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NORMS OF ELEMENTARY OPERATORS

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ABSTRACT. Let A_i and B_i, $1 \le i \le n$, be bounded linear operators acting on a separable Hilbert space $\mathcal{H}$. In this note, we prove that $\sup\{\|\sum_{i=1}^n A_iXB_i\| : X \in \mathcal{B}(\mathcal{H}), \|X\| \le 1\} = \sup\{\|\sum_{i=1}^n A_iUB_i\| : UU^* = U^*U = I, U \in \mathcal{B}(\mathcal{H})\}$. Moreover, we prove that there exists an operator X_0 with $\| X_0 \| = 1$ such that $\| \sum_{i=1}^n A_iX_0B_i \| = \sup\{\| \sum_{i=1}^n A_iXB_i \| : X \in \mathcal{B}(\mathcal{H}), \| X \| \le 1\}$ if and only if there exists a unitary $U_0 \in \mathcal{B}(\mathcal{H})$ such that $\| \sum_{i=1}^n A_iU_0B_i \| = \sup\{\|\sum_{i=1}^n A_iXB_i\| : X \in \mathcal{B}(\mathcal{H}), \|X\| \le 1\}$.

1. Introduction and statement of the main theorem

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. The unit ball $\{A : A \in \mathcal{B}(\mathcal{H})$ and $\| A \| \le 1\}$ and the unitary group $\{U : U \in \mathcal{B}(\mathcal{H})$ and $UU^* = U^*U = I\}$ of $\mathcal{B}(\mathcal{H})$ are denoted, respectively, by $\mathcal{B}(\mathcal{H})_1$ and $\mathcal{U}(\mathcal{H})$. For $A_i, B_i \in \mathcal{B}(\mathcal{H})$, $i = 1, 2, \ldots, n$, the n-tuples $\tilde{A}$ and $\tilde{B}$ are defined, respectively, by

$$\tilde{A} = (A_1, A_2, \ldots, A_n) \text{ and } \tilde{B} = (B_1, B_2, \ldots, B_n).$$

The elementary operator $\delta_{\tilde{A},\tilde{B}}$ on $\mathcal{B}(\mathcal{H})$ induced by $\tilde{A}$ and $\tilde{B}$ is defined by

$$\delta_{\tilde{A},\tilde{B}}X = \sum_{i=1}^{n} A_iXB_i, \text{ for } X \in \mathcal{B}(\mathcal{H}). \tag{1}$$

The norm $\| \delta_{\tilde{A},\tilde{B}} \|$ of the elementary operator $\delta_{\tilde{A},\tilde{B}}$ is defined by

$$\|\delta_{\tilde{A},\tilde{B}}\| = \sup\{\|\sum_{i=1}^{n} A_iXB_i\| : X \in \mathcal{B}(\mathcal{H})_1\}. \tag{2}$$

The elementary operator as an operator on the Banach space $\mathcal{B}(\mathcal{H})$ has attracted much attention of many mathematicians. Some interesting results about the spectra, the ranges and the norms of elementary operators have been obtained (see [1]-[5]).

About the discussion of the norms of elementary operators, one can trace back to Stampfli's theorem in 1970 (see [6]).

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Theorem S (see [5], [6]). Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H})_1\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbb{C}\}.$$

In the present paper, the inspiration originated from the main result obtained recently by Choi and Li in [1].

Theorem C-L (Theorem 2.1 in [1]). Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\sup\{\|U^*AU + V^*BV\| : U, V \in \mathcal{U}(\mathcal{H})\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbb{C}\}.$$

Moreover, the quantity in the above equality is the same as

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H})_1\}.$$

In this note, we shall concentrate on the norms of elementary operators. The main result in this paper is the following.

Theorem 1.1. *Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{B}(\mathcal{H})$ and let $\delta_{\widetilde{A},\widetilde{B}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined by $\delta_{\widetilde{A},\widetilde{B}}(X) = \sum_{i=1}^n A_i X B_i$ for $X \in \mathcal{B}(\mathcal{H})$. Then*

$$\|\delta_{\widetilde{A},\widetilde{B}}\| = \sup_{U \in \mathcal{U}(\mathcal{H})} \|\delta_{\widetilde{A},\widetilde{B}}(U)\|.$$

Moreover, there is a contraction $X \in \mathcal{B}(\mathcal{H})_1$ such that $\|\delta_{\widetilde{A},\widetilde{B}}(X)\| = \|\delta_{\widetilde{A},\widetilde{B}}\|$ if and only if there is a unitary $U \in \mathcal{U}(\mathcal{H})$ such that $\|\delta_{\widetilde{A},\widetilde{B}}(U)\| = \|\delta_{\widetilde{A},\widetilde{B}}\|$.

An elementary operator $\delta_{\widetilde{A},\widetilde{B}}$ is said to be norm-attainable if there is a contraction $X \in \mathcal{B}(\mathcal{H})_1$ such that $\|\delta_{\widetilde{A},\widetilde{B}}(X)\| = \|\delta_{\widetilde{A},\widetilde{B}}\|$.

