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GENERALIZED BI-CIRCULAR PROJECTIONS ON MINIMAL IDEALS OF OPERATORS

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ABSTRACT. We characterize generalized bi-circular projections on $\mathcal{I}(\mathcal{H})$, a minimal norm ideal of operators in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

1. INTRODUCTION

The existence of classes of projections on a given Banach space, as well as their characterizations, are basic problems in Banach Space Theory; see [2], [3], [9], [13], and [16]. Recently, a class of projections, namely bi-circular projections, was proposed by Stachó and Zalar in [17]. A projection is called a bi-circular projection if $e^{i\alpha}P + e^{i\beta}(I - P)$ is an isometry for all α, $\beta \in R$. Such projections have been studied in a variety of settings by Stachó and Zalar; see [18]. Fosner, Ilisevic, and C. K. Li, in [7], considered a generalization of this concept by requiring that $P + \lambda(I - P)$ is an isometry, for some λ with $|\lambda| = 1$. We call these projections generalized bi-circular projections. In this paper, we completely characterize generalized bi-circular projections on minimal norm ideals of Hilbert space operators. It is an easy consequence of our characterization that these projections are bi-contractive. We start by recalling the basic definitions and results to be used throughout the paper.

Definition 1.1. We consider a Banach space X with the norm $\|\cdot\|$. The operator Q (on X) is said to be a generalized bi-circular projection if and only if $Q^2 = Q$ and there exists $\lambda \in C$, $\lambda \neq 1$, and $|\lambda| = 1$, for which $Q + \lambda(I - Q)$ is an isometry of X, denoted by τ.

We observe that τ is a surjective isometry. In fact, if $\omega \in X$, there exists $z \in X$, namely $z = Q(\omega) + \frac{1}{\lambda}(\omega - Q(\omega))$, such that $\tau(z) = \omega$.

Let $\mathcal{H}$ be a complex separable Hilbert space of infinite dimension and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. A *symmetric norm ideal*, $(\mathcal{I}, \nu)$, in $\mathcal{B}(\mathcal{H})$ consists of a two-sided proper ideal $\mathcal{I}$ together with a norm ν on $\mathcal{I}$ satisfying the conditions:

(i) $\nu(A) = \|A\|$, for every rank 1 operator A.

(ii) $\nu(UAV) = \nu(A)$, for every $A \in \mathcal{I}$ and unitary operators U and V on $\mathcal{H}$.

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If the set of finite rank operators is dense in $\mathcal{I}$, then $\mathcal{I}$ is a minimal norm ideal; see [11].

The isometries of minimal norm ideals were characterized by Sourour, in [12]. For completeness of exposition we state Sourour's result. First, we recall the concept of transpose of an operator relative to a fixed orthonormal basis, $\{e_i\}$, for the Hilbert space. We denote by T^t the transpose of the operator T.

Definition 1.2. Given $T \in B(\mathcal{H})$, the transpose T^t is defined to be the unique operator in $B(\mathcal{H})$ such that

$$\langle T^t e_i, e_j \rangle = \langle T e_j, e_i \rangle.$$

Theorem 1.3 (Sourour [12])**.** *If $\mathcal{I}$ is a minimal ideal in $B(\mathcal{H})$ different from $C_2(\mathcal{H})$, and $\mathcal{U}$ is a linear transformation on $\mathcal{I}$, then $\mathcal{U}$ is a surjective isometry of $\mathcal{I}$ if and only if there exist unitaries U and V on $\mathcal{H}$ such that*

$$\mathcal{U}(T) = UTV \quad or \quad \mathcal{U} = UT^tV$$

for every $T \in \mathcal{I}$.

In addition to the result of Sourour, the following theorem due to Fong and Sourour (cf. [15]) will also be used in our proofs.

