PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 136, Number 4, April 2008, Pages 1205–1211
S 0002-9939(07)09128-9
Article electronically published on November 23, 2007

ON FALTINGS' ANNIHILATOR THEOREM

TAKESI KAWASAKI

(Communicated by Bernd Ulrich)

Dedicated to Professor Shiro Goto on the occasion of his sixtieth birthday

ABSTRACT. In the present article, the author shows that Faltings' annihilator theorem holds for any Noetherian ring A if A is universally catenary; all the formal fibers of all the localizations of A are Cohen-Macaulay; and the Cohen-Macaulay locus of each finitely generated A-algebra is open.

1. INTRODUCTION

Throughout the present article, A always denotes a commutative Noetherian ring. We say that the annihilator theorem holds for A if it satisfies the following proposition [4].

The Annihilator Theorem. *Let M be a finitely generated A-module, n an integer and Y, Z subsets of $\operatorname{Spec} A$ which are stable under specialization. Then the following statements are equivalent*:

(1) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$ *for any* $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ *and* $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$;
(2) *there is an ideal $\mathfrak{b}$ in A such that $V(\mathfrak{b}) \subset Y$ and $\mathfrak{b}$ annihilates the local cohomology modules $H^0_Z(M), \ldots, H^{n-1}_Z(M)$.*

Faltings [3] proved that the annihilator theorem holds for A if A has a dualizing complex or if A is a homomorphic image of a regular ring and that (2) always implies (1). Several authors [1, 2, 9, 10, 11] tried to extend Faltings' result. In this article, the author shows the following

Theorem 1.1. *The annihilator theorem holds for A if*

(C1) *A is universally catenary*;
(C2) *all the formal fibers of all the localizations of A are Cohen-Macaulay*; *and*
(C3) *the Cohen-Macaulay locus of each finitely generated A-algebra is open.*

These conditions are not only sufficient but also necessary for the annihilator theorem. Indeed, Faltings [4] showed that A satisfies (C1)–(C3) whenever the annihilator theorem holds for each A-algebra essentially of finite type.

Received by the editors September 15, 2006 and, in revised form, January 11, 2007.
1991 *Mathematics Subject Classification.* Primary 13D45; Secondary 13C15, 14B15.
Key words and phrases. Annihilators of local cohomologies, Cousin complex, Artin-Rees theorem, Briançon-Skoda theorem.
This work was supported by the Japan Society for the Promotion of Science (the Grant-in-Aid for Scientific Researches (C)(2) 16540032).

These conditions are also related to the uniform Artin-Rees theorem and the uniform Briançon-Skoda theorem. We give an affirmative answer to the conjecture of Huneke [7, Conjecture 2.13] in the last section.

2. Preliminaries

First we recall the definition of the local cohomology functor. A subset Z of $\operatorname{Spec} A$ is said to be stable under specialization if $\mathfrak{p} \in Z$ implies $V(\mathfrak{p}) \subset Z$. Let M be an A-module and Z a subset of $\operatorname{Spec} A$ which is stable under specialization. Then we put

$$H^0_Z(M) = \{m \in M \mid \operatorname{Supp} Am \subset Z\}.$$

This is an A-submodule of M, and $H^0_Z(-)$ is a left exact functor.

Definition 2.1 ([5, p. 223])**.** The local cohomology functor $H^p_Z(-)$ with respect to Z is the right derived functor of $H^0_Z(-)$.

If $\mathfrak{b}$ is an ideal, then $Z = V(\mathfrak{b})$ is stable under specialization and $H^p_Z(-)$ coincides with the ordinary local cohomology functor $H^p_{\mathfrak{b}}(-)$.