By Theorem 1.1 and Theorem S, one can deduce Theorem C-L as follows. In fact, in Theorem 1.1, let $n = 2$, $A_1 = A$, $A_2 = I$, $B_1 = I$ and $B_2 = B$. Then

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H}), \|X\| \leq 1\} = \sup\{\|AU + UB\| : U \in \mathcal{U}(\mathcal{H})\}.$$

It is clear that

$$\begin{aligned} \sup\{\|AU + UB\| : U \in \mathcal{U}(\mathcal{H})\} &= \sup\{\|AUV^* + UV^*B\| : U, V \in \mathcal{U}(\mathcal{H})\} \\ &= \sup\{\|U^*AU + V^*BV\| : U, V \in \mathcal{U}(\mathcal{H})\}. \end{aligned}$$

From Theorem S, we have

$$\sup\{\|AX + XB\| : X \in \mathcal{B}(\mathcal{H})_1\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbb{C}\}.$$

Hence,

$$\sup\{\|U^*AU + V^*BV\| : U, V \in \mathcal{U}(\mathcal{H})\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbb{C}\}.$$

2. Proof of the main theorem and auxiliary results

In this section, we begin with some notation and terminology.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. If A is positive, then the unique positive square root of A is denoted by $A^{\frac{1}{2}}$. The spectrum, the null-space and the range of $A \in \mathcal{B}(\mathcal{H})$ are denoted by $\sigma(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. An operator $V \in \mathcal{B}(\mathcal{H})$ is said to be an isometry (or co-isometry) if $V^*V = I$ (or $VV^* = I$). For a subspace $M \subseteq \mathcal{H}$, if $\dim M^\perp$ is infinite, then M is said to be infinite co-dimensional, where $\dim K$ denotes the dimension of a subspace $K \subset \mathcal{H}$ and $K^\perp$ denotes the orthogonal complement of K. The orthogonal projection on M is denoted by P_M.

To complete the proof of Theorem 1.1, we need some auxiliary results.

Lemma 2.1. *If $A \in \mathcal{B}(\mathcal{H})_1$, then there exist two isometries or co-isometries V_1 and V_2 in $\mathcal{B}(\mathcal{H})_1$ such that*

$$A = \frac{1}{2}(V_1 + V_2). \tag{3}$$

Moreover, if $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^*)$, then V_1 and V_2 can be taken to be unitaries.

Proof. Let $A \in \mathcal{B}(\mathcal{H})_1$ and $A = VP$ be the polar decomposition of A. Since $A \in \mathcal{B}(\mathcal{H})_1$, P is a positive contraction in $\mathcal{B}(\mathcal{H})_1$, so $I - P^2$ is also a positive contraction in $\mathcal{B}(\mathcal{H})_1$. Define operators U_1 and U_2 by

$$U_1 = P + i(I - P^2)^{\frac{1}{2}} \text{ and } U_2 = P - i(I - P^2)^{\frac{1}{2}},$$

respectively. Noting that $U_1^* = U_2$ and directly checking, $U_1 U_1^* = U_1^* U_1 = I$ and $U_2 U_2^* = U_2^* U_2 = I$, so U_1 and U_2 are unitaries and

$$P = \frac{1}{2}(U_1 + U_2).$$

If $\dim \mathcal{N}(A) < \dim \mathcal{N}(A^*)$, then V can be taken to be an isometry. Take $V_1 = VU_1$ and $V_2 = VU_2$. Then V_1 and V_2 are isometries and

$$A = VP = V\left(\frac{1}{2}(U_1 + U_2)\right) = \frac{1}{2}(V_1 + V_2). \tag{4}$$

If $\dim \mathcal{N}(A) > \dim \mathcal{N}(A^*)$, then V can taken to be a co-isometry. So V_1 and V_2 in (4) can be taken to be co-isometries.

If $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^*)$, then V can be taken to be a unitary. So V_1 and V_2 in (4) can be taken to be unitaries. □

Corollary 2.2. *If the elementary operator $\delta_{\tilde{A},\tilde{B}}$ is norm-attainable, then there exists an isometry or a co-isometry V_0 such that*

$$\| \delta_{\tilde{A},\tilde{B}}\| = \|\sum_{i=1}^{n} A_i V_0 B_i\| .$$

Proof. If the elementary operator $\delta_{\tilde{A},\tilde{B}}$ is norm-attainable, then there exists an operator $X_0 \in \mathcal{B}(\mathcal{H})_1$ such that

$$\|\delta_{\tilde{A},\tilde{B}}\| = \|\sum_{i=1}^{n} A_i X_0 B_i\| .$$

To complete the proof, it is sufficient to show that X_0 can be represented as an average of two isometries or two co-isometries. By (3) in Lemma 2.1, it is obvious. □

Corollary 2.3. *If $A \in \mathcal{B}(\mathcal{H})_1$ is of finite rank, then there exist two unitaries U_1 and U_2 in $\mathcal{U}(\mathcal{H})$ such that*

$$A = \frac{1}{2}(U_1 + U_2).$$

Proof. Since $A \in \mathcal{B}(\mathcal{H})_1$ is of finite rank, which implies that $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^*)$, the corollary is evident by Lemma 2.1. □

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $x_0 \in \mathcal{H}$ such that $\|Ax_0\| = \|A\|$.

