PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 4, April 2008, Pages 1375–1384
S 0002-9939(07)09113-7
Article electronically published on November 23, 2007

ON THE COMPACTNESS OF THE PRODUCT OF HANKEL OPERATORS ON THE SPHERE

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(Communicated by Joseph A. Ball)

ABSTRACT. Consider Hankel operators H_φ and H_ψ on the unit sphere in $\mathbf{C}^n$. If $n = 1$, then a necessary condition for $H_\varphi^* H_\psi$ to be compact is $\lim_{|z|\uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| = 0$. We show that when $n \geq 2$, this condition is no longer necessary for $H_\varphi^* H_\psi$ to be compact.

1. INTRODUCTION

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. Furthermore we normalize σ such that $\sigma(S) = 1$. The Hardy space $H^2(S)$ is the norm closure in $L^2(S, d\sigma)$ of the collection of polynomials in the complex variables $z_1, ..., z_n$ [3, Section 5.6]. Let $P : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. For each $\varphi \in L^\infty(S, d\sigma)$, the Toeplitz operator $T_\varphi : H^2(S) \to H^2(S)$ and the Hankel operator $H_\varphi : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$ are respectively defined by the formulas

$$T_\varphi h = P\varphi h \quad \text{and} \quad H_\varphi h = (1 - P)\varphi h,$$

$h \in H^2(S)$. As usual, let k_z denote the normalized reproducing kernel function for $H^2(S)$. That is, for each $z \in \mathbf{C}^n$ with $|z| < 1$, we write

$$k_z(w) = \frac{(1 - |z|^2)^{n/2}}{(1 - \langle w, z\rangle)^n}, \qquad |w| \leq 1.$$

The main motivation for this investigation comes from the following sufficient condition for the compactness of $H_\varphi^* H_\psi$ due to D. Zheng.

Theorem 1.1 ([6, Theorem 3]). *Let φ and ψ be in BMO. Then the operator $H_\varphi^* H_\psi$ is compact if*

$$\lim_{|z|\uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| = 0. \tag{1.1}$$

Also see the comments on page 22 of [6]. This raises the obvious

Question 1.2. Is (1.1) a necessary condition for the compactness of $H_\varphi^* H_\psi$?

In the case $n = 1$, the answer to this question is affirmative. Indeed Zheng proved

Received by the editors November 30, 2006 and, in revised form, February 14, 2007.
2000 *Mathematics Subject Classification.* Primary 47B07, 47B35.
This work was supported in part by National Science Foundation grant DMS-0456448.

Theorem 1.3 ([5, Theorem 2]). *Suppose that $n = 1$. If the operator $H^*_\varphi H_\psi$ is compact, then*

$$\lim_{|z|\uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| = 0.$$

Moreover, for any complex dimension n, if $H^*_\varphi H_\varphi$ is compact, then one trivially has

$$\lim_{|z|\uparrow 1} \|H_\varphi k_z\|^2 = 0.$$

Given these two facts, and the fact that (1.1) is such a natural-looking condition, one might be tempted to "extrapolate" that the answer to Question 1.2 is affirmative for all $n \in \mathbf{N}$. The purpose of this paper is to report that that is not the case. In other words, Theorem 1.3 is actually something of an anomaly; in the case $n \geq 2$, (1.1) is *not* a necessary condition for the compactness of $H^*_\varphi H_\psi$. More precisely, we will prove

Theorem 1.4. *For each complex dimension $n \geq 2$, there exists a pair of functions φ and ψ in $L^\infty(S, d\sigma)$ such that*

$$\limsup_{|z|\uparrow 1} \|H_\varphi k_z\|\|H_\psi k_z\| > 0 \tag{1.2}$$

and such that the operator $H^*_\varphi H_\psi$ is compact.

This result tells us something that is somewhat anti-intuitive: while Theorem 1.1 cannot be improved in the case $n = 1$, for $n \geq 2$ one should try to prove the compactness of $H^*_\varphi H_\psi$ under a condition that is weaker than (1.1)! This leads to the following question for future investigations.

Question 1.5. For $n \geq 2$, what is a necessary and sufficient condition for the compactness of $H^*_\varphi H_\psi$?