Let Z be a subset of $\operatorname{Spec} A$ which is stable under specialization. If $\mathfrak{b}$, $\mathfrak{b}'$ are ideals such that $V(\mathfrak{b})$, $V(\mathfrak{b}') \subset Z$, then $V(\mathfrak{b} \cap \mathfrak{b}') \subset Z$. Therefore the set $\mathcal{F}$ of all ideals $\mathfrak{b}$ such that $V(\mathfrak{b}) \subset Z$ is a directed set with respect to the opposite inclusion. If $\mathfrak{b}$, $\mathfrak{b}' \in \mathcal{F}$ are such that $\mathfrak{b}' \subset \mathfrak{b}$, then there is a natural transformation $\operatorname{Ext}^p_A(A/\mathfrak{b}, -) \to \operatorname{Ext}^p_A(A/\mathfrak{b}', -)$. Since $H^0_Z(-) = \operatorname{inj\,lim}_{\mathfrak{b} \in \mathcal{F}} \operatorname{Hom}(A/\mathfrak{b}, -)$, we obtain the natural isomorphism

$$H^p_Z(-) = \operatorname*{inj\,lim}_{\mathfrak{b} \in \mathcal{F}} \operatorname{Ext}^p_A(A/\mathfrak{b}, -). \tag{2.1.1}$$

The following lemma was essentially given by Raghavan [11, p. 491].

Lemma 2.2. *Let M be a finitely generated A-module. Then $\mathcal{L} = \{H^0_Z(M) \mid Z \subset \operatorname{Spec} A$ is stable under specialization$\}$ is a finite set.*

Proof. Let $\operatorname{Ass} M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ and $0 = M_1 \cap \cdots \cap M_r$ be a primary decomposition of 0 in M where $\operatorname{Ass} M/M_i = \{\mathfrak{p}_i\}$ for all i. Then $H^0_Z(M) = \bigcup_{V(\mathfrak{b}) \subset Z} 0 :_M \mathfrak{b} = \bigcap_{\mathfrak{p}_i \notin Z} M_i$. Therefore $\#\mathcal{L} \leq 2^r$. □

We need Cousin complexes to prove Theorem 1.1.

Let M be a finitely generated A-module. For a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$, the M-height of $\mathfrak{p}$ is defined to be $\operatorname{ht}_M \mathfrak{p} = \dim M_{\mathfrak{p}}$. If $\mathfrak{b}$ is an ideal in A such that $M \neq \mathfrak{b} M$, let $\operatorname{ht}_M \mathfrak{b} = \inf\{\operatorname{ht}_M \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M \cap V(\mathfrak{b})\}$.

Definition 2.3 ([12])**.** The Cousin complex $(M^\bullet, d^\bullet_M)$ of M is defined as follows:

Let $M^{-2} = 0$, $M^{-1} = M$ and $d^{-2}_M \colon M^{-2} \to M^{-1}$ be the zero map. If $p \geq 0$ and $d^{p-2}_M \colon M^{p-2} \to M^{p-1}$ is given, then we put

$$M^p = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Supp} M \\ \operatorname{ht}_M \mathfrak{p} = p}} (\operatorname{Coker} d^{p-2}_M)_{\mathfrak{p}}.$$

If $\xi \in M^{p-1}$ and $\bar{\xi}$ is the image of ξ in $\operatorname{Coker} d^{p-2}_M$, then the component of $d^p_M(\xi)$ in $(\operatorname{Coker} d^{p-2}_M)_{\mathfrak{p}}$ is $\bar{\xi}/1$.

The following theorem contains [6, Theorems 11.4 and 11.5].

Theorem 2.4. *Assume that A satisfies* (C1)–(C3) *and let M be a finitely generated A-module satisfying*

(QU) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{ht}_M \mathfrak{q} = \operatorname{ht}_M \mathfrak{p}$ *for any* $\mathfrak{p}$, $\mathfrak{q} \in \operatorname{Supp} M$ *such that* $\mathfrak{p} \supset \mathfrak{q}$.

Then there is an ideal $\mathfrak{a}$ in A satisfying the following properties:

(1) $V(\mathfrak{a})$ *is the non-Cohen-Macaulay locus of M. In particular,* $\operatorname{ht}_M \mathfrak{a} > 0$.