Lemma 2.4. *If $A \in \mathcal{B}(\mathcal{H})$ is not norm-attainable, then there exists an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of orthonormal vectors such that* $\lim_{n\to\infty} \|Ax_n\| = \|A\|$, $Ax_i \neq 0$ *for all* $1 \leq i < \infty$ *and* $Ax_i \perp Ax_j$, $i \neq j$.

Proof. If $A = UP$ is the polar decomposition of A, where U is a partial isometry from $\overline{\mathcal{R}(A^*)}$ onto $\overline{\mathcal{R}(A)}$, then $\|Ax\| = \|Px\|$ for all $x \in \mathcal{H}$. This shows that A is norm-attainable if and only if P is norm-attainable. Denote the spectral decomposition of P by

$$P = \int_0^{\|P\|} \lambda dE_\lambda$$

and suppose A is not norm-attainable. Then P is not norm-attainable either. Hence, $\|P\|$ is not an isolated point of $\sigma(P)$. So we can choose a sequence $\{\alpha_n\}$ of positive numbers such that $\{\alpha_n\}$ is strictly increasing, $\lim_{n\to\infty} \alpha_n = \|P\|$ and the spectral projection $E_n = E([\alpha_n, \alpha_{n+1})) \neq 0$, $1 \leq n < \infty$. For this situation, $[\alpha_n, \alpha_{n+1}) \cap \sigma(P) \neq \emptyset$ and $E_n\mathcal{H}$ is an invariant subspace of P for $1 \leq n < \infty$. Taking a unit vector $x_n \in E_n\mathcal{H}$, it is clear that $\{x_n\}$ is a sequence of orthonormal vectors with $A^*Ax_n = P^2x_n \in E_n\mathcal{H}$; thus $(A^*Ax_i, x_j) = 0$ since $E_iE_j = E_jE_i = 0$ for $i \neq j$, $1 \leq i, j < \infty$. So $(Ax_i, Ax_j) = 0$ for $i \neq j$, $1 \leq i, j < \infty$. In this case,

$$\begin{aligned}\|Ax_n\|^2 &= (A^*Ax_n, x_n) = (P^2x_n, x_n) = (\int_0^{\|P\|} \lambda^2 dE_\lambda x_n, x_n) \\ &= (\int_{\alpha_n}^{\alpha_{n+1}} \lambda^2 dE_\lambda x_n, x_n) \geq \alpha_n^2 \to \| P \|^2 = \|A\|^2.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|Ax_n\| = \| A \| . \qquad \square$$

Lemma 2.5. *For an operator $A \in \mathcal{B}(\mathcal{H})$, the operator A is norm-attainable if and only if its adjoint A^* is norm-attainable.*

Proof. It is enough to show that $A \neq 0$ is norm-attainable implies that A^* is norm-attainable. If A is norm-attainable, then there exists a unit vector x_0 such that $\| Ax_0\| = \|A\|$. That is,

$$A^*Ax_0 = \|A\|^2 x_0.$$

Denote $y_0 = \frac{Ax_0}{\|A\|}$. Then y_0 is a unit vector and $\|A^*y_0\| = \|A\| = \|A^*\|$. Hence A^* is norm-attainable. $\square$

Lemma 2.6. *Let V be an isometry and let $\mathcal{M}$ be an infinite co-dimensional subspace of $\mathcal{H}$. If the restriction of V on $\mathcal{M}$ is denoted by $V\,|_{\mathcal{M}}$, where $V\,|_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ is defined by*

$$V\,|_{\mathcal{M}} = \begin{cases} Vx, & x \in \mathcal{M}, \\ 0, & x \in \mathcal{M}^\perp, \end{cases}$$

then $\dim \mathcal{N}(V\,|_{\mathcal{M}}) = \dim \mathcal{N}((V\,|_{\mathcal{M}})^*) = \infty$.

Proof. It is clear that $\dim \mathcal{N}(V\,|_{\mathcal{M}}) = \infty$.

By the assumption that V is an isometry, if $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$, then $(Vx, Vy) = (V^*Vx, y) = (x, y) = 0$. Hence $\mathcal{R}(V\,|_{\mathcal{M}}) \perp \mathcal{R}(V\,|_{\mathcal{M}^\perp})$. Observing that $\dim \mathcal{R}(V\,|_{\mathcal{M}^\perp}) = \dim \mathcal{M}^\perp = \infty$, so $\dim \mathcal{N}((V\,|_{\mathcal{M}})^*) \geq \dim \mathcal{R}(V\,|_{\mathcal{M}^\perp}) = \infty$. Therefore

$$\dim \mathcal{N}(V\,|_{\mathcal{M}}) = \dim \mathcal{N}((V\,|_{\mathcal{M}})^*) = \infty. \qquad \square$$

Lemma 2.7. *Let* $\{y_i\}_{i=1}^{\infty}$ *be a sequence of unit vectors with* $|(y_i, y_j)| < \frac{1}{2^{max\{i,j\}+4}}$ *for* $i \neq j$, $1 \leq i, j < \infty$. *Then, for each* $k \in \mathbb{N}$, y_k *is not contained in the closed subspace spanned by* $\{y_j, j \neq k\}$.