The operators $\{A_i\}_{i=1,\dots,m}$ and $\{B_i\}_{i=1,\dots,m}$ are bounded operators on the Banach space X and Φ acts on $B(X)$ as follows:

$$\Phi(T) = A_1TB_1 + A_2TB_2 + \cdots + A_mTB_m.$$

Theorem 1.4 (Fong and Sourour [15])**.** *If $\Phi(T) = O$, for all $T \in B(X)$, then $\{B_1, B_2, \cdots, B_m\}$ is linearly dependent. Furthermore, if $\{B_1, B_2, \cdots, B_n\}$ $(n \leq m)$ is linearly independent, and the (c_{kj}) denote constants for which*

$$B_j = \sum_{k=1}^{n} c_{kj} B_k, \quad n+1 \leq j \leq m,$$

then $\Phi(T) \equiv 0$ for all $T \in B(X)$ if and only if

$$A_k = -\sum_{j=n+1}^{m} c_{kj} A_j \;, \quad 1 \leq k \leq n.$$

If $n = m$, then $A_1 = A_2 = \cdots = A_m = 0$.

2. Generalized bi-circular projections on ideals of Hilbert space operators

In [4], the authors have shown that generalized bi-circular projections on certain Banach spaces are the average of the identity with an isometric reflection, i.e. an isometry R such that $R^2 = Id$. We show in this paper that a similar characterization holds for ideals of Hilbert space operators. In anticipation of this result, we now give a simple characterization of isometric reflections on minimal norm ideals.

Lemma 2.1. *The operator τ on $\mathcal{I}$ is an isometric reflection if and only if either*

(1) *$\tau(T) = UT^tV$ with U and V unitary operators on $\mathcal{H}$ so that $V = \pm(U^t)^*$, or*

(2) *$\tau(T) = UTV$, with U and V isometries of the form $U = \sqrt{\overline{\alpha}}P_0 - \sqrt{\overline{\alpha}}(Id - P_0)$ and $V = \sqrt{\alpha}P_1 - \sqrt{\alpha}(Id - P_1)$, where P_0 and P_1 are projections onto closed subspaces of $\mathcal{H}$ and α is a complex number of modulus 1.*

Proof. This lemma is a straightforward consequence of Fong and Sourour's theorem. If $\tau(T) = U\,T^t\,V$, then $\tau^2 = Id$ if and only if $UV^tTU^tV - T = 0$, for all $T \in \mathcal{I}$. Therefore, $U^tV = \alpha Id$, for some complex number α of modulus 1, and $VU^t = \overline{\alpha} Id$. This implies that $\alpha = \overline{\alpha}$ and hence $\alpha = \pm 1$. Consequently $V = \pm (U^t)^*$.

If $\tau(T) = U\,T\,V$, then $\tau^2 = Id$ if and only if $U^2TV^2 - T = 0$, for all $T \in \mathcal{I}$. This implies that $V^2 = \alpha Id$, with $|\alpha| = 1$, and $U^2 = \overline{\alpha} Id$. The second statement now follows from the spectral theorem applied to U and V; see [14]. □

Proposition 2.2. *Let $(\mathcal{I}, \nu)$ be a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class) and let Q be a generalized bi-circular projection on $\mathcal{I}$. If Q is associated with a surjective isometry τ on $\mathcal{I}$ of the form $\tau(T) = UT^tV$ (U and V unitary operators on $\mathcal{H}$), then Q is the average of the identity with an isometric reflection.*

Proof. If Q is a generalized bi-circular projection, then

$$Q(T) = \frac{1}{1-\lambda}\left[-\lambda T + UT^tV\right]$$

and

$$\lambda T - (\lambda+1)UT^tV + UV^tTU^tV = O, \tag{2.1}$$

for every $T \in \mathcal{I}$. We first observe that for $\lambda = -1$, the equation (2.1) reduces to

$$-T + U\,V^t\,T\,U^t\,V = O, \ \text{ for all } T \in \mathcal{I}.$$

Fong and Sourour's theorem implies that $U^tV = \alpha Id$ and $Id = \alpha UV^t$, for some modulus 1 complex number α. Therefore $\alpha^2 = 1$ and $V = \pm\,U^{*t}$. The projection Q is the average of the identity with the isometric reflection: $R(T) = \pm UT^tU^{*t}$. Now, we consider $\lambda \neq -1$. We show that there are no unitary operators U and V for which the equation (2.1) holds for every $T \in \mathcal{I}$.