The basic idea behind Theorem 1.4 is the following. First of all, the problem can be converted to a problem for the product of Toeplitz operators. That is, if f and g are real valued and have disjoint supports, then $H^*_f H_g = -T_f T_g$. If there is a positive distance between the supports of f and g, then $T_f T_g$ is compact. Furthermore, if f and g depend only on $|z_1|, ..., |z_{n-1}|$, then T_f and T_g are diagonal operators with respect to the standard orthonormal basis $\{e_i : i \in \mathbf{Z}^n_+\}$ in $H^2(S)$. Therefore $T_f T_g$ is a diagonal operator with eigenvalues $\langle fe_i, e_i\rangle\langle ge_i, e_i\rangle$, $i \in \mathbf{Z}^n_+$. Thus in order for $\|T_f T_g\|$ to be small, it suffices if one of the two factors $|\langle fe_i, e_i\rangle|$, $|\langle ge_i, e_i\rangle|$ is small for each $i \in \mathbf{Z}^n_+$. The fact that we have two factors to manipulate allows us to construct f and g such that $\|T_f T_g\|$ is arbitrarily small while $\|H_f k_0\|\|H_g k_0\|$ has a predetermined lower bound. The desired functions φ and ψ are then obtained from a sequence of such f's, a sequence of such g's, and Möbius transforms. The lower bound for $\|H_f k_0\|\|H_g k_0\|$ translates into (1.2), and the smallness of $\|T_f T_g\|$ results in the compactness of $H^*_\varphi H_\psi$.

The rest of the paper consists of the details of what we described above. Specifically, Section 2 contains the key step, Lemma 2.2. Section 3 recalls Möbius tranforms and associated unitary operators. The proof of Theorem 1.4 is completed in Section 4.

For the rest of the paper we assume $n \geq 2$.

2. Two Toeplitz operators

Let Q denote the "first quadrant" of the closed unit ball in $\mathbf{R}^{n-1}$. In other words,

$$Q = \{(x_1, ..., x_{n-1}) \in \mathbf{R}^{n-1} : x_1^2 + ... + x_{n-1}^2 \leq 1 \text{ and } x_1 \geq 0, ..., x_{n-1} \geq 0\}.$$

On the compact set Q we define the measure $d\mu$ by the formula

$$d\mu(x_1, ..., x_{n-1}) = (n-1)! 2^{n-1} x_1 ... x_{n-1} dx_1 ... dx_{n-1}.$$

Using the technique described on page 17 of [3], it is straightforward to verify that

$$\int_Q x_1^{2i_1} ... x_{n-1}^{2i_{n-1}} d\mu(x_1, ..., x_{n-1}) = \frac{(n-1)! i_1! ... i_{n-1}!}{(n-1+i_1+...+i_{n-1})!}$$

for all integers $i_1, ..., i_{n-1}$ in $\mathbf{Z}_+$. But for such $i_1, ..., i_{n-1}$ we also have

$$\int_S |z_1|^{2i_1} ... |z_{n-1}|^{2i_{n-1}} d\sigma(z_1, ..., z_{n-1}, z_n) = \frac{(n-1)! i_1! ... i_{n-1}!}{(n-1+i_1+...+i_{n-1})!}$$

[3, Proposition 1.4.9]. Hence, by the Stone-Weierstrass approximation theorem, we have

$$\int_S \xi(|z_1|, ..., |z_{n-1}|) d\sigma(z_1, ..., z_{n-1}, z_n) = \int_Q \xi(x_1, ..., x_{n-1}) d\mu(x_1, ..., x_{n-1}) \tag{2.1}$$

for every $\xi \in C(Q)$.

We will use the usual multi-index notation [3, page 3]. For each $i \in \mathbf{Z}_+^n$, define

$$c_i = \left\{ \frac{(n-1+|i|)!}{(n-1)! i!} \right\}^{1/2}$$

and the function

$$e_i(z) = c_i z^i, \quad z \in S.$$

Then $\{e_i : i \in \mathbf{Z}_+^n\}$ is the standard orthonormal basis for $H^2(S)$ [3, Proposition 1.4.9].

For the rest of the paper, we set

$$\delta = \frac{1}{200(n-1)^{1/2}}.$$

With this fixed δ, we define the subsets A and B of S as follows:

$$A = \{(z_1, ..., z_{n-1}, z_n) \in S : \delta < |z_j| < 2\delta \text{ for } 1 \leq j \leq n-1\},$$

$$B = \{(z_1, ..., z_{n-1}, z_n) \in S : (n-1)^{-1/2} - \delta < |z_j| < (n-1)^{-1/2} \text{ for } 1 \leq j \leq n-1\}.$$

Lemma 2.1. (i) *Let $f \in C(S)$ be such that $\|f\|_\infty \leq 1$. Furthermore, suppose that the support of f is contained in A. Then for every $i = (i_1, ..., i_{n-1}, i_n)$ in $\mathbf{Z}_+^n$ satisfying the condition*

$$i_1 + ... + i_{n-1} \geq i_n$$

we have $|\langle f e_i, e_i \rangle| \leq 2^{-|i|/2}$.

(ii) *Let $g \in C(S)$ be such that $\|g\|_\infty \leq 1$. Furthermore, suppose that the support of g is contained in B. Then for every $i = (i_1, ..., i_{n-1}, i_n)$ in $\mathbf{Z}_+^n$ satisfying the condition*

$$i_1 + ... + i_{n-1} \leq i_n$$

we have $|\langle g e_i, e_i \rangle| \leq 2^{n-1}(10/3)^{-|i|/4}$.