Proof. Let $\{y_i\}_{i=1}^{\infty}$ be a sequence of unit vectors with $|(y_i, y_j)| < \frac{1}{2^{max\{i,j\}+4}}$ for $i \neq j$. Firstly, we shall show that if $y_{i_0} = \sum_{j \neq i_0} \lambda_j y_j$, then the set $\{|\lambda_j| : j \neq i_0\}$ is bounded. Moreover, $|\lambda_j| \leq 4, 1 \leq j < \infty$.

On the contrary, assume that there exists a k_0 with $|\lambda_{k_0}| > 4$. If $\epsilon < \frac{1}{2}$ is small enough, then there exists an n_0 with $n_0 > k_0$ such that

$$\|y_{i_0} - \sum_{j \neq i_0, j=1}^{n_0} \lambda_j y_j \| < \epsilon.$$

Denote

$$x_{i_0,n_0} = y_{i_0} - \sum_{j \neq i_0, j=1}^{n_0} \lambda_j y_j$$

and $|\lambda_{k_0}| = \max\{|\lambda_j| : 1 \leq j \leq n_0\}$, where $1 \leq k_0 \leq n_0$. Then $y_{i_0} = \sum_{j \neq i_0, j=1}^{n_0} \lambda_j y_j + x_{i_0,n_0}$ and $|\lambda_{k_0}| > 4$. So

$$\begin{aligned}
|(y_{i_0}, y_{k_0})| &= |(\sum_{j \neq i_0, j=1}^{n_0} \lambda_j y_j + x_{i_0,n_0}, y_{k_0})| \\
&\geq |\lambda_{k_0}| - (\sum_{j=1, j \neq i_0, j \neq k_0}^{n_0} \frac{1}{2^{max\{k_0,j\}+4}} |\lambda_j| + \epsilon) \\
&\geq |\lambda_{k_0}| (1 - \sum_{j=1, j \neq i_0, j \neq k_0}^{n_0} \frac{1}{2^{max\{k_0,j\}+4}}) - \epsilon \\
&\geq \frac{1}{2} |\lambda_{k_0}| - \varepsilon \\
&\geq \frac{1}{4} |\lambda_{k_0}| .
\end{aligned}$$

Moreover,

$$\frac{1}{2^{\max\{i_0,k_0\}+4}} \geq |(y_{i_0}, y_{k_0})| \geq \frac{1}{4} |\lambda_{k_0}| \geq 1.$$

This is a contradiction.

Secondly, we shall show that, for $k \in \mathbb{N}$, y_k is not contained in the closed subspace spanned by $\{y_j, j \neq k\}$.

On the contrary again, assume that there exists a vector y_{k_0} such that

$$y_{k_0} = \sum_{j=1, j \neq k_0} \lambda_j y_j.$$

Observing that $|\lambda_j| \leq 4$ for $j \neq k_0$, then

$$\begin{aligned} 1 &= (y_{k_0}, y_{k_0}) \\ &= |\,(\sum_{j=1, j\neq k_0} \lambda_j y_j, y_{k_0})\,| \\ &= |\sum_{j=1, j\neq k_0} (\lambda_j y_j, y_{k_0})\,| \\ &\leq \sum_{j=1, j\neq k_0} \frac{1}{2^{\max\{k_0, j\}+4}} |\lambda_j| \\ &\leq 4(\frac{1}{2^3}(\frac{k_0}{2^{k_0}} + \frac{1}{2^{k_0}})) \leq \frac{1}{2} \ (|\lambda_k| \leq 4). \end{aligned}$$

This is a contradiction. □

From Lemma 2.7, we have the following definition: A sequence $\{y_i\}_{i=1}^{\infty}$ of unit vectors is said to be almost-orthonormal if $|(y_i, y_j)| < \frac{1}{2^{\max\{i,j\}+4}}$ for all $1 \leq i, j < \infty$ and $i \neq j$.

By Lemma 2.7, all the vectors in an almost-orthonormal sequence are linearly independent.

Lemma 2.8. *Let $\{x_i\}_{i=1}^{\infty} \subset \mathcal{H}$ be an infinite sequence of orthonormal vectors. If $\{y_j\}_{j=1}^{n} \subset \mathcal{H}$ is a finite sequence, then, for an arbitrary $\epsilon > 0$, there exists a positive integer i_0 such that*

$$|(y_j, x_i)| < \epsilon, \ \textit{for all } 1 \leq j \leq n \textit{ and } i > i_0.$$

Proof. This is obvious. □

We shall devote the next section to a proof of Theorem 1.1.