(2) *Let Z be a subset of* $\operatorname{Spec} A$ *which is stable under specialization and let n be an integer. If* $\operatorname{ht}_M \mathfrak{p} \geq n$ *for any* $\mathfrak{p} \in Z \cap \operatorname{Supp} M$, *then* $\mathfrak{a} H^p_Z(M) = 0$ *for each* $p < n$.

(3) *Let $x_1, \ldots, x_n \in A$ be a sequence. If* $\operatorname{ht}_M(x_1, \ldots, x_n)A \geq n$, *then $\mathfrak{a}$ annihilates the Koszul cohomology module $H^p(x_1, \ldots, x_n; M)$ of M with respect to x_1, $\ldots$, x_n for any $p < n$.*

Proof. Let $M^\bullet$ be the Cousin complex of M and $\mathfrak{a}$ the product of all the annihilators of all the non-zero cohomologies of $M^\bullet$. Then the ideal $\mathfrak{a}$ is well-defined and satisfies (1). See [8, Corollary 6.4].

We prove (2). Because of (2.1.1), it is enough to show that $\mathfrak{a} \operatorname{Ext}^p(A/\mathfrak{b}, M) = 0$ for any ideal $\mathfrak{b}$ such that $V(\mathfrak{b}) \subset Z$ and for any $p < n$. Let $\mathfrak{b}$ be such an ideal and let $F_\bullet$ be a free resolution of $A/\mathfrak{b}$. The double complex $\operatorname{Hom}(F_\bullet, M^\bullet)$ gives two spectral sequences

$$'E_2^{pq} = \operatorname{Ext}^p(A/\mathfrak{b}, H^q(M^\bullet)) \Rightarrow H^{p+q}(\operatorname{Hom}(F_\bullet, M^\bullet)),$$
$$''E_2^{pq} = H^p(\operatorname{Ext}^q(A/\mathfrak{b}, M^\bullet)) \Rightarrow H^{p+q}(\operatorname{Hom}(F_\bullet, M^\bullet)).$$

The first spectral sequence tells us that $\mathfrak{a} H^k(\operatorname{Hom}(F_\bullet, M^\bullet)) = 0$ for any k.

On the other hand, $''E_2^{pq} = 0$ if $p < -1$ or if $q < 0$. Let $0 \leq p < n$ be an integer and $\mathfrak{p} \in \operatorname{Supp} M$ such that $\operatorname{ht}_M \mathfrak{p} = p$. Since $\mathfrak{b} \not\subset \mathfrak{p}$, we find that $\operatorname{Hom}(F_\bullet, (\operatorname{Coker} d_M^{p-2})_\mathfrak{p})$ is exact. Hence $\operatorname{Hom}(F_\bullet, M^p)$ is also exact. Thus $''E_2^{pq} = 0$ if $0 \leq p < n$ and $''E_2^{-1,q} = \operatorname{Ext}^q(A/\mathfrak{b}, M)$. If $k < n$, then $''E_2^{p,k-p-1} = {}''E_2^{p,k-p} = 0$ whenever $p \neq -1$. Therefore $H^{k-1}(\operatorname{Hom}(F_\bullet, M^\bullet)) = {}''E_2^{-1,k} = \operatorname{Ext}^k(A/\mathfrak{b}, M)$ is annihilated by $\mathfrak{a}$.