Proof. (i) Since $\|f\|_\infty \le 1$ and the support of f is contained in A, it is an easy consequence of (2.1) that for every $i = (i_1, ..., i_{n-1}, i_n) \in \mathbf{Z}_+^n$ we have

$$|\langle fe_i, e_i\rangle| \le \int_A |e_i|^2 d\sigma = c_i^2 \int_{\tilde{A}} x_1^{2i_1} ... x_{n-1}^{2i_{n-1}} (1 - x_1^2 - ... - x_{n-1}^2)^{i_n} d\mu(x_1, ..., x_{n-1}),$$

where

$$\tilde{A} = \{(x_1, ..., x_{n-1}) : \delta < x_j < 2\delta \ \text{ for } \ 1 \le j \le n-1\}, \tag{2.2}$$

which is contained in Q. By the definition of δ, Q also contains the set

$$\tilde{C} = \{(y_1, ..., y_{n-1}) : 4\delta < y_j < 5\delta \ \text{ for } \ 1 \le j \le n-1\}.$$

Also by the definition of δ, if $(y_1, ..., y_{n-1}) \in \tilde{C}$, then $y_1^2 + ... + y_{n-1}^2 < 1/5$. On the other hand, if $(x_1, ..., x_{n-1}) \in \tilde{A}$, then $x_j + 3\delta > x_j + x_j = 2x_j$ for every $1 \le j \le n-1$. Hence

$$\begin{aligned}
1 &= \int_S |e_i|^2 d\sigma = c_i^2 \int_Q y_1^{2i_1} ... y_{n-1}^{2i_{n-1}} (1 - y_1^2 - ... - y_{n-1}^2)^{i_n} d\mu(y_1, ..., y_{n-1}) \\
&\ge c_i^2 \int_{\tilde{C}} y_1^{2i_1} ... y_{n-1}^{2i_{n-1}} (4/5)^{i_n} d\mu(y_1, ..., y_{n-1}) \\
&= (4/5)^{i_n} c_i^2 (n-1)! 2^{n-1} \int_{\tilde{C}} y_1^{2i_1+1} ... y_{n-1}^{2i_{n-1}+1} dy_1 ... dy_{n-1} \\
&= (4/5)^{i_n} c_i^2 (n-1)! 2^{n-1} \int_{\tilde{A}} (x_1 + 3\delta)^{2i_1+1} ... (x_{n-1} + 3\delta)^{2i_{n-1}+1} dx_1 ... dx_{n-1} \\
&\ge (4/5)^{i_n} c_i^2 (n-1)! 2^{n-1} \int_{\tilde{A}} 2^{2(i_1 + ... + i_{n-1})} x_1^{2i_1+1} ... x_{n-1}^{2i_{n-1}+1} dx_1 ... dx_{n-1} \\
&= (4/5)^{i_n} 2^{2(i_1 + ... + i_{n-1})} c_i^2 \int_{\tilde{A}} x_1^{2i_1} ... x_{n-1}^{2i_{n-1}} d\mu(x_1, ..., x_{n-1}) \\
&\ge (4/5)^{i_n} 2^{2(i_1 + ... + i_{n-1})} |\langle fe_i, e_i\rangle|.
\end{aligned}$$

If $i_1 + ... + i_{n-1} \ge i_n$, then $(4/5)^{i_n} 2^{2(i_1 + ... + i_{n-1})} \ge 2^{i_1 + ... + i_{n-1}} \ge 2^{|i|/2}$. This proves (i).

(ii) Since $\|g\|_\infty \le 1$ and the support of g is contained in B, it is an easy consequence of (2.1) that for every $i = (i_1, ..., i_{n-1}, i_n) \in \mathbf{Z}_+^n$ we have

$$|\langle ge_i, e_i\rangle| \le \int_B |e_i|^2 d\sigma = c_i^2 \int_{\tilde{B}} x_1^{2i_1} ... x_{n-1}^{2i_{n-1}} (1 - x_1^2 - ... - x_{n-1}^2)^{i_n} d\mu(x_1, ..., x_{n-1}),$$

where

$$\tilde{B} = \{(x_1, ..., x_{n-1}) : (n-1)^{-1/2} - \delta < x_j < (n-1)^{-1/2} \ \text{ for } \ 1 \le j \le n-1\}, \tag{2.3}$$

which is contained in Q. By the definition of δ, Q also contains the set

$$\tilde{D} = \{(y_1, ..., y_{n-1}) : (n-1)^{-1/2} - 6\delta < y_j < (n-1)^{-1/2} - 5\delta \ \text{ for } \ 1 \le j \le n-1\}.$$