Proof of Theorem 1.1. First, we shall show that

$$\|\delta_{\tilde{A},\tilde{B}}\| = \sup\{\|\sum_{i=1}^{n} A_i U B_i\| : U \in \mathcal{U}(\mathcal{H})\}.$$

By the definition of the operator norm, there exists an operator sequence $\{X_m\}_{m=1}^{\infty} \subseteq \mathcal{B}(\mathcal{H})_1$ such that

$$\|\delta_{\tilde{A},\tilde{B}}\| = \lim_{m\to\infty} \|\delta_{\tilde{A},\tilde{B}} X_m\| = \lim_{m\to\infty} \|\sum_{i=1}^{n} A_i X_m B_i\|\,.$$

For each $m \in \mathbb{N}$, there exists a unit vector $x_m \in \mathcal{H}$ such that

$$\|\sum_{i=1}^{n} A_i X_m B_i\| - \|\sum_{i=1}^{n} A_i X_m B_i x_m\| < \frac{1}{m}, 1 \leq i \leq n.$$

Define an operator X_m^0 for $m \in \mathbb{N}$ by

$$\begin{cases} X_m^0 B_i x_m = X_m B_i x_m, & 1 \leq i \leq n; \\ X_m^0 y = 0, & \text{if } y \in (\bigvee\{B_i x_m, 1 \leq i \leq n\})^{\perp}. \end{cases}$$

It is obvious that $\|X_m^0\| \leq \|X_m\| \leq 1$. That is, $X_m^0 \in \mathcal{B}(\mathcal{H})_1$ and

$$\|\sum_{i=1}^{n} A_i X_m B_i\| - \|\sum_{i=1}^{n} A_i X_m^0 B_i x_m\| < \frac{1}{m}, m \in \mathbb{N}.$$

From the definition of X_m^0, X_m^0 is of finite rank and $X_m^0 \in \mathcal{B}(\mathcal{H})_1$, so by Corollary 2.3 there exist two unitaries $U_m^{(1)}$ and $U_m^{(2)}$ for each $m \in \mathbb{N}$ such that

$$X_m^0 = \frac{1}{2}(U_m^{(1)} + U_m^{(2)}), \text{ for } m \in \mathbb{N}.$$

Observing that

$$\begin{aligned}\|\sum_{i=1}^n A_i X_m^0 B_i x_m\| &= \| \sum_{i=1}^n \frac{1}{2} A_i (U_m^{(1)} + U_m^{(2)}) B_i x_m\| \\ &\leq \frac{1}{2}(\|\sum_{i=1}^n A_i U_m^{(1)} B_i x_m\| + \|\sum_{i=1}^n A_i U_m^{(2)} B_i x_m \|)\end{aligned}$$

for $m \in \mathbb{N}$, this shows that at least one of the numbers $\|\sum_{i=1}^n A_i U_m^{(1)} B_i x_m\|$ and $\|\sum_{i=1}^n A_i U_m^{(2)} B_i x_m\|$ is greater than or equal to $\|\sum_{i=1}^n A_i X_m^0 B_i x_m\|$. Without loss of generality, we can assume that

$$\|\sum_{i=1}^n A_i U_m^{(1)} B_i x_m\| \geq \|\sum_{i=1}^n A_i X_m^0 B_i x_m \|, \text{ for } m \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}\| \delta_{\tilde{A},\tilde{B}} \| &\geq \| \sum_{i=1}^n A_i U_m^{(1)} B_i\| \\ &\geq \| \sum_{i=1}^n A_i U_m^{(1)} B_i x_m \| \\ &\geq \| \sum_{i=1}^n A_i X_m^0 B_i x_m \| \\ &\geq \| \sum_{i=1}^n A_i X_m B_i\| - \frac{1}{m}.\end{aligned}$$

So

$$\begin{aligned}\| \delta_{\tilde{A},\tilde{B}}\| &\geq \lim_{m\to\infty} \|\sum_{i=1}^n A_i U_m^{(1)} B_i\| \\ &\geq \lim_{m\to\infty} (\|\sum_{i=1}^n A_i X_m B_i\| - \frac{1}{m}) \\ &= \lim_{m\to\infty} \|\sum_{i=1}^n A_i X_m B_i\| \\ &= \| \delta_{\tilde{A},\tilde{B}} \| .\end{aligned}$$

That is,

$$\| \delta_{\tilde{A},\tilde{B}}\| = \lim_{m\to\infty} \|\sum_{i=1}^n A_i U_m^{(1)} B_i\| .$$

Second, we shall prove that there is a contraction $X \in \mathcal{B}(\mathcal{H})_1$ such that $\| \delta_{\widetilde{A},\widetilde{B}}(X) \| = \| \delta_{\widetilde{A},\widetilde{B}} \|$ if and only if there is a unitary $U \in \mathcal{U}(\mathcal{H})$ such that $\| \delta_{\widetilde{A},\widetilde{B}}(U) \| = \| \delta_{\widetilde{A},\widetilde{B}} \|$.

"$\Leftarrow$" is obvious.

It is enough to consider that "⇒".

For convenience, we divide the part of the proof into three cases.