Next we consider (3). Let $K_\bullet$ be the Koszul complex of A with respect to x_1, $\ldots, x_n$. By considering the double complex $\operatorname{Hom}(K_\bullet, M^\bullet)$, instead of $\operatorname{Hom}(F_\bullet, M^\bullet)$, we obtain the assertion. □

3. The proof of Theorem 1.1

Before the proof of Theorem 1.1, we fix some notation. Let $\mathfrak{X}$ be the free Abelian group with basis $\operatorname{Spec} A$ and let $\mathfrak{X}_+ = \{\sum k_\mathfrak{p} \mathfrak{p} \mid k_\mathfrak{p} \geq 0 \text{ for all } \mathfrak{p}\}$. If $\alpha = k_1\mathfrak{p}_1 + \cdots + k_n\mathfrak{p}_n$ and $\beta = l_1\mathfrak{p}_1 + \cdots + l_n\mathfrak{p}_n$ where $\mathfrak{p}_i \neq \mathfrak{p}_j$ whenever $i \neq j$, then we put

$$\alpha \vee \beta = \sum_{i=1}^n \max\{k_i, l_i\}\mathfrak{p}_i.$$

It is clear that $(\alpha \vee \beta) + \gamma = (\alpha + \gamma) \vee (\beta + \gamma)$. Let $\alpha = k_1\mathfrak{p}_1 + \cdots + k_n\mathfrak{p}_n \in \mathfrak{X}_+$ and let Y be a subset of $\operatorname{Spec} A$ which is stable under specialization. Then we put $\mathfrak{b}(\alpha, Y) = \prod_{\mathfrak{p}_i \in Y} \mathfrak{p}_i^{k_i}$. Since $V(\mathfrak{b}(\alpha, Y)) \subset Y$, Theorem 1.1 is contained in the following

Theorem 3.1. *Assume that A satisfies* (C1)–(C3). *If M is a finitely generated A-module, then there is $\alpha(M) \in \mathfrak{X}_+$ satisfying the following property*:

Let Y, Z be subsets of $\operatorname{Spec} A$ *which are stable under specialization and let n be an integer. If*

(A) $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$ *for any* $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ *and* $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$,

then

(B) $\mathfrak{b}(\alpha(M), Y)$ *annihilates* $H^0_Z(M), \ldots, H^{n-1}_Z(M)$.

We prove this theorem by Noetherian induction on $\operatorname{Supp} M$ and induction on the number of associated primes of M.

If $M = 0$, then $\alpha(M) = 0$ obviously satisfies the assertion. Assume that $M \neq 0$ and that, for any finitely generated A-module M', there is $\alpha(M')$ satisfying the assertion of Theorem 3.1 if $\operatorname{Supp} M' \subsetneq \operatorname{Supp} M$ or if $\operatorname{Supp} M' = \operatorname{Supp} M$ and $\# \operatorname{Ass} M' < \# \operatorname{Ass} M$. We first prove the following claim.

Claim. There is $\alpha'(M) \in \mathfrak{X}_+$ satisfying the following property:

Let Y, Z be subsets of $\operatorname{Spec} A$ which are stable under specialization and let n be an integer. If $Y \cap \operatorname{Ass} M = \emptyset$ and (A) holds, then (B) holds, too.

Proof. Let $\operatorname{Ass} M = \{P_1, \ldots, P_r\}$. We may assume that $P_1 \not\subset P_2, \ldots, P_r$ without loss of generality. There is an exact sequence

$$0 \to L \to M \to N \to 0$$

such that $\operatorname{Ass} L = \{P_2, \ldots, P_r\}$ and $\operatorname{Ass} N = \{P_1\}$. Since A is universally catenary and N has the unique minimal prime, N satisfies (QU). Let $\mathfrak{a}$ be the ideal obtained by applying Theorem 2.4 to N. Then $P_1 \subsetneq \mathfrak{a}$. Since $P_1 \not\subset P_2, \ldots, P_r$, we find that $\mathfrak{a} \not\subset P_2, \ldots, P_r$. Let $x'' \in \mathfrak{a} \setminus (P_1 \cup \cdots \cup P_r)$.