The choice of δ ensures that if $(x_1, ..., x_{n-1}) \in \tilde{B}$, then

$$\begin{aligned}
&1 - x_1^2 - ... - x_{n-1}^2 \le 1/100, \\
&1 - (x_1 - 5\delta)^2 - ... - (x_{n-1} - 5\delta)^2 \ge 1/30, \quad \text{and} \\
&x_j - 5\delta \ge (9/10)x_j \quad \text{for } \ 1 \le j \le n-1.
\end{aligned}$$

Therefore

$$\begin{aligned}
1 &= \int_S |e_i|^2 d\sigma \geq c_i^2 \int_{\tilde{D}} y_1^{2i_1} ... y_{n-1}^{2i_{n-1}} (1 - y_1^2 - ... - y_{n-1}^2)^{i_n} d\mu(y_1, ..., y_{n-1}) \\
&= c_i^2 (n-1)! 2^{n-1} \int_{\tilde{D}} y_1^{2i_1+1} ... y_{n-1}^{2i_{n-1}+1} (1 - y_1^2 - ... - y_{n-1}^2)^{i_n} dy_1 ... dy_{n-1} \\
&= c_i^2 (n-1)! 2^{n-1} \int_{\tilde{B}} \prod_{j=1}^{n-1} (x_j - 5\delta)^{2i_j+1} \cdot \left(1 - \sum_{j=1}^{n-1} (x_j - 5\delta)^2 \right)^{i_n} dx_1 ... dx_{n-1} \\
&\geq (9/10)^{2(i_1 + ... + i_{n-1}) + n - 1} (10/3)^{i_n} \\
&\qquad \times c_i^2 (n-1)! 2^{n-1} \int_{\tilde{B}} x_1^{2i_1+1} ... x_{n-1}^{2i_{n-1}+1} (1 - x_1^2 - ... - x_{n-1}^2)^{i_n} dx_1 ... dx_{n-1} \\
&= (9/10)^{2(i_1 + ... + i_{n-1}) + n - 1} (10/3)^{i_n} \\
&\qquad \times c_i^2 \int_{\tilde{B}} x_1^{2i_1} ... x_{n-1}^{2i_{n-1}} \left(1 - \sum_{j=1}^{n-1} x_j^2 \right)^{i_n} d\mu(x_1, ..., x_{n-1}) \\
&\geq 2^{-(n-1)} (9/10)^{2(i_1 + ... + i_{n-1})} (10/3)^{i_n} |\langle g e_i, e_i \rangle|.
\end{aligned}$$

Since $(9/10)^2 (10/3)^{1/2} > 1$, if $i_n \geq i_1 + ... + i_{n-1}$, then $(9/10)^{2(i_1+...+i_{n-1})}(10/3)^{i_n} \geq (10/3)^{i_n/2} \geq (10/3)^{|i|/4}$. This completes the proof. □

For each $f \in L^2(S, d\sigma)$, denote

$$\mathrm{Var}(f) = \int \left| f - \int f d\sigma \right|^2 d\sigma.$$

Lemma 2.2. *For any given $\epsilon > 0$, there exist real-valued $\tilde{f}, \tilde{g} \in C(Q)$ with $\|\tilde{f}\|_\infty \leq 1$ and $\|\tilde{g}\|_\infty \leq 1$ such that the functions f and g defined by the formulas*

(2.4)

$$f(z_1, ..., z_{n-1}, z_n) = \tilde{f}(|z_1|, ..., |z_{n-1}|) \quad \textit{and} \quad g(z_1, ..., z_{n-1}, z_n) = \tilde{g}(|z_1|, ..., |z_{n-1}|),$$

$(z_1, ..., z_{n-1}, z_n) \in S$, have the following properties:

(α) *The support of f is contained in A and the support of g is contained in B.*

(β) *Var$(f) \geq (1/3)\delta^{2(n-1)}$ and Var$(g) \geq (1/3)\delta^{2(n-1)}$.*

(γ) *$\|T_f T_g\| \leq \epsilon$.*

Proof. Given $\epsilon > 0$, let $N \in \mathbf{N}$ be such that $2^{n-1}(10/3)^{-N/4} \leq \epsilon$. We first show that there is a real-valued $\tilde{f} \in C(Q)$ with $\|\tilde{f}\|_\infty \leq 1$ such that the function f defined by (2.4) has the following properties:

(i) The support of f is contained in A.

(ii) $|\langle f e_i, e_i \rangle| \leq \epsilon$ if $|i| \leq N$.

(iii) $\mathrm{Var}(f) \geq (1/3)\delta^{2(n-1)}$.