Case 1. There exists an operator $X_0 \in \mathcal{B}(\mathcal{H})_1$ with $\dim \mathcal{N}(X_0) = \dim \mathcal{N}(X_0^*)$ such that

$$\| \delta_{\tilde{A},\tilde{B}} \| = \| \delta_{\tilde{A},\tilde{B}} X_0 \| = \| \sum_{i=1}^{n} A_i X_0 B_i \| .$$

In this case, by Lemma 2.1, there exist two unitaries U_0^1 and U_0^2 such that

$$X_0 = \frac{1}{2}(U_0^1 + U_0^2).$$

Therefore,

$$\begin{array}{rcl} \| \delta_{\tilde{A},\tilde{B}} \| & = & \| \delta_{\tilde{A},\tilde{B}} X_0 \| \\ & = & \| \delta_{\tilde{A},\tilde{B}} \frac{1}{2}(U_0^1 + U_0^2) \| \\ & \le & \frac{1}{2}(\| \delta_{\tilde{A},\tilde{B}} U_0^1 \| + \| \delta_{\tilde{A},\tilde{B}} U_0^2 \|) \\ & \le & \| \delta_{\tilde{A},\tilde{B}} \| . \end{array}$$

So $\frac{1}{2}(\| \delta_{\tilde{A},\tilde{B}} U_0^1 \| + \| \delta_{\tilde{A},\tilde{B}} U_0^2 \|) = \| \delta_{\tilde{A},\tilde{B}} \|$. That is,

$$\| \delta_{\tilde{A},\tilde{B}} U_0^1 \| = \| \delta_{\tilde{A},\tilde{B}} U_0^2 \| = \| \delta_{\tilde{A},\tilde{B}} \| .$$

Case 2. There exists an operator $X_0 \in \mathcal{B}(\mathcal{H})_1$ such that

$$\| \delta_{\tilde{A},\tilde{B}} \| = \| \delta_{\tilde{A},\tilde{B}} X_0 \| = \| \sum_{i=1}^{n} A_i X_0 B_i \|$$

and $\sum_{i=1}^{n} A_i X_0 B_i \in \mathcal{B}(\mathcal{H})$ is also norm-attainable.

In such a case, there exists a unit vector $x_0 \in \mathcal{B}(\mathcal{H})$ such that

$$\| \sum_{i=1}^{n} A_i X_0 B_i x_0 \| = \| \sum_{i=1}^{n} A_i X_0 B_i \| .$$

Define an operator X_0^0 by

$$\begin{cases} X_0^0 B_i x_0 = X_0 B_i x_0, & 1 \le i \le n; \\ X_0^0 y = 0, & y \in (\bigvee \{B_i x_0, 1 \le i \le n\})^{\perp}. \end{cases}$$

It is obvious that $\| X_0^0 \| \le \| X_0 \| \le 1$, so $X_0^0 \in \mathcal{B}(\mathcal{H})_1$. In this case,

$$\begin{aligned} \| \delta_{\tilde{A},\tilde{B}} \| &= \| \sum_{i=1}^{n} A_i X_0 B_i \| \\ &= \| \sum_{i=1}^{n} A_i X_0 B_i x_0 \| \\ &= \| \sum_{i=1}^{n} A_i X_0^0 B_i x_0 \| \\ &\le \| \sum_{i=1}^{n} A_i X_0^0 B_i \| \\ &\le \| \delta_{\tilde{A},\tilde{B}} \| . \end{aligned}$$

That is, $\|\delta_{\tilde{A},\tilde{B}}\| = \|\sum_{i=1}^{n} A_i X_0^0 B_i\|$. By the definition of X_0^0, X_0^0 is of finite rank. Similar to the proof of Case 1 and by Corollary 2.3, there exists a unitary U_{00} such that $\|\delta_{\tilde{A},\tilde{B}}\| = \|\sum_{i=1}^{n} A_i U_{00} B_i\|$.

Case 3. There exists an operator $X_0 \in \mathcal{B}(\mathcal{H})_1$ with $\dim \mathcal{N}(X_0) \neq \dim \mathcal{N}(X_0^*)$ and $\sum_{i=1}^{n} A_i X_0 B_i \in \mathcal{B}(\mathcal{H})$ is not norm-attainable such that

$$\|\delta_{\tilde{A},\tilde{B}}\| = \|\delta_{\tilde{A},\tilde{B}} X_0\| = \|\sum_{i=1}^{n} A_i X_0 B_i\| .$$

In this case, if $\dim \mathcal{N}(X_0) < \dim \mathcal{N}(X_0^*)$, by Lemma 2.1, we can assume that there exists an isometry V_0 such that

$$\|\delta_{\tilde{A},\tilde{B}}\| = \|\delta_{\tilde{A},\tilde{B}} V_0\| = \|\sum_{i=1}^{n} A_i V_0 B_i\| .$$

If $\sum_{i=1}^{n} A_i V_0 B_i \in \mathcal{B}(\mathcal{H})$ is norm-attainable, by Case 2, there is nothing to do. So, in the next case, we assume that the operator $\sum_{i=1}^{n} A_i V_0 B_i \in \mathcal{B}(\mathcal{H})$ is not norm-attainable.

