{p^\alpha} \sideset{}{'}\sum_{v=1}^{p^{\alpha-\gamma}} \\
&= 2 \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1 \\ p|(u+1)^k-1}}^{p^\alpha} \sum_{v=1}^{p-1} + 2 \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1}}^{p^\alpha} = 2(p-1) \sideset{}{'}\sum_{\substack{u=1 \\ (u+1,p)=1 \\ (u+1)^k\equiv 1 (\bmod p)}}^{p^\alpha} + 2p^{\alpha-1}(p-2).
\end{aligned}$$

Noting that

$$\sideset{}{'}\sum_{\substack{u=1\\(u+1,p)=1\\(u+1)^k\equiv 1(\bmod p)}}^{p^\alpha} = p^{\alpha-1}\sum_{\substack{u=1\\(u+1)^k\equiv 1(\bmod p)}}^{p-2} = p^{\alpha-1}\sum_{\substack{u=2\\u^k\equiv 1(\bmod p)}}^{p-1} = p^{\alpha-1}\left((k,p-1)-1\right),$$

then we have

$$\Gamma_2 = 2\phi(p^\alpha)\left((k,p-1)-1\right) + 2p^{\alpha-1}(p-2). \tag{2.3}$$

Now combining (2.1), (2.2) and (2.3) we immediately get

$$\Psi_2(k,p^\alpha) = (\alpha - 1 + 2(k,p-1))\,\phi(p^\alpha) - p^{\alpha-1}, \quad \text{if} \quad \alpha \geq 2.$$

For $\alpha = 1$, we easily get

$$\begin{aligned}\Psi_2(k,p) = \sum_{\substack{a=1\\p|(a^k-1)(b^k-1)}}^{p-1}\sum_{b=1}^{p-1} &= \sum_{\substack{a=1\\p|a^k-1}}^{p-1}\sum_{b=1}^{p-1} + \sum_{\substack{a=1\\p|b^k-1}}^{p-1}\sum_{b=1}^{p-1} - \sum_{\substack{a=1\\p|a^k-1\\p|b^k-1}}^{p-1}\sum_{b=1}^{p-1}\\ &= 2(p-1)(k,p-1) - (k,p-1)^2.\end{aligned}$$

□

Lemma 2.5. *Let p be a prime, and let k and α be positive integers. Define*

$$\Psi_3(k,p^\alpha) := \sideset{}{'}\sum_{\substack{a=1\\p^{\alpha-1}|(a^k-1)(b^k-1)\\p^\alpha|(a-1)(b-1)}}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} \quad \textit{and} \quad \Psi_4(k,p^\alpha) := \sideset{}{'}\sum_{\substack{a=1\\p^{\alpha-1}|(a^k-1)(b^k-1)\\p^{\alpha-1}|(a-1)(b-1)}}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}.$$

Then we have

$$\Psi_3(k,p^\alpha) = (\alpha+1)\phi(p^\alpha) - p^{\alpha-1} \tag{2.4}$$

and

$$\Psi_4(k,p^\alpha) = \begin{cases} \alpha\phi(p^{\alpha+1}) - p^\alpha, & \alpha \geq 2;\\ (p-1)^2, & \alpha = 1.\end{cases} \tag{2.5}$$

Proof. First we prove (2.4). Since $p^\alpha \mid (a-1)(b-1)$, then $p^{\alpha-1} \mid (a^k-1)(b^k-1)$. So from the proof of Lemma 2.3 we easily get

$$\Psi_3(k,p^\alpha) = \sideset{}{'}\sum_{\substack{a=1\\p^\alpha|(a-1)(b-1)}}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} = (\alpha+1)\phi(p^\alpha) - p^{\alpha-1}.$$