We fix λ, with modulus 1 and different from -1. If there exists a pair of unitary operators $(U,\,V)$ so that equation (2.1) holds for every $T \in \mathcal{I}$, then

$$T^t = \frac{1}{\lambda+1}U^*\left[\lambda T + UV^tTU^tV\right]V^*$$

and

$$T = \frac{1}{\lambda+1}V^{*t}\left[\lambda T^t + V^t\,U\,T^t\,V\,U^t\right]U^{*t}.$$

Therefore

$$T = \frac{1}{(\lambda+1)^2}\left[\lambda^2V^{*t}U^*TV^*U^{*t} + 2\lambda T + U\,V^tTU^tV\right],$$

or equivalently

$$-(\lambda^2+1)T + \lambda^2V^{*t}U^*TV^*U^{*t} + UV^tTU^tV = O, \ \ \forall T \in \mathcal{I}. \tag{2.2}$$

Fong and Sourour's theorem implies that $\{Id,\ V^*U^{*t},\ U^tV\}$ is linearly dependent and we consider the following cases:

I. $U^t\,V = \alpha Id$; then $V^*\,U^{*t} = \overline{\alpha} Id$ and $V = \alpha U^{*t}$. Equation (2.1) becomes

$$(\lambda+\alpha^2)T - (\lambda+1)\alpha UT^tU^{*t} = O.$$

This also implies that $\left(\frac{\lambda+\alpha^2}{(\lambda+1)\alpha}\right)^2 = 1$ and $TU^t = \pm\,UT^t,\ \ \forall\,T \in \mathcal{I}$. We show that this is impossible. We first consider $T = e_i \otimes e_j$, with $i \neq j$. Therefore, we have that $TU^t(e_k) = \langle\,U^t(e_k),\,e_j\rangle\,e_i = \pm U\,T^t(e_k) = \pm\langle e_k,\,e_i\rangle\,U(e_j)$. Therefore for $k \neq i$

we have $\langle U(e_j), e_k\rangle = 0$, and then $U(e_j) = \nu_j e_i$, for some modulus 1 complex number ν_j. On the other hand, if $T = e_i \otimes e_i$, we have $TU^t(e_k) = \langle U^t(e_k), e_i\rangle e_i = \pm U T^t(e_k) = \pm\langle e_k, e_i\rangle U(e_i)$. This implies that $\langle U(e_i), e_k\rangle = 0$, for every $k \neq i$. This implies that, for every i, $U(e_i) = \mu_i e_i$. Then, given $j \neq i$, we have that $U(e_j) = \nu_j e_i = \mu_j e_j$, which is impossible. Therefore there exist no U and V (unitary operators) so that $U^t V = \alpha Id$, and the equation (2.1) holds for every $T \in \mathcal{I}$.

II. $U^tV = \alpha Id + \beta V^*U^{*t}$ $(\beta \neq 0)$. We also have that $\alpha UV^t = (\lambda^2 + 1)Id$ and $-\beta UV^t = \lambda^2 V^{*t}U^*$. These equations imply that $U^{*t} = -\frac{\beta}{\lambda^2} V U^t V$, and $\left(1 + \frac{\beta^2}{\lambda^2}\right) U^tV = \alpha Id$. We observe that $\alpha = 0$; then $\lambda^2 = -1$ and $\beta = \pm 1$. Therefore $U^tV = \pm V^*U^{*t}$, $(UV^t)^2 = Id$, and $(U^tV)^2 = \pm Id$. As previously considered, we let $T = e_i \otimes e_j$ $(i \neq j)$; then $(\lambda + 1)\langle V(e_i), e_i\rangle U(e_j) = \langle U^tV(e_i), e_j\rangle UV^t(e_i)$. If there exists i so that $\langle V(e_i), e_i\rangle \neq 0$, then $V^t(e_i) = \frac{(\lambda+1)\langle V(e_i), e_i\rangle}{\langle U^tV(e_i), e_j\rangle} e_j$, for $j \neq i$. Let $\alpha_j = \frac{(\lambda+1)\langle V(e_i), e_i\rangle}{\langle U^tV(e_i), e_j\rangle}$; then $|\alpha_j| \geq |\lambda + 1||\langle V(e_i), e_i\rangle| \neq 0$. Then the series $\sum_k |\langle V^t(e_i), e_k\rangle|^2$ diverges which would contradict Parseval's identity. Therefore, we must have that for every i, $\langle V(e_i), e_i\rangle = 0$. This implies that $\langle U^tV(e_i), e_j\rangle = 0$, for $j \neq i$ and $U^tV(e_i) = \alpha_i e_i$. Since $(U^tV)^2 = \pm Id$ we have that $\alpha_i^2 = \pm 1$.