In this case, by Lemma 2.4, there exists an orthonormal sequence $\{x_m\}_{m=1}^{\infty} \subseteq \overline{\mathcal{R}(\sum_{i=1}^{n} B_i^* V_0^* A_i^*)}$ of vectors such that

$$\lim_{m\to\infty} \|\sum_{i=1}^{n} A_i V_0 B_i x_m\| = \|\sum_{i=1}^{n} A_i V_0 B_i\|, \tag{5}$$

and $\sum_{i=1}^{n} A_i V_0 B_i x_j \perp \sum_{i=1}^{n} A_i V_0 B_i x_k$, $1 \le j,k < \infty$, and $j \neq k$.

Observing that if $\|\delta_{\tilde{A},\tilde{B}}\| = 0$, the discussion is trivial, so we assume that $\|\delta_{\tilde{A},\tilde{B}}\| \neq 0$ in the next case. If $\|\delta_{\tilde{A},\tilde{B}}\| \neq 0$, by Lemma 2.4, there exist a positive number $\alpha > 0$ and $m_0 \in \mathbb{N}$ such that $\|\sum_{i=1}^{n} A_i V_0 B_i x_m\| > \alpha$ for each $m > m_0$ and $\{\sum_{i=1}^{n} A_i V_0 B_i x_m, 1 \le m < \infty\}$ is not contained in a finite-dimensional subspace of $\mathcal{H}$. This shows that $\{B_i x_m\}_{1\le i\le n, 1\le m<\infty}$ is not contained in a finite-dimensional subspace of $\mathcal{H}$. Furthermore, if $\|\delta_{\tilde{A},\tilde{B}}\| \neq 0$, then there does not exist a subsequence $\{z_k\}_{k=1}^{\infty} \subseteq \{x_m\}_{m=1}^{\infty}$ such that $\lim_{k\to\infty} B_i z_k = 0$ for all $1 \le i \le n$. This implies that there exist a subsequence $\{y_j\}_{j=1}^{\infty} \subseteq \{x_m\}_{m=1}^{\infty}$ and $1 \le i_0 \le n$ such that $\{B_{i_0} y_j\}_{j=1}^{\infty}$ is bounded below. Without loss of generality, we can assume that $\{x_m\} = \{y_j\}$ and $i_0 = 1$. Hence, $\{B_1 x_m\}_{m=1}^{\infty}$ is bounded below. That is, there exists a $\delta > 0$ such that $\|B_1 x_m\| > \delta$ for all $1 \le m < \infty$.

Since $\|B_1 x_m\| > \delta$, we have

$$|\,(\frac{B_1 x_i}{\|B_1 x_i\|}, \frac{B_1 x_j}{\|B_1 x_j\|})\,| \le \delta^{-2}\,|\,(B_1^* B_1 x_i, x_j)\,| .$$

Moreover, by Lemma 2.8, there exists a subsequence $\{x_{m_j}\}_{j=1}^{\infty} \subseteq \{x_m\}_{m=1}^{\infty}$ such that

$$|\,(\frac{B_1 x_{m_j}}{\|B_1 x_{m_j}\|}, \frac{B_1 x_{m_k}}{\|B_1 x_{m_k}\|})\,| < \frac{1}{2^{\max\{j,k\}+4}},$$

since $\{x_m\}_{m=1}^{\infty}$ is an infinite orthonormal sequence. This means that $\{\frac{B_1 x_{m_j}}{\|B_1 x_{m_j}\|}\}_{j=1}^{\infty}$ is almost orthonormal. For the sake of convenience, we can think of the subsequence $\{x_{m_j}\}_{j=1}^{\infty}$ as being the same as the sequence $\{x_m\}_{m=1}^{\infty}$. By Lemma 2.7, $B_1 x_{2m}$, $1 \le m < \infty$, are not contained in the closed subspace spanned by $\{B_1 x_{2m-1}\}_{m=1}^{\infty}$. This shows that the closed subspace spanned by $\{B_1 x_{2m-1}\}_{m=1}^{\infty}$ is infinite co-dimensional.

Denote by P the orthogonal projection on the closed subspace $\bigvee\{B_1 x_{2m-1}\}_{m=1}^{\infty}$. Then $P^{\perp}\mathcal{H}$ is an infinite-dimensional subspace.

Next, we shall divide the remainder of the proof into two subcases.

Subcase 1. Suppose that there exists a subsequence $\{z_k\}_{k=1}^{\infty} \subseteq \{x_{2m-1}\}_{m=1}^{\infty}$ such that $P^{\perp}B_i z_k \to 0$ as $k \to \infty$ for every $2 \le i \le n$. Denote by $\mathcal{M}$ the closed subspace $\bigvee\{B_1 z_k : 1 \le k \le \infty\}$. Since $\bigvee\{B_1 z_k : 1 \le k \le \infty\} \subseteq \bigvee\{B_1 x_{2m-1} : 1 \le m \le \infty\}$, $\mathcal{M}$ is infinite co-dimensional. Define $D_0 := V_0 P$. It is clear that $\| D_0 \| \le \| V_0 \| = 1$. Observing that