Since $\operatorname{Supp} L \subsetneq \operatorname{Supp} M$ or since $\operatorname{Supp} L = \operatorname{Supp} M$ and $\# \operatorname{Ass} L < \# \operatorname{Ass} M$, there is $\alpha(L) \in \mathfrak{X}_+$ satisfying the assertion of Theorem 3.1. Let $\alpha(L) = k_1 Q_1 + \cdots + k_s Q_s$. We may assume that $Q_1, \ldots, Q_{s_0} \not\subset P_1 \cup \cdots \cup P_r$ and $Q_{s_0+1}, \ldots, Q_s \subset P_1 \cup \cdots \cup P_r$. Let $x' \in Q_1^{k_1} \cdots Q_{s_0}^{k_{s_0}} \setminus P_1 \cup \cdots \cup P_r$ and $x = x'x''$.

Since x is an M-non-zero divisor, $\operatorname{Supp} M/xM \subsetneq \operatorname{Supp} M$. We want to show that $\alpha'(M) = \alpha(M/xM)$ satisfies the assertion of the claim.

Let Y, Z be subsets of $\operatorname{Spec} A$ which are stable under specialization and let n be an integer. Assume that $Y \cap \operatorname{Ass} M = \emptyset$ and $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. If $\mathfrak{p} \in Z \cap \operatorname{Supp} N$, then $\operatorname{ht} \mathfrak{p}/P_1 + \operatorname{depth} M_{P_1} \geq n$ because $\operatorname{Supp} N = V(P_1)$ and $P_1 \notin Y$. Since $\operatorname{depth} M_{P_1} = 0$, we have

$$\operatorname{ht}_N \mathfrak{p} = \operatorname{ht} \mathfrak{p}/P_1 \geq n \quad \text{for any } \mathfrak{p} \in Z \cap \operatorname{Supp} N. \tag{3.1.1}$$

By using Theorem 2.4 (2), we find that $x'' H^p_Z(N) = 0$ for any $p < n$.

Let $\mathfrak{q} \in \operatorname{Spec} A \setminus (Y \cup V(x''A))$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. Since $x'' \notin \mathfrak{q}$, $N_{\mathfrak{q}}$ is Cohen-Macaulay. If $N_{\mathfrak{q}} \neq 0$, then $\mathfrak{p} \in Z \cap \operatorname{Supp} N$ and hence

$$\begin{aligned} \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} N_{\mathfrak{q}} &= \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \dim N_{\mathfrak{q}} \\ &= \operatorname{ht}_N \mathfrak{p} \geq n. \end{aligned}$$

Here we used (3.1.1). If $N_{\mathfrak{q}} = 0$, then $\operatorname{depth} N_{\mathfrak{q}} = \infty$ and hence $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} N_{\mathfrak{q}} \geq n$. Since $\mathfrak{q} \notin Y$, the assumption tells us that $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_{\mathfrak{q}} \geq n$. Therefore $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} L_{\mathfrak{q}} \geq n$. Because of the induction hypothesis,

$$\mathfrak{b}(\alpha(L), Y \cup V(x''A)) H^p_Z(L) = 0$$

for $p < n$.

Since $x'' \notin P_1 \cup \cdots \cup P_r$, $P_1, \ldots, P_r \notin Y$ and $Q_{s_0+1}, \ldots, Q_s \subset P_1 \cup \cdots \cup P_r$, we have $Q_{s_0+1}, \ldots, Q_s \notin Y \cup V(x''A)$. Therefore $x' \in Q_1^{k_1} \cdots Q_{s_0}^{k_{s_0}} \subset \mathfrak{b}(\alpha(L), Y \cup V(x''A))$ and hence $x' H_Z^p(L) = 0$ if $p < n$. Since $H_Z^p(L) \to H_Z^p(M) \to H_Z^p(N)$ is exact, $x H_Z^p(M) = 0$ if $p < n$.