Now we prove (2.5). First we suppose that $\alpha \geq 2$. Then from the proof of Lemma 2.3 we have

$$\Psi_4(k,p^\alpha) = p^2 \sideset{}{'}\sum_{\substack{a=1\\p^{\alpha-1}|(a^k-1)(b^k-1)\\p^{\alpha-1}|(a-1)(b-1)}}^{p^{\alpha-1}}\sideset{}{'}\sum_{b=1}^{p^{\alpha-1}} = p^2\left(\alpha\phi(p^{\alpha-1}) - p^{\alpha-2}\right) = \alpha\phi(p^{\alpha+1}) - p^\alpha.$$

For $\alpha = 1$, we immediately get

$$\Psi_4(k,p) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} = (p-1)^2.$$

□

Lemma 2.6. *Let p be a prime, and let k and α be positive integers with $(k,p)=1$. Then for any integer n_0 with $(n_0,p)=1$, we have*

$$\sum_{\chi \bmod p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}|C(m,n_0,k,\chi;p^\alpha)|^4$$
$$=\begin{cases} p^{2\alpha}\phi^2(p^\alpha)\left(\alpha+1-\dfrac{\alpha+1+2(k,p-1)}{p}\right), & \alpha\geq 2;\\ p^2(p-1)^2\left(2-\dfrac{2\left((k,p-1)+1\right)}{p}-\dfrac{1}{p^2}+\dfrac{(k,p-1)^2}{p(p-1)}\right), & \alpha=1.\end{cases}$$

Proof. By the properties of reduced residue systems we have

$$\begin{aligned}|C(m,n,k,\chi;p^\alpha)|^2 &= \sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)\overline{\chi}(b)\mathrm{e}\left(\frac{m(a^k-b^k)+n(a-b)}{p^\alpha}\right)\\ &= \sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\chi(a)\mathrm{e}\left(\frac{mb^k(a^k-1)+nb(a-1)}{p^\alpha}\right).\end{aligned}$$

Then from Lemma 2.2 we get

$$\sum_{\chi \bmod p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}|C(m,n_0,k,\chi;p^\alpha)|^4=\frac{1}{\phi(p^\alpha)}\sum_{\chi \bmod p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}\sideset{}{'}\sum_{n=1}^{p^\alpha}|C(m,n,k,\chi;p^\alpha)|^4$$
$$=\frac{1}{\phi(p^\alpha)}\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sideset{}{'}\sum_{c=1}^{p^\alpha}\sideset{}{'}\sum_{d=1}^{p^\alpha}\sum_{\chi \bmod p^\alpha}\chi(a)\overline{\chi}(c)$$
$$\times\sideset{}{'}\sum_{m=1}^{p^\alpha}\mathrm{e}\left(\frac{m\left(b^k(a^k-1)-d^k(c^k-1)\right)}{p^\alpha}\right)\sideset{}{'}\sum_{n=1}^{p^\alpha}\mathrm{e}\left(\frac{n\left(b(a-1)-d(c-1)\right)}{p^\alpha}\right)$$
$$=\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sideset{}{'}\sum_{d=1}^{p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}\mathrm{e}\left(\frac{m(a^k-1)(b^k-d^k)}{p^\alpha}\right)\sideset{}{'}\sum_{n=1}^{p^\alpha}\mathrm{e}\left(\frac{n(a-1)(b-d)}{p^\alpha}\right)$$
$$=\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sideset{}{'}\sum_{d=1}^{p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}\mathrm{e}\left(\frac{md^k(a^k-1)(b^k-1)}{p^\alpha}\right)\sideset{}{'}\sum_{n=1}^{p^\alpha}\mathrm{e}\left(\frac{nd(a-1)(b-1)}{p^\alpha}\right)$$
$$=\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}\sideset{}{'}\sum_{d=1}^{p^\alpha}\sideset{}{'}\sum_{m=1}^{p^\alpha}\mathrm{e}\left(\frac{m(a^k-1)(b^k-1)}{p^\alpha}\right)\sideset{}{'}\sum_{n=1}^{p^\alpha}\mathrm{e}\left(\frac{n(a-1)(b-1)}{p^\alpha}\right)$$
$$=\phi(p^\alpha)\sideset{}{'}\sum_{a=1}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha}C_{p^\alpha}\left((a^k-1)(b^k-1)\right)C_{p^\alpha}\left((a-1)(b-1)\right),$$