To construct such an f, let $F = \{i \in \mathbf{Z}_+^n : |i| \leq N\}$ and let dm_{n-1} denote the standard Lebesgue measure on $\mathbf{R}^{n-1}$. For each $i = (i_1, ..., i_{n-1}, i_n)$ in F, define the function

$$u_i(x_1, ..., x_{n-1}) = c_i^2 (n-1)! 2^{n-1} x_1^{2i_1+1} ... x_{n-1}^{2i_{n-1}+1} (1 - x_1^2 - ... - x_{n-1}^2)^{i_n} \tag{2.5}$$

on Q. Since each u_i is continuous and since card$(F) < \infty$, for the given ϵ we can decompose the cube $\tilde{A}$ defined by (2.2) as the union of a finite family of pairwise disjoint subcubes $\{\tilde{A}_j : j \in J\}$ such that for each $j \in J$ and each $i \in F$, we have

$$|u_i(x) - u_i(y)| \le \epsilon \quad \text{for all } \ x, y \in \tilde{A}_j. \tag{2.6}$$

Now, for each $j \in J$, it is elementary to construct a real-valued function $\tilde{f}_j \in C(Q)$ which has the following properties:

(a) The support of $\tilde{f}_j$ is contained in the interior of $\tilde{A}_j$.
(b) $-1 \le \tilde{f}_j \le 1$.
(c) $m_{n-1}(\{x : \tilde{f}_j(x) = 1\}) \ge (1/3)m_{n-1}(\tilde{A}_j)$.
(d) $m_{n-1}(\{x : \tilde{f}_j(x) = -1\}) \ge (1/3)m_{n-1}(\tilde{A}_j)$.
(e) $\int_{\tilde{A}_j} \tilde{f}_j dm_{n-1} = 0$.

Define $\tilde{f} = \sum_{j\in J} \tilde{f}_j$. Then $\tilde{f} \in C(Q)$, $-1 \le f \le 1$, and the support of $\tilde{f}$ is contained in $\tilde{A}$. Hence the support of f is contained in A, verifying (i).

To verify (ii), apply (2.1), (2.5) and (e). For each $i \in F$ we have

$$\langle fe_i, e_i\rangle = \int_Q u_i \tilde{f} dm_{n-1} = \sum_{j\in J} \int_{\tilde{A}_j} u_i \tilde{f}_j dm_{n-1} = \sum_{j\in J} \int_{\tilde{A}_j} (u_i - u_i(a_j))\tilde{f}_j dm_{n-1},$$

where a_j is any chosen point in $\tilde{A}_j$. Combining this with (2.6), we conclude that

$$|\langle fe_i, e_i\rangle| \le \sum_{j\in J} m_{n-1}(\tilde{A}_j)\epsilon = m_{n-1}(\tilde{A})\epsilon \le \epsilon,$$

$i \in F$. To prove (iii), observe that (c) and (d) together give us the estimate

$$\int_{\tilde{A}} |\tilde{f} - c|^2 dm_{n-1} \ge \sum_{j\in J} \frac{1}{3} m_{n-1}(\tilde{A}_j) = \frac{1}{3} m_{n-1}(\tilde{A}) = \frac{1}{3}\delta^{n-1}$$

for every $c \in \mathbf{C}$. By (2.1) and the fact that $x_1...x_{n-1} \ge \delta^{n-1}$ if $(x_1, ..., x_{n-1}) \in \tilde{A}$, we have

$$\int |f - c|^2 d\sigma = \int_Q |\tilde{f} - c|^2 d\mu \ge \delta^{n-1} \int_{\tilde{A}} |\tilde{f} - c|^2 dm_{n-1} \ge \frac{1}{3}\delta^{2(n-1)}.$$

This proves (iii) and complets the construction of f.

To construct $\tilde{g}$, consider the cube $\tilde{B}$ defined by (2.3). It is elementary that there is a real-valued $\tilde{g} \in C(Q)$ which has the following properties:

(1) The support of $\tilde{g}$ is contained in the interior of $\tilde{B}$.
(2) $-1 \le \tilde{g} \le 1$.
(3) $m_{n-1}(\{x : \tilde{g}(x) = 1\}) \ge (1/3)m_{n-1}(\tilde{B})$.
(4) $m_{n-1}(\{x : \tilde{g}(x) = -1\}) \ge (1/3)m_{n-1}(\tilde{B})$.

Then (3) and (4) together imply that

$$\int_{\tilde{B}} |\tilde{g} - c|^2 dm_{n-1} \ge \frac{1}{3} m_{n-1}(\tilde{B}) = \frac{1}{3}\delta^{n-1}$$

for every $c \in \mathbf{C}$. By (2.3) and the definition of δ, if $(x_1, ..., x_{n-1}) \in \tilde{B}$, then $x_1...x_{n-1} \ge \{(n-1)^{-1/2} - \delta\}^{n-1} \ge \delta^{n-1}$. Thus it follows from (2.4) and (2.1) that

$$\int |g - c|^2 d\sigma = \int_Q |\tilde{g} - c|^2 d\mu \ge \delta^{n-1} \int_{\tilde{B}} |\tilde{g} - c|^2 dm_{n-1} \ge \frac{1}{3}\delta^{2(n-1)}.$$

This establishes (α) and (β).