We evaluate $\lambda T - (\lambda + 1)UT^tV + UV^tTU^tV = O$ with $T = e_i \otimes e_j$ at $V^*(e_i)$, and obtain that

$$\lambda\langle V^*(e_i), e_j\rangle e_i - (\lambda + 1)U(e_j) + \alpha_i \langle U(e_j), e_i\rangle e_i = O.$$

Therefore

$$\lambda\langle V^*(e_i), e_j\rangle \langle e_i, U(e_j)\rangle - (\lambda + 1) + \alpha_i \langle U(e_j), e_i\rangle \langle e_i, U(e_j)\rangle = 0,$$

which implies that $(\lambda\overline{\alpha}_i + \alpha_i)|\langle e_i, U(e_j)\rangle|^2 = \lambda + 1$, and $|\langle e_i, U(e_j)\rangle|^2 = \frac{\lambda+1}{\lambda\overline{\alpha}_i+\alpha_i)}$. Clearly $\lambda\overline{\alpha}_i + \alpha_i \neq 0$ because $\lambda \neq -1$ and $\alpha_i^2 = \pm 1$. Similarly to the previous case we have $|\langle U^*(e_i), e_k\rangle|$ must converge to zero. Thus, $\alpha = 0$ yields no solution and it is left to analyze the case where $\alpha \neq 0$ and $\beta \neq 0$. This will imply that $UV^t = \frac{\lambda^2+1}{\alpha} Id$ and hence $U^tV = \frac{\lambda^2+1}{\alpha} Id$. For $T = e_i \otimes e_j$ $(i \neq j)$, we have that $\lambda T(e_k) - (\lambda + 1)UT^t(V(e_k)) + \frac{\lambda^2+1}{\alpha} UV^tT(e_k) = 0$. If $k \neq j$, then $\langle V(e_k), e_i\rangle = 0$. This implies that $\langle V^*(e_i), e_k\rangle = 0$ for all k. This contradiction completes the proof. □

Proposition 2.3. *Let $(\mathcal{I}, \nu)$ be a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class) and let Q be a generalized bi-circular projection on $\mathcal{I}$. If Q is associated with a surjective isometry τ on $\mathcal{I}$ of the form $\tau(T) = UTV$ (U and V unitary operators on $\mathcal{H}$) and $\lambda \neq -1$, then Q is given as*

$$Q(T) = P_F T \text{ or } Q(T) = TP_F,$$

where P_F represents a projection on $\mathcal{H}$ onto a closed subspace F.

Proof. Since Q is a generalized bi-circular projection then

$$Q(T) = \frac{1}{1 - \lambda} [-\lambda T + UTV]$$

and

$$\lambda T - (\lambda + 1)UTV + U^2TV^2 = O, \tag{2.3}$$

for every $T \in \mathcal{I}$.

It follows from Fong and Sourour's theorem that $\{Id, V, V^2\}$ must be linearly dependent. Therefore we have two cases to analyze: **1)** $V = \alpha Id$ and $V^2 = \alpha^2 Id$, and **2)** $V^2 = \alpha Id + \beta V$, for some complex numbers α and β.

1) If $V = \alpha Id$ and $V^2 = \alpha^2 Id$, then $|\alpha| = 1$ and $\lambda Id = \alpha(1+\lambda)U - \alpha^2 U^2$. The spectral representation of U is therefore of the form

$$U = \overline{\alpha}\lambda P_{ker\{\alpha U - \lambda Id\}} + \overline{\alpha} P_{ker\{\alpha U - Id\}},$$

where $P_{ker\{\alpha U - \lambda Id\}}$ represents the projection onto $ker\{\alpha U - \lambda Id\}$. Therefore $Q(T) = P_{ker\{\alpha U - Id\}} T$.