Since x is an M-non-zero divisor, $H_Z^0(M) = 0$ and

$$0 \to H_Z^{p-1}(M) \to H_Z^{p-1}(M/xM) \to H_Z^p(M) \to 0$$

is exact for $p < n$ and $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth}(M/xM)_\mathfrak{q} \geq n-1$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and any $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$. Therefore $\mathfrak{b}(\alpha'(M), Y) = \mathfrak{b}(\alpha(M/xM), Y)$ annihilates $H_Z^p(M)$ if $p < n$. □

Next we construct $\alpha(M)$. Let $\operatorname{Ass} M = \{P_1, \ldots, P_r\}$ and $0 = M_1 \cap \cdots \cap M_r$ be a primary decomposition of 0 in M such that $\operatorname{Ass} M/M_i = \{P_i\}$. Then there are integers $k_1, \ldots, k_r$ such that $P_i^{k_i} M \subset M_i$ for each i.

Let $\{H_Z^0(M) \mid Y \subset \operatorname{Spec} A$ is stable under specialization$\} = \{L_1, \ldots, L_s\}$. Assume that $L_1 = 0$ and $L_2, \ldots, L_s \neq 0$. Since $\operatorname{Supp} M/L_i \subsetneq \operatorname{Supp} M$ or $\operatorname{Supp} M/L_i = \operatorname{Supp} M$, $\# \operatorname{Ass} M/L_i < \# \operatorname{Ass} M$, there is $\alpha(M/L_i) \in \mathfrak{X}_+$ satisfying the assertion of Theorem 3.1 for each $i = 2, \ldots, s$. We put $\alpha(M) = \alpha'(M) \vee [\sum k_i P_i + \alpha(M/L_2) \vee \cdots \vee \alpha(M/L_s)]$. Then $\alpha(M)$ has the required property.

Indeed, let Y, Z be subsets of $\operatorname{Spec} A$ which are stable under specialization and let n be an integer. If $H_Y^0(M) = 0$, then $Y \cap \operatorname{Ass} M = \emptyset$ and hence $\mathfrak{b}(\alpha'(M), Y)$ annihilates $H_Z^0(M), \ldots, H_Z^{n-1}(M)$. Assume that $H_Y^0(M) = L_j$ for some $2 \leq j \leq s$. If $\mathfrak{q} \in \operatorname{Spec} A \setminus Y$ and $\mathfrak{p} \in V(\mathfrak{q}) \cap Z$, then $(L_j)_\mathfrak{q} = 0$ and hence $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth}(M/L_j)_\mathfrak{q} = \operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_\mathfrak{q} \geq n$. Therefore $\mathfrak{b}(\alpha(M/L_j), Y)$ annihilates $H_Z^0(M/L_j), \ldots, H_Z^{n-1}(M/L_j)$. On the other hand, since there is a monomorphism

$$L_j = \bigcap_{P_i \notin Y} M_i \hookrightarrow \bigoplus_{P_i \in Y} M/M_i,$$

we find that $\mathfrak{b}(\sum k_i P_i, Y) L_j = 0$. Since $H_Z^p(L_j) \to H_Z^p(M) \to H_Z^p(M/L_j)$ is exact, $\mathfrak{b}(\sum k_i P_i + \alpha(M/L_j), Y)$ annihilates $H_Z^0(M), \ldots, H_Z^{n-1}(M)$. Thus (B) holds.

If $L_1, \ldots, L_s$ are all non-zero, we put $\alpha(M) = \sum k_i P_i + \alpha(M/L_1) \vee \cdots \vee \alpha(M/L_s)$. We can show that $\alpha(M)$ satisfies the assertion of Theorem 3.1 in the same way as above. The proof of Theorem 1.1 is completed.

The following corollary is an improvement of [11, Theorem 3.1].

Corollary 3.2. *Assume that A satisfies* (C1)–(C3). *If M is a finitely generated A-module, then there is a positive integer k satisfying the following property*:

Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in A and let n be an integer. If $\operatorname{ht} \mathfrak{p}/\mathfrak{q} + \operatorname{depth} M_\mathfrak{q} \geq n$ *for any* $\mathfrak{q} \in \operatorname{Spec} A \setminus V(\mathfrak{b})$ *and* $\mathfrak{p} \in V(\mathfrak{a} + \mathfrak{q})$, *then* $\mathfrak{b}^k H_\mathfrak{a}^p(M) = 0$ *for all* $p < n$.