To prove (γ), note that (2.4) implies $\langle fe_i, e_{i'}\rangle = 0 = \langle ge_i, e_{i'}\rangle$ for all $i \neq i'$ in $\mathbf{Z}_+^n$. Thus the Toeplitz operators T_f and T_g are diagonal operators with respect to the orthonormal basis $\{e_i : i \in \mathbf{Z}_+^n\}$. Consequently

$$T_fT_g = \sum_{i\in\mathbf{Z}_+^n} \langle fe_i, e_i\rangle\langle ge_i, e_i\rangle e_i \otimes e_i.$$

By Lemma 2.1, $|\langle fe_i, e_i\rangle\langle ge_i, e_i\rangle| \le 2^{n-1}(10/3)^{-|i|/4}$ for every $i \in \mathbf{Z}_+^n$. By the choice of N, this gives us $|\langle fe_i, e_i\rangle\langle ge_i, e_i\rangle| \le \epsilon$ in the case $|i| \ge N$. But when $|i| \le N$, it follows from property (ii) for f that $|\langle fe_i, e_i\rangle\langle ge_i, e_i\rangle| \le |\langle fe_i, e_i\rangle| \le \epsilon$. Hence $\|T_fT_g\| \le \epsilon$. $\square$

3. Möbius transform

For each $z \in \mathbf{C}^n$ with $0 < |z| < 1$, define the Möbius transform

$$\varphi_z(w) = \frac{1}{1-\langle w, z\rangle}\left\{z - \frac{\langle w, z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(w - \frac{\langle w, z\rangle}{|z|^2}z\right)\right\}, \qquad |w| \le 1.$$

Then φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = id$ [3, Theorem 2.2.2]. Recall that the formula

$$(U_zf)(w) = f(\varphi_z(w))k_z(w) \tag{3.1}$$

defines a unitary operator on $L^2(S, d\sigma)$ with the property $[U_z, P] = 0$ [4, Section 6]. Therefore for any $f \in L^\infty(S, d\sigma)$, $\|H_{f\circ\varphi_z}k_z\| = \|(1-P)U_zf\| = \|(1-P)f\|$. If f is a real-valued function, then $2\|(1-P)f\|^2 \ge \mathrm{Var}(f)$ [4, (6.3)]. Thus we conclude that

$$\|H_{f\circ\varphi_z}k_z\|^2 \ge \frac{1}{2}\mathrm{Var}(f) \tag{3.2}$$

for every real-valued $f \in L^\infty(S, d\sigma)$.

Recall that the formula $d(u, v) = |1 - \langle u, v\rangle|^{1/2}$, $u, v \in S$, defines a metric on S [3, page 66]. For $u \in S$ and $a > 0$, let $B(u, a)$ denote the open ball with respect to the metric d. That is, we write

$$B(u, a) = \{v \in S : |1 - \langle u, v\rangle|^{1/2} < a\}.$$

Lemma 3.1. *Let* $0 < a < 1$. *If* $1 - (a^4/4) < r < 1$, *then for every* $u \in S$ *we have*

$$\varphi_{ru}(S\backslash B(u, a)) \subset B(u, a).$$

Proof. It is easy to see that $\varphi_{ru}(-u) = u$ if $0 < r < 1$ and $u \in S$. For such r and u, it follows from [3, Theorem 2.2.2] that

$$1 - \langle\varphi_{ru}(w), u\rangle = 1 - \langle\varphi_{ru}(w), \varphi_{ru}(-u)\rangle = \frac{(1-r)(1+\langle w, u\rangle)}{1 - r\langle w, u\rangle}.$$

It is elementary that if $|c| \le 1$ and $0 < r < 1$, then $2|1 - rc| \ge |1 - c|$. Hence

$$|1 - \langle\varphi_{ru}(w), u\rangle| \le \frac{4(1-r)}{|1 - \langle w, u\rangle|}$$

for $0 < r < 1$ and $w, u \in S$. Therefore if $1 - r < a^4/4$ and $|1 - \langle w, u\rangle| \ge a^2$, then $|1 - \langle\varphi_{ru}(w), u\rangle| < a^2$. That is, if $1 - (a^4/4) < r < 1$ and $w \in S\backslash B(u, a)$, then $\varphi_{ru}(w) \in B(u, a)$. $\square$