where $C_q(n)=\sideset{}{'}\sum_{a=1}^{q}\mathrm{e}\left(\frac{an}{q}\right)$ is the Ramanujan sum. By Theorem 8.6 of [1] we know that $C_q(n)=\sum_{d|(q,n)}d\mu\left(\frac{q}{d}\right)$, where μ is the Möbius function. Then for any integer n,

$$C_{p^\alpha}(n) = \sum_{d\mid(p^\alpha,n)} d\mu\left(\frac{p^\alpha}{d}\right) = \begin{cases} \phi(p^\alpha), & \text{if } p^\alpha \mid n; \\ -p^{\alpha-1}, & \text{if } p^{\alpha-1} \parallel n; \\ 0, & \text{if } p^{\alpha-1} \nmid n. \end{cases}$$

Therefore,

$$\begin{aligned}
&\sum_{\chi \bmod p^\alpha} {\sum_{m=1}^{p^\alpha}}' |C(m,n_0,k,\chi;p^\alpha)|^4 \\
&= \phi(p^\alpha)\Bigg[\phi^2(p^\alpha) \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^\alpha\mid(a^k-1)(b^k-1)\\ p^\alpha\mid(a-1)(b-1)}} - p^{\alpha-1}\phi(p^\alpha) \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^\alpha\mid(a^k-1)(b^k-1)\\ p^{\alpha-1}\parallel(a-1)(b-1)}} \\
&\qquad - p^{\alpha-1}\phi(p^\alpha) \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^{\alpha-1}\parallel(a^k-1)(b^k-1)\\ p^\alpha\mid(a-1)(b-1)}} + p^{2\alpha-2} \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^{\alpha-1}\parallel(a^k-1)(b^k-1)\\ p^{\alpha-1}\parallel(a-1)(b-1)}}\Bigg] \\
&= \phi(p^\alpha)\Bigg[p^{2\alpha} \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^\alpha\mid(a^k-1)(b^k-1)\\ p^\alpha\mid(a-1)(b-1)}} - p^{2\alpha-1} \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^\alpha\mid(a^k-1)(b^k-1)\\ p^{\alpha-1}\mid(a-1)(b-1)}} \\
&\qquad - p^{2\alpha-1} \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^{\alpha-1}\mid(a^k-1)(b^k-1)\\ p^\alpha\mid(a-1)(b-1)}} + p^{2\alpha-2} \mathop{{\sum_{a=1}^{p^\alpha}}'{\sum_{b=1}^{p^\alpha}}'}_{\substack{p^{\alpha-1}\mid(a^k-1)(b^k-1)\\ p^{\alpha-1}\mid(a-1)(b-1)}}\Bigg] \\
&= p^{2\alpha}\phi(p^\alpha)\Psi_1(k,p^\alpha) - p^{2\alpha-1}\phi(p^\alpha)\Psi_2(k,p^\alpha) \\
&\quad - p^{2\alpha-1}\phi(p^\alpha)\Psi_3(k,p^\alpha) + p^{2\alpha-2}\phi(p^\alpha)\Psi_4(k,p^\alpha).
\end{aligned}$$

Then from Lemma 2.3, Lemma 2.4 and Lemma 2.5 we have

$$\begin{aligned}
&\sum_{\chi \bmod p^\alpha} {\sum_{m=1}^{p^\alpha}}' |C(m,n_0,k,\chi;p^\alpha)|^4 \\
&= \begin{cases} p^{2\alpha}\phi^2(p^\alpha)\left(\alpha+1-\dfrac{\alpha+1+2(k,p-1)}{p}\right), & \alpha \ge 2; \\ p^2(p-1)^2\left(2-\dfrac{2\left((k,p-1)+1\right)}{p}-\dfrac{1}{p^2}+\dfrac{(k,p-1)^2}{p(p-1)}\right), & \alpha = 1. \end{cases} \qquad \square
\end{aligned}$$