Proof. Let $\alpha(M) = k_1 \mathfrak{p}_1 + \cdots + k_r \mathfrak{p}_r$ and $k = k_1 + \cdots + k_r$. Then $\mathfrak{b}(\alpha(M), V(\mathfrak{b})) \supset \mathfrak{b}^k$. □

4. A conjecture of Huneke

The following theorem is an affirmative answer to Conjecture 2.13 of [7]. Its proof is similar to that of Theorem 2.4.

Theorem 4.1. *Assume that A satisfies* (C1)–(C3) *and let M be a finitely generated A-module satifying* (QU). *Then there is an ideal $\mathfrak{a}$ in A which satisfies the following requirements*:

(1) $\mathrm{ht}_M\, \mathfrak{a} > 0$.
(2) *If*

$$0 \longrightarrow F^{-n} \xrightarrow{f^{-n}} F^{-n+1} \longrightarrow \cdots \longrightarrow F^{-1} \xrightarrow{f^{-1}} F^0$$

is any complex of finitely generated free A-modules such that
(a) $\operatorname{rank} f^{-n} = \operatorname{rank} F^{-n}$,
(b) $\operatorname{rank} F^i = \operatorname{rank} f^i + \operatorname{rank} f^{i-1}$ *for each* $-n < i < 0$,
(c) $\mathrm{ht}_M\, I_{r_i}(f^i) \geq -i$ *for each* $-n \leq i < 0$ *where* $r_i = \operatorname{rank} f_i$ *for each* i,
then $\mathfrak{a}H^p(F^\bullet \otimes M) = 0$ *for all* $p < 0$. *Here* $I_{r_i}(f^i)$ *denotes the ideal generated by all the* r_i*-minors of the representation matrix of* f^i.

Proof. Let $M^\bullet$ be the Cousin complex of M and let $\mathfrak{a}$ be the product of all the annihilators of all the non-zero cohomologies of $M^\bullet$. Then $\mathfrak{a}$ satisfies (1). The double complex $F^\bullet \otimes M^\bullet$ gives a spectral sequence

$$'E_2^{pq} = H^p(F^\bullet \otimes H^q(M^\bullet)) \Rightarrow H^{p+q}(F^\bullet \otimes M^\bullet),$$

which tells us that $\mathfrak{a}H^p(F^\bullet \otimes M^\bullet) = 0$ for all p. On the other hand, $F^\bullet \otimes M^\bullet$ gives another spectral sequence $''E_2^{pq} \Rightarrow H^{p+q}(F^\bullet \otimes M^\bullet)$ where $''E_2^{pq}$ is the cohomology of

$$H^q(F^\bullet \otimes M^{p-1}) \to H^q(F^\bullet \otimes M^p) \to H^q(F^\bullet \otimes M^{p+1}).$$

If $0 \leq p < n$ and $\mathfrak{p} \in \operatorname{Supp} M$ such that $p = \mathrm{ht}_M\, \mathfrak{p}$, then

$$0 \longrightarrow (F^{-n})_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (F^{-p})_{\mathfrak{p}}$$

is split exact and hence $H^q(F^\bullet \otimes M^p) = 0$ if $q < -p$. Therefore $''E_2^{pq} = 0$ if $p > 0$ and $p + q < 0$. Furthermore $''E_2^{-1,q} = H^q(F^\bullet \otimes M)$ for each $q < 0$. Of course, $''E_2^{pq} = 0$ if $p < -1$. Thus $H^p(F^\bullet \otimes M) = {''E_2^{-1,p}} = H^{p-1}(F^\bullet \otimes M^\bullet)$ is annihilated by $\mathfrak{a}$ if $p < 0$. □

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Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

E-mail address: kawasaki@tmu.ac.jp