4. Proof of Theorem 1.4

Let A, B be the same as in Section 2. Then the open set $V = S \backslash (\bar{A} \cup \bar{B})$ is obviously not empty. Thus there exist a sequence of points $\{u_j\}_{j=1}^{\infty}$ in V and a sequence of positive numbers $\{a_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} a_j = 0$ such that $B(u_j, a_j) \subset V$ for every j and

$$B(u_j, 2a_j) \cap B(u_{j'}, 2a_{j'}) = \emptyset \quad \text{if } \; j \neq j'. \tag{4.1}$$

For each $j \in \mathbf{N}$, pick an $r_j \in (1 - (a_j^4/4), 1)$. Then $\lim_{j\to\infty} r_j = 1$. Define $z(j) = r_j u_j$, $j \in \mathbf{N}$. Then Lemma 3.1 tells us that

$$\varphi_{z(j)}(S \backslash B(u_j, a_j)) \subset B(u_j, a_j) \tag{4.2}$$

for every j.

By Lemma 2.2, for each $j \in \mathbf{N}$ there exist real-valued $f_j, g_j \in C(S)$ such that

(1) $\|f_j\|_\infty \leq 1$ and $\|g_j\|_\infty \leq 1$;

(2) the support of f_j is contained in A and the support of g_j is contained in B;

(3) $\mathrm{Var}(f_j) \geq (1/3)\delta^{2(n-1)}$ and $\mathrm{Var}(g_j) \geq (1/3)\delta^{2(n-1)}$;

(4) $\|T_{f_j} T_{g_j}\| \leq 2^{-j}$.

By (4.2) and the fact that $B(u_j, a_j) \subset V = S \backslash (\bar{A} \cup \bar{B})$, the supports of $f_j \circ \varphi_{z(j)}$ and $g_j \circ \varphi_{z(j)}$ are contained in $B(u_j, a_j)$. Combining this with (4.1), we have

$$f_j \circ \varphi_{z(j)} \cdot f_{j'} \circ \varphi_{z(j')} = 0 = g_j \circ \varphi_{z(j)} \cdot g_{j'} \circ \varphi_{z(j')} \quad \text{if } \; j \neq j'. \tag{4.3}$$

Since $f_j g_j = 0$, we also have

$$f_j \circ \varphi_{z(j)} \cdot g_{j'} \circ \varphi_{z(j')} = 0 \quad \text{for all } \; j, j' \in \mathbf{N}. \tag{4.4}$$

Denote $c = (1/6)\delta^{2(n-1)}$. Since f_j, g_j are real-valued, (3.2) tells us that

$$\|H_{f_j \circ \varphi_{z(j)}} k_{z(j)}\| \|H_{g_j \circ \varphi_{z(j)}} k_{z(j)}\| \geq \frac{1}{2}\{\mathrm{Var}(f_j)\mathrm{Var}(g_j)\}^{1/2} \geq c, \tag{4.5}$$

$j \in \mathbf{N}$.

It is well known that $\sigma(B(u, a)) \leq A_0 a^{2n}$ [3, Proposition 5.1.4]. Since $\|f_j\|_\infty \leq 1$ and $\|g_j\|_\infty \leq 1$ and the supports of $f_j \circ \varphi_{z(j)}$ and $g_j \circ \varphi_{z(j)}$ are contained in $B(u_j, a_j)$, we have

$$\lim_{j\to\infty} \|M_{f_j \circ \varphi_{z(j)}} h\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|M_{g_j \circ \varphi_{z(j)}} h\| = 0 \tag{4.6}$$

for every $h \in L^2(S, d\sigma)$. By (4.1) and a trivial estimate using the Cauchy integral formula for P [3, Section 3.2],

$$\lim_{j\to\infty} \|M_{f_j \circ \varphi_{z(j)}} P M_{g_\nu \circ \varphi_{z(\nu)}}\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|M_{f_\nu \circ \varphi_{z(\nu)}} P M_{g_j \circ \varphi_{z(j)}}\| = 0 \tag{4.7}$$

for every $\nu \in \mathbf{N}$. Since $f_\nu \circ \varphi_{z(\nu)}, g_\nu \circ \varphi_{z(\nu)} \in C(S)$, the Hankel operators $H_{f_\nu \circ \varphi_{z(\nu)}}$ and $H_{g_\nu \circ \varphi_{z(\nu)}}$ are compact. Therefore for every $\nu \in \mathbf{N}$ we also have

$$\lim_{j\to\infty} \|H_{f_\nu \circ \varphi_{z(\nu)}} k_{z(j)}\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|H_{g_\nu \circ \varphi_{z(\nu)}} k_{z(j)}\| = 0. \tag{4.8}$$