$$\begin{aligned}
&\lim_{k\to\infty} \|\sum_{i=1}^{n} A_i D_0 B_i z_k \| \\
&= \lim_{k\to\infty} \|\sum_{i=1}^{n} A_i V_0 P B_i z_k \| \\
&= \lim_{k\to\infty} \| \sum_{i=1}^{n} A_i V_0 P B_i z_k + \sum_{i=2}^{n} A_i V_0 P^{\perp} B_i z_k \| \\
&= \lim_{k\to\infty} \|\sum_{i=1}^{n} A_i V_0 P B_i z_k + \sum_{i=1}^{n} A_i V_0 P^{\perp} B_i z_k\| \\
&= \lim_{k\to\infty} \|\sum_{i=1}^{n} A_i V_0 B_i z_k \| \\
&= \lim_{m\to\infty} \| \sum_{i=1}^{n} A_i V_0 B_i x_{2m-1} \| \\
&=\|\sum_{i=1}^{n} A_i V_0 B_i\| = \| \delta_{\tilde{A},\tilde{B}} \|,
\end{aligned}$$

then

$$\| \sum_{i=1}^{n} A_i D_0 B_i \| = \| \delta_{\tilde{A},\tilde{B}} \| .$$

From the definition of D_0, $\dim \mathcal{N}(D_0) = \dim \mathcal{N}(D_0^*) = \infty$ by Lemma 2.6. Furthermore, by Case 1, there exists a unitary U_0 such that

$$\| \sum_{i=1}^{n} A_i U_0 B_i \| = \| \delta_{\tilde{A},\tilde{B}} \| .$$

Subcase 2. Assume that there does not exist a subsequence $\{z_k\}_{k=1}^{\infty} \subseteq \{x_{2m-1}\}_{m=1}^{\infty}$ such that $P^{\perp}B_i z_k \to 0$ as $k \to \infty$ for every $2 \le i \le n$. Then it is obvious that $\bigvee\{P^{\perp}B_i z_k : 2 \le i \le n, 1 \le k < \infty\}$ is infinite dimensional. Denote $C_i = P^{\perp}B_i Q$, $2 \le i \le n$. In such a case, instead of $(B_1, B_2, \ldots, B_n)$, we use $(C_2, C_3, \ldots, C_n)$ and repeat the programme as Subcase 1. Then in at most by $n-1$ steps we can get a subsequence $\{y_l\}_{l=1}^{\infty} \subset \{x_{2m-1}\}_{m=1}^{\infty}$ such that the closed subspace $\bigvee\{B_i y_l : 1 \le i \le n_0 \le n, 1 \le l < \infty\}$ is infinite co-dimensional and $P_{\mathcal{M}_1}^{\perp} B_j y_l \to 0$, as $l \to \infty$ and $n_0 < j \le n$, where if it is necessary that we can change the order of the n-tuple

$(B_1, B_2, \ldots, B_n)$ and let $\mathcal{M}_1$ denote the closed subspace

$$\bigvee\{B_i y_l : 1 \leq i \leq n_0, 1 \leq l < \infty\}.$$

Similar to Subcase 1, define $D_0^2 := V_0 P_{\mathcal{M}_1}$. Then

$$\begin{aligned}
&\lim_{l\to\infty} \|\sum_{i=1}^{n} A_i D_0^2 B_i y_l\| \\
&= \lim_{l\to\infty} \|\sum_{i=1}^{n} A_i V_0 P_{\mathcal{M}_1} B_i y_l \| \\
&= \lim_{l\to\infty} \| \sum_{i=1}^{n} A_i V_0 P_{\mathcal{M}_1} B_i y_l + \sum_{i=n_0+1}^{n} A_i V_0 P_{\mathcal{M}_1}^{\perp} B_i y_l \| \\
&= \lim_{l\to\infty} \| \sum_{i=1}^{n} A_i V_0 P_{\mathcal{M}_1} B_i y_l + \sum_{i=1}^{n} A_i V_0 P_{\mathcal{M}_1}^{\perp} B_i y_l \| \\
&= \lim_{l\to\infty} \| \sum_{i=1}^{n} A_i V_0 B_i y_l\| \\
&= \lim_{m\to\infty} \|\sum_{i=1}^{n} A_i V_0 B_i x_{2m-1} \| = \|\sum_{i=1}^{n} A_i V_0 B_i\| = \| \delta_{\tilde{A},\tilde{B}} \| .
\end{aligned}$$

So there exists a unitary U_0 such that

$$\|\sum_{i=1}^{n} A_i U_0 B_i\| = \| \delta_{\tilde{A},\tilde{B}} \| .$$

If $\dim \mathcal{N}(X_0) > \dim \mathcal{N}(X_0^*)$, by Lemma 2.5, we shall consider the operator $\sum_{i=1}^{n} B_i^* X_0^* A_i^*$ by an argument similar to the above. Then there exists a unitary W such that

$$\| \delta_{\tilde{A},\tilde{B}}\| = \|\sum_{i=1}^{n} B_i^* X_0^* A_i^*\| = \|\sum_{i=1}^{n} B_i^* W A_i^*\| = \|\sum_{i=1}^{n} A_i W^* B_i\| .$$

The proof is completed. □

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