2) If $V^2 = \alpha Id + \beta V$, for some complex numbers α and β, then $\lambda Id = -\alpha U^2$ and $(\lambda+1)U = \beta U^2$. This implies that $|\alpha| = 1$ and that $\alpha = -\frac{\lambda\beta^2}{(\lambda+1)^2}$. Therefore the spectral theorem applied to V implies the following representation:

$$V = \frac{\beta\lambda}{\lambda+1} P_{Ker(V - \frac{\beta\lambda}{\lambda+1} Id)} + \frac{\beta}{\lambda+1} P_{Ker(V - \frac{\beta}{\lambda+1} Id)}.$$

We also notice that $P_{V - \frac{\beta}{\lambda+1} Id} + P_{V - \frac{\beta\lambda}{\lambda+1} Id} = Id$. Therefore, if we denote $Ker(V - \frac{\beta}{\lambda+1} Id)$ by F, we have that

$$Q(T) = \frac{1}{1-\lambda}\left[-\lambda T + UTV\right] = TP_F.$$

This completes the proof of the proposition. □

Corollary 2.4. *If $Q(T) = P_F T$ or $Q(T) = TP_F$, where P_F represents a projection on $\mathcal{H}$, then Q is the average of the identity with an isometric reflection.*

Proof. We consider $Q(T) = P_F T$ and show that Q is the average of the identity with an isometric reflection. If λ is a modulus 1 complex number, let U_λ be defined on $\mathcal{H}$ by $U_\lambda(v) = P_F(v) + \lambda(Id - P_F)(v)$. It is easy to see that U_λ is a surjective isometry. In fact, given $v \in \mathcal{H}$ we have that

$$\|U_\lambda(v)\|^2 = \|P_F(v)\|^2 + \|(Id - P_F)(v)\|^2 = \|v\|^2.$$

The surjectivity follows, since $U_\lambda(P_F(v) + \lambda^{-1}(Id - P_F)(v)) = v$. On the other hand, we have that $\tau(T) = 2P_F T - T$ is an isometry of $\mathcal{I}$ since it can be written as $\tau(T) = U_{-1} T Id$. It follows that $Q(T) = \frac{1}{2}(Id + \tau)(T)$. If $Q(T) = TP_F$ the proof follows similarly. □

Proposition 2.5. *Let $(\mathcal{I}, \nu)$ be a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class) and let Q be a generalized bi-circular projection on $\mathcal{I}$. If Q is associated with a surjective isometry τ on $\mathcal{I}$ of the form $\tau(T) = UTV$ (U and V unitary operators on $\mathcal{H}$) and $\lambda = -1$, then Q is the average of the identity with an isometric reflection on $\mathcal{H}$.*

Proof. The operator $Q(T) = \frac{1}{2}\left[T + UTV\right]$ is a projection if and only if $T = U^2 T V^2$ for every operator $T \in \mathcal{I}$. Therefore we have that $V^2 = \alpha Id$ and $U^2 = \overline{\alpha} Id$. □

Theorem 2.6. *If $(\mathcal{I}, \nu)$ is a separable minimal norm ideal different from $\mathcal{C}_2$ (the Hilbert-Schmidt class), then Q is a generalized bi-circular projection on $\mathcal{I}$ if and only if Q is the average of the identity with an isometric reflection.*

Proof. If a projection Q is the average of the identity with an isometric reflection, denoted by R, then $R = Q - (Id - Q)$ and Q is a generalized bi-circular projection with $\lambda = -1$. Conversely, if Q is a generalized bi-circular projection the statement in this theorem follows from Proposition 2.3, Corollary 2.4, and Proposition 2.5. □

The following corollary is an immediate consequence of the previous results.

Corollary 2.7. *Every generalized bi-circular projection on $\mathcal{I}$ is a bi-contractive projection, i.e. $\|P\| \leq 1$ and $\|Id - P\| \leq 1$.*

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