Using (4.6), (4.7), (4.8) and a standard induction argument, we can select a strictly increasing sequence of natural numbers $j_1 < ... < j_m < ...$ such that the inequalities

$$\sum_{i=1}^{m-1}(\|M_{f_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_i)}\| + \|M_{g_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_i)}\|) \le 2^{-m}, \tag{4.9}$$

$$\sum_{i=1}^{m-1}(\|M_{f_{j_m}\circ\varphi_{z(j_m)}}PM_{g_{j_i}\circ\varphi_{z(j_i)}}\| + \|M_{f_{j_i}\circ\varphi_{z(j_i)}}PM_{g_{j_m}\circ\varphi_{z(j_m)}}\|) \le 2^{-m}, \tag{4.10}$$

$$\sum_{i=1}^{m-1}(\|H_{f_{j_i}\circ\varphi_{z(j_i)}}k_{z(j_m)}\| + \|H_{g_{j_i}\circ\varphi_{z(j_i)}}k_{z(j_m)}\|) \le 2^{-m} \tag{4.11}$$

hold for every $m \ge 2$.

To prove Theorem 1.4, we define

$$\varphi = \sum_{m=1}^{\infty} f_{j_m}\circ\varphi_{z(j_m)} \quad \text{and} \quad \psi = \sum_{m=1}^{\infty} g_{j_m}\circ\varphi_{z(j_m)}. \tag{4.12}$$

By (4.3) and the fact that $\|f_j\|_\infty \le 1$ and $\|g_j\|_\infty \le 1$, we have $\|\varphi\|_\infty \le 1$ and $\|\psi\|_\infty \le 1$. For each $m \ge 2$, it follows from (4.11) and (4.9) that

$$\begin{aligned}
\|H_\varphi k_{z(j_m)}\| &\ge \|H_{f_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\| - \sum_{i=1}^{m-1}\|H_{f_{j_i}\circ\varphi_{z(j_i)}}k_{z(j_m)}\| \\
&\quad - \sum_{i=m+1}^{\infty}\|H_{f_{j_i}\circ\varphi_{z(j_i)}}k_{z(j_m)}\| \\
&\ge \|H_{f_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\| - 2^{-m} - \sum_{i=m+1}^{\infty} 2^{-i} \\
&= \|H_{f_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\| - 2^{-m+1}.
\end{aligned}$$

Similarly, $\|H_\psi k_{z(j_m)}\| \ge \|H_{g_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\| - 2^{-m+1}$. Combining this with (4.5), we have

$$\begin{aligned}
\|H_\varphi k_{z(j_m)}\|\|H_\psi k_{z(j_m)}\| &\ge \|H_{f_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\|\,\|H_{g_{j_m}\circ\varphi_{z(j_m)}}k_{z(j_m)}\| - 2^{-m+2} \\
&\ge c - 2^{-m+2}
\end{aligned}$$

for $m \ge 2$. Since $|z(j_m)| = r_{j_m}$ and $\lim_{m\to\infty} r_{j_m} = 1$, this proves (1.2).

To prove that $H_\varphi^* H_\psi$ is compact, observe that (4.4) gives us $\varphi\psi = 0$. Thus $H_\varphi^* H_\psi = -T_{\bar\varphi}T_\psi = -T_\varphi T_\psi$. Hence it suffices to show that $T_\varphi T_\psi$ is compact. By (4.4) and the fact that $f_j, g_{j'}$ are continuous, the operator $T_{f_j\circ\varphi_{z(j)}}T_{g_{j'}\circ\varphi_{z(j')}}$ is compact for all $j, j' \in \mathbf{N}$. Thus, by (4.12), to prove that $T_\varphi T_\psi$ is compact, it suffices to show that

$$\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\|T_{f_{j_\ell}\circ\varphi_{z(j_\ell)}}T_{g_{j_m}\circ\varphi_{z(j_m)}}\| < \infty. \tag{4.13}$$

We write the above sum as $X + Y$, where

$$X = \sum_{m=1}^{\infty} \|T_{f_{j_m}\circ\varphi_{z(j_m)}} T_{g_{j_m}\circ\varphi_{z(j_m)}}\|,$$

$$Y = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} (\|T_{f_{j_{m+i}}\circ\varphi_{z(j_{m+i})}} T_{g_{j_m}\circ\varphi_{z(j_m)}}\| + \|T_{f_{j_m}\circ\varphi_{z(j_m)}} T_{g_{j_{m+i}}\circ\varphi_{z(j_{m+i})}}\|).$$

By (4.10), we have $Y \le \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} 2^{-(m+i)} < \infty$. Recalling (3.1), we have

$$U_{z(j)} T_{f_j} T_{g_j} U^*_{z(j)} = T_{f_j\circ\varphi_{z(j)}} T_{g_j\circ\varphi_{z(j)}}.$$

Hence $\|T_{f_j\circ\varphi_{z(j)}} T_{g_j\circ\varphi_{z(j)}}\| = \|T_{f_j}T_{g_j}\| \le 2^{-j}$, which leads to the conclusion $X < \infty$. This proves (4.13) and completes the proof of Theorem 1.4.

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