3. Proofs of Theorem 1.1 and Corollary 1.3

First we prove Theorem 1.1. Let $q = \prod i = 1^r p_i^{\alpha_i}$, $m = \sum_{i=1}^r m_i \frac{q}{p_i^{\alpha_i}}$. It is clear that if m_i $(i = 1, 2, \cdots, r)$ runs through a reduced residue system modulo $p_i^{\alpha_i}$, then m runs through a reduced residue system modulo q. So from Lemma 2.1 and Lemma 2.6 we have

$$\begin{aligned}
&\sum_{\chi \bmod q} \sum_{m=1}^{q}{}' |C(m, n, k, \chi; q)|^4 \\
&= \prod_{i=1}^{r} \left[\sum_{\chi \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}}{}' \left| C\left(m_i \frac{q}{p_i^{\alpha_i}} \left(\frac{q}{p_i^{\alpha_i}} \right)^{k-1}, n, k, \chi_i; p_i^{\alpha_i} \right) \right|^4 \right] \\
&= \prod_{i=1}^{r} \left[\sum_{\chi \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}}{}' \left| C\left(m_i, n, k, \chi_i; p_i^{\alpha_i}\right) \right|^4 \right] \\
&= \prod_{\substack{i=1 \\ \alpha_i \geq 2}}^{r} \left[\sum_{\chi \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}}{}' \left| C\left(m_i, n, k, \chi_i; p_i^{\alpha_i}\right) \right|^4 \right] \\
&\quad \times \prod_{\substack{i=1 \\ \alpha_i = 1}}^{r} \left[\sum_{\chi \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}}{}' \left| C\left(m_i, n, k, \chi_i; p_i^{\alpha_i}\right) \right|^4 \right] \\
&= \prod_{\substack{p^\alpha \| q \\ \alpha \geq 2}} \left[p^{2\alpha} \phi^2(p^\alpha) \left(\alpha + 1 - \frac{\alpha + 1 + 2(k, p-1)}{p} \right) \right] \\
&\quad \times \prod_{p \| q} \left[p^2 (p-1)^2 \left(2 - \frac{2\left((k, p-1) + 1\right)}{p} - \frac{1}{p^2} + \frac{(k, p-1)^2}{p(p-1)} \right) \right] \\
&= q^2 \phi^2(q) \prod_{\substack{p^\alpha \| q \\ \alpha \geq 2}} \left(\alpha + 1 - \frac{\alpha + 1 + 2(k, p-1)}{p} \right) \\
&\quad \times \prod_{p \| q} \left(2 - \frac{2\left((k, p-1) + 1\right)}{p} - \frac{1}{p^2} + \frac{(k, p-1)^2}{p(p-1)} \right).
\end{aligned}$$

This proves Theorem 1.1.

Now we prove Corollary 1.3. Taking $k \equiv -1 (\bmod \phi(q))$, we have

$$\begin{aligned}
C(m, n, k, \chi; q) &= \sum_{a=1}^{q}{}' \chi(a) \mathrm{e}\left(\frac{m\overline{a} + na}{q} \right) \\
&= \sum_{a=1}^{q}{}' \overline{\chi}(a) \mathrm{e}\left(\frac{ma + n\overline{a}}{q} \right) = K(m, n, \overline{\chi}, q).
\end{aligned}$$

Noting that $(k, \phi(q)) = 1$, then from Theorem 1.1 we immediately get

$$\sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |K(m, n, \chi; q)|^4 = \sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |K(m, n, \overline{\chi}; q)|^4$$
$$= \sum_{\chi \bmod q} \sideset{}{'}\sum_{m=1}^{q} |C(m, n, k, \chi; q)|^4$$
$$= q^2\phi^2(q) \prod_{\substack{p^\alpha \| q \\ \alpha \geq 2}} \left(\alpha + 1 - \frac{\alpha+3}{p}\right) \prod_{p \| q} \left(2 - \frac{4}{p} - \frac{1}{p^2} + \frac{1}{p(p-1)}\right).$$

This completes the proof of Corollary 1.